Research Article

Eigenvalue of Coupled Systems for Hammerstein Integral Equation with Two Parameters

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By using the fixed-point index theory, we discuss the existence, multiplicity, and nonexistence of positive solutions for the coupled systems of Hammerstein integral equation with parameters.

1. Introduction

In recent years, the study of solutions for Hammerstein integral equations has been an interesting topic, since the solution of some boundary value problems for differential equations are usually equivalent to solutions of Hammerstein integral equations [1–6]. And many results concerning the existence of solutions for Hammerstein integral equations have been obtained by many authors [1, 7–9]. For example, in [1] the Hammerstein integral equation:

$$\psi(x) = \int_G k(x, y) f(y, \psi(y)) dy, \quad (1.1)$$

was considered, where $G \subset \mathbb{R}^n$ is a bounded domain. When the nonlinear term is of the form $f(u)$ or $f(x, u) = \sum_{i=1}^n a_i(x) u_i^{\alpha_i}$, $\alpha_i > 0$; $i = 1, 2, \dots, n$, some existence results of nonnegative solutions in $C(G)$ for (1.1) were obtained; when the nonlinear term is a general $f(x, u)$, some multiple results for (1.1) in space $L_p(G)$ ($p \geq 1$) were derived. In [8], by means of the decomposition of the operator and the critical point theory, the existence of infinitely many solutions for (1.1) was considered. In [7] the integral equation

$$\psi(x) = \lambda \int_{[x, x+T]} k(x, y) f(y, \psi(y - \tau(y))) dy \quad (1.2)$$

was studied, where λ is a parameter. Using the Leggett-Williams fixed-point theorem, when λ belongs to some intervals, the existence of triple positive solutions for (1.2) was proved.

More recently, in [9] the integral equation

$$u(x) = \lambda \int_0^1 k(x, y) f(y, u(y)) dy, \quad (1.3)$$

was studied and obtained that there exists a $\lambda^* > 0$ such that (1.3) has at least two, one and no positive solutions for $\lambda \in (0, \lambda^*)$, $\lambda = \lambda^*$, and $\lambda > \lambda^*$, respectively.

Motivated by the papers mentioned above, we consider the coupled systems of Hammerstein integral equation (1.4) in this paper,

$$\begin{aligned} u(x) &= \lambda \int_0^1 k_1(x, y) f_1(y, u(y), v(y)) dy, \\ v(x) &= \mu \int_0^1 k_2(x, y) f_2(y, u(y), v(y)) dy, \end{aligned} \quad (1.4)$$

where $\lambda, \mu \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$, $\mathbb{R}_+ = [0, +\infty)$. Under some new assumptions, we show that there exists a continuous curve Γ separating $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (1.4) has at least two, one and no positive solutions for $\mathcal{O}_1$, Γ and $\mathcal{O}_2$, respectively. Proofs of our results are mainly based on the fixed-point index theory, for this type of results see [10–12].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence, multiplicity, and nonexistence of positive solution of the systems (1.4).

The vector (u^*, v^*) is said to be a positive solution of problem (1.4) if and only if (u^*, v^*) satisfies problem (1.4) and $u^*(x) \geq 0, v^*(x) > 0$, or $u^*(x) > 0, v^*(x) \geq 0$ for any $x \in [0, 1]$.

2. Preliminaries and Lemmas

In the rest of the paper, we always suppose the following assumptions hold:

- (H₁) $f_i \in C[[0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $i = 1, 2$;
- (H₂) $k_i \in C[[0, 1] \times [0, 1], \mathbb{R}_+]$, $i = 1, 2$;
- (H₃) there exist $\sigma, \alpha, \beta \in (0, 1)$, $\alpha < \beta$ such that $k_i(x, y) \geq \sigma k_i(y, y)$ for $y \in (0, 1)$ and $x \in [\alpha, \beta]$, $i = 1, 2$;
- (H₄) f_i are nondecreasing on $\mathbb{R}_+^2$ for $x \in [0, 1]$, that is, $f_i(x, u_1, v_1) \leq f_i(x, u_2, v_2)$, $i = 1, 2$ whenever $(u_1, v_1) \leq (u_2, v_2)$, where the inequality on $\mathbb{R}_+^2$ can be understood componentwise and $f_1(x, 0, 0) > 0$ or $f_2(x, 0, 0) > 0$ for all $x \in [0, 1]$;
- (H₅) there exist constants $c_{f_i} > 0$, $i = 1, 2$, such that $f_i(x, u, v) \geq c_{f_i}(u + v)$ for all $x \in [0, 1]$;
- (H₆) $\lim_{\|(u, v)\| \rightarrow \infty} (f_i(x, u, v)/(u + v)) = \infty$, $i = 1, 2$ uniformly for $x \in [0, 1]$.

We will consider the Banach space $E = C[0, 1] \times C[0, 1]$ equipped with the standard norm

$$\|(u, v)\| = \|u\| + \|v\| = \max_{0 \leq x \leq 1} |u(x)| + \max_{0 \leq x \leq 1} |v(x)|, \quad (u, v) \in C[0, 1] \times C[0, 1]. \quad (2.1)$$

Define

$$P = \left\{ (u, v) \in E : u(x) \geq 0, v(x) \geq 0, \min_{x \in [\alpha, \beta]} (u(x) + v(x)) \geq \sigma \|(u, v)\| \right\}. \quad (2.2)$$

It is easy to see that P is a cone in E .

We define the operators $T_\lambda, T_\mu : P \rightarrow C[0, 1]$ and $T : P \rightarrow C[0, 1] \times C[0, 1]$ by

$$\begin{aligned} T_\lambda(u, v)(x) &= \lambda \int_0^1 k_1(x, y) f_1(x, u(y), v(y)) dy, \\ T_\mu(u, v)(x) &= \mu \int_0^1 k_2(x, y) f_2(y, u(y), v(y)) dy, \\ T(u, v) &= (T_\lambda(u, v), T_\mu(u, v)), \quad \forall (u, v) \in P. \end{aligned} \quad (2.3)$$

Obviously, the existence of a positive solution of problem (1.4) is equivalent to the existence of a nontrivial fixed point of T in P . It is easy to prove that the following lemma is true.

Lemma 2.1. *Assume that (H_1) – (H_3) hold. Then $T(P) \subset P$ and $T : P \rightarrow P$ is completely continuous.*

Finally we list two lemmas, which are crucial to prove our main results.

Lemma 2.2 (see [13]). *Let E be a Banach space, K a cone and Ω an open bounded subset of E . Let $\theta \in \Omega$ and $T : K \cap \overline{\Omega} \rightarrow K$ be a completely continuous mapping. Suppose that $Tx \neq \varpi x$ for all $x \in K \cap \partial\Omega$ and all $\varpi \geq 1$. Then $i(T, K \cap \Omega, K) = 1$.*

Lemma 2.3 (see [13]). *Let E be a Banach space and K a cone in E . Assume that $T : \overline{K_r} \rightarrow K$ ($K_r = \{x \in K, \|x\| < r, r > 0\}$) is a compact map such that $Tx \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for all $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*

3. Main Results

In this section, we consider the existence of positive solutions for (1.4) in E .

Lemma 3.1. *Assume that (H_1) – (H_6) hold, and let Σ be a compact subset of $\mathbb{R}_+^2 \setminus \{(0, 0)\}$. Then there exists a constant $C_\Sigma > 0$ such that for all $(\lambda, \mu) \in \Sigma$ and all possible positive solutions $(u(x), v(x))$ of (1.4) at (λ, μ) , one has $\|(u, v)\| \leq C_\Sigma$.*

Proof. Suppose by contradiction that there is a sequence (u_n, v_n) of positive solutions of (1.4) at (λ_n, μ_n) such that $(\lambda_n, \mu_n) \in \Sigma$ for all n and $\|(u_n, v_n)\| \rightarrow \infty$. Then $(u_n, v_n) \in P$ and thus

$$\min_{x \in [\alpha, \beta]} (u_n(x) + v_n(x)) \geq \sigma \|(u_n, v_n)\|. \quad (3.1)$$

Since Σ is compact, the sequence $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty}$ has a convergent subsequence which we denote without loss of generality still by $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda^*$, $\lim_{n \rightarrow \infty} \mu_n = \mu^*$, and at least one $\lambda^* > 0$ or $\mu^* > 0$, hence for n sufficiently large, we have $\lambda_n \geq \lambda^*/2 > 0$.

Then by (H_6) , there exists $R_{f_1} > 0$ such that

$$f_1(t, u, v) \geq K_{f_1}(u + v), \quad \forall u + v \geq R_{f_1}, \quad (3.2)$$

where K_{f_1} satisfies

$$\frac{\lambda^* \sigma^2 K_{f_1}}{2} \int_{\alpha}^{\beta} k_1(y, y) dy > 1. \quad (3.3)$$

Thus for $x \in [\alpha, \beta]$ and $\forall u_n + v_n \geq R_{f_1}$, by using (3.1) and (3.2), we get

$$\begin{aligned} \|u_n\| &\geq u_n(x) \\ &= \lambda_n \int_0^1 k_1(x, y) f_1(y, u_n(y), v_n(y)) dy \\ &\geq \frac{\lambda^*}{2} \int_{\alpha}^{\beta} k_1(x, y) f_1(y, u_n(y), v_n(y)) dy \\ &\geq \frac{\lambda^* K_{f_1}}{2} \int_{\alpha}^{\beta} k_1(x, y) (u_n(y) + v_n(y)) dy \\ &\geq \frac{\lambda^* \sigma^2 K_{f_1}}{2} \int_{\alpha}^{\beta} k_1(y, y) (\|u_n\| + \|v_n\|) dy \\ &> \|u_n\| + \|v_n\| \geq \|u_n\| \end{aligned} \quad (3.4)$$

for all n sufficiently large. This is a contraction. $\square$

Lemma 3.2. Assume that (H_1) – (H_6) hold, and let (1.4) have a positive solution at $(\bar{\lambda}, \bar{\mu})$. Then the problem also has a positive solution at (λ, μ) for all $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$.

Proof. Let $(\bar{u}, \bar{v})$ be a positive solution of (1.4) at $(\bar{\lambda}, \bar{\mu})$, and let $(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$ with $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$. First, we assume $\lambda \in (0, \bar{\lambda})$, and $\mu \in (0, \bar{\mu})$, then

$$\begin{aligned} u_{\bar{\lambda}}(x) &= \bar{\lambda} \int_0^1 k_1(x, y) f_1(y, u_{\bar{\lambda}}(y), v_{\bar{\mu}}(y)) dy \geq \lambda \int_0^1 k_1(x, y) f_1(y, u_{\bar{\lambda}}(y), v_{\bar{\mu}}(y)) dy, \\ v_{\bar{\mu}}(x) &= \bar{\mu} \int_0^1 k_2(x, y) f_2(y, u_{\bar{\lambda}}(y), v_{\bar{\mu}}(y)) dy \geq \mu \int_0^1 k_2(x, y) f_2(y, u_{\bar{\lambda}}(y), v_{\bar{\mu}}(y)) dy. \end{aligned} \quad (3.5)$$

Set

$$\begin{aligned} T_\lambda(u, v)(x) &= \lambda \int_0^1 k_1(x, y) f_1(y, u(y), v(y)) dy, \\ T_\mu(u, v)(x) &= \mu \int_0^1 k_2(x, y) f_2(y, u(y), v(y)) dy, \end{aligned} \quad (3.6)$$

and let $u_0(x) = u_{\bar{\lambda}}(x)$, $v_0(x) = v_{\bar{\mu}}(x)$, $u_n(x) = T_\lambda(u_{n-1}(x), v_{n-1}(x))$, $v_n(x) = T_\mu(u_{n-1}(x), v_{n-1}(x))$, $T(u_n(x), v_n(x)) = (T_\lambda(u_n(x), v_n(x)), T_\mu(u_n(x), v_n(x)))$, $n = 1, 2, \dots$. Then

$$\begin{aligned} u_0 &\geq u_1 \geq \dots \geq u_n \geq \dots \geq \sigma \lambda c_{f_1} K_1 > \theta, \\ v_0 &\geq v_1 \geq \dots \geq v_n \geq \dots \geq \sigma \mu c_{f_2} K_2 > \theta, \end{aligned} \quad (3.7)$$

where $K_i = \int_\alpha^\beta k_i(y, y) dy$, $n = 1, 2$. By the compactness of the operator $T_\lambda(u, v)(x)$, $T_\mu(u, v)(x)$, Lemma 2.1, and the Lebesgue dominated convergence theorem, the sequence $\{u_n\}_{n=0}^\infty = \{T_\lambda(u_{n-1}, v_{n-1})\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty = \{T_\mu(u_{n-1}, v_{n-1})\}_{n=0}^\infty$ converges to u_* and v_* , respectively. It is clear that $(u_*, v_*) \in P \setminus \{(0, 0)\}$ and is a solution (1.4) at (λ, μ) . Proof of the case $\mu = 0$, and $\lambda \in (0, \bar{\lambda})$ or $\lambda = 0$, and $\mu \in (0, \bar{\lambda})$ can be done similarly. The proof is complete. $\square$

Lemma 3.3. Assume that (H_1) – (H_4) hold. Then there exists $(\lambda_*, \mu_*) > (0, 0)$ such that (1.4) has a positive solution for all $(\lambda, \mu) \leq (\lambda_*, \mu_*)$.

Proof. Let $u_0(x) = \int_0^1 k_1(x, y) dy$, $v_0(x) = \int_0^1 k_2(x, y) dy$, then $(u_0(x), v_0(x)) \in P \setminus \{(0, 0)\}$. Take $M_{f_i} = \max_{x \in [0, 1]} f_i(x, u_0(x), v_0(x))$, $i = 1, 2$. Then $M_{f_i} > 0$ and at $(\lambda_*, \mu_*) = (1/M_{f_1}, 1/M_{f_2})$, we get

$$\begin{aligned} T_{\lambda_*}(u, v)(x) &= \lambda_* \int_0^1 k_1(x, y) f_1(y, u(y), v(y)) dy, \\ T_{\mu_*}(u, v)(x) &= \mu_* \int_0^1 k_2(x, y) f_2(y, u(y), v(y)) dy. \end{aligned} \quad (3.8)$$

This shows that $(\lambda_*, \mu_*) > (0, 0)$ and

$$\begin{aligned} u_0(x) &\geq \lambda_* \int_0^1 k_1(x, y) f_1(y, u_0(y), v_0(y)) dy, \\ v_0(x) &\geq \mu_* \int_0^1 k_2(x, y) f_2(y, u_0(y), v_0(y)) dy \end{aligned} \quad (3.9)$$

for $x \in [0, 1]$. Set

$$\begin{aligned} u_n(x) &= T_{\lambda_*}(u_{n-1}(x), v_{n-1}(x)), \quad v_n(x) = T_{\mu_*}(u_{n-1}(x), v_{n-1}(x)), \\ T(u_n(x), v_n(x)) &= (T_{\lambda_*}(u_n(x), v_n(x)), T_{\mu_*}(u_n(x), v_n(x))), \quad n = 1, 2, \dots \end{aligned} \quad (3.10)$$

for $x \in [0, 1]$. Then

$$\begin{aligned} u_0 \geq u_1 \geq \cdots \geq u_n \geq \cdots \geq \sigma \lambda_* \int_0^1 k_1(y, y) dy &> \theta, \\ v_0 \geq v_1 \geq \cdots \geq v_n \geq \cdots \geq \sigma \mu_* \int_0^1 k_2(y, y) dy &> \theta. \end{aligned} \quad (3.11)$$

We conclude the proof similarly to Lemma 3.2. $\square$

Define $\mathcal{P} = \{(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0, 0)\} : (1.4) \text{ has a positive solution at } (\lambda, \mu)\}$; then by Lemma 3.3, $\mathcal{P} \neq \emptyset$, and it is easy to see that $(\mathcal{P}, \leq)$ is a partially ordered set.

Lemma 3.4. *Assume that (H_1) – (H_6) hold. Then $(\mathcal{P}, \leq)$ is bounded above.*

Proof. Suppose to the contrary that there exists a fixed-point sequence $\{(u_n, v_n)\}$ ($n = 1, 2, \dots$) of $T(u, v)$ at (λ_n, μ_n) such that $\lim_{n \rightarrow \infty} \|(\lambda_n, \mu_n)\| = \infty$. Considering a subsequence if necessary, we assume $\lim_{n \rightarrow \infty} \lambda_n = \infty$. The proof for the case $\lim_{n \rightarrow \infty} \mu_n = \infty$ can be shown by an analogous way. Then there are two cases to be considered:

- (i) there exists a constant $H > 0$ such that $\|(u_n, v_n)\| \leq H$, $n = 0, 1, 2, \dots$;
- (ii) there exists a subsequence $\{(u_{n_i}, v_{n_i})\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} \|(u_{n_i}, v_{n_i})\| = \infty$,

which is impossible by Lemma 3.1. So we only consider (i). In view of (H_4) – (H_6) we can choose $l_0 > 0$ such that $f_1(y, 0, 0) > l_0 H$ or $f_2(y, 0, 0) > l_0 H$ and further $f_1(y, u_n, v_n) > l_0 H$ or $f_2(y, u_n, v_n) > l_0 H$ for $y \in [0, 1]$. Thus for $x \in [\alpha, \beta]$, we know

$$\begin{aligned} u_n(x) + v_n(x) &= \lambda_n \int_0^1 k_1(x, y) f_1(y, u_n(y), v_n(y)) dy \\ &\quad + \mu_n \int_0^1 k_2(x, y) f_2(y, u_n(y), v_n(y)) dy \\ &\geq \lambda_n \sigma \int_\alpha^\beta k_1(y, y) f_1(y, u_n(y), v_n(y)) dy \\ &\quad + \mu_n \sigma \int_\alpha^\beta k_2(y, y) f_2(y, u_n(y), v_n(y)) dy. \end{aligned} \quad (3.12)$$

Now we will distinguish two cases.

Case 1. If $f_1(y, u_n, v_n) > l_0 H$ and $f_2(y, u_n, v_n) > l_0 H$, we have

$$u_n(x) + v_n(x) \geq \lambda_n \sigma l_0 H K_1 + \mu_n \sigma l_0 H K_2 \geq (\lambda_n + \mu_n) \sigma l_0 H K_0, \quad (3.13)$$

where $K_0 = \min(K_1, K_2)$, which implies that $H \geq (\lambda_n + \mu_n) \sigma l_0 H K_1$ or $(\lambda_n + \mu_n) \leq 1 / \sigma l_0 K_1$, which is a contradiction.

Case 2. If that one of $f_1(y, u_n, v_n) > l_0 H$ and $f_2(y, u_n, v_n) > l_0 H$ is satisfied, without loss of generality, we assume $f_1(y, u_n, v_n) > l_0 H$. The proof for the case $f_2(y, u_n, v_n) > l_0 H$ can be shown by an analogous way. By (H_5) we have

$$\begin{aligned} u_n(x) + v_n(x) &\geq \lambda_n \sigma l_0 H K_1 + \mu_n \sigma^2 \|(u_n, v_n)\| c_{f_2} K_2 \\ &\geq \lambda_n \sigma l_0 \|(u_n, v_n)\| K_1 + \mu_n \sigma^2 \|(u_n, v_n)\| c_{f_2} K_2 \\ &\geq (\lambda_n + \mu_n) K_3 \|(u_n, v_n)\|, \end{aligned} \quad (3.14)$$

where $K_3 = \min(\sigma l_0 K_1, \sigma^2 c_{f_2} K_2)$; which implies that $\|(u_n, v_n)\| \geq (\lambda_n + \mu_n) K_3 \|(u_n, v_n)\|$ or $(\lambda_n + \mu_n) \leq 1/K_3$, which is a contradiction. The proof is complete. $\square$

Lemma 3.5. Assume that (H_1) – (H_6) hold. Then every chain in $\mathcal{P}$ has a unique supremum in $\mathcal{P}$.

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Since $\mathcal{P}$ is a partially ordered set, it is enough to show that $\mathcal{C}$ has an upper bound in $\mathcal{P}$. Without loss of generality, we may choose a distinct sequence $\{(\lambda_n, \mu_n)\} \in \mathcal{C}$ such that $(\lambda_n, \mu_n) \leq (\lambda_{n+1}, \mu_{n+1})$, $n = 1, 2, \dots$. By Lemma 3.4, two sequences $\{\lambda_n\}$ and $\{\mu_n\}$ converge to, say, λ_c and μ_c , respectively. If $(\lambda_c, \mu_c) \in \mathcal{P}$, then the proof is done. Since the sequence $\{(\lambda_n, \mu_n)\}$ is bounded above, we may assume that the sequence belongs to a compact rectangle in $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ and Lemma 3.1 implies that the corresponding solutions $\{(u_n, v_n)\}$ are uniformly bounded in E . By the compactness of the integral operators T_λ and T_μ , the sequence $\{(u_n, v_n)\}$ has a subsequence converging to, say, $\{(u_c, v_c)\} \in E$. We can easily show, by the Lebesgue convergence theorem, that (u_c, v_c) is a solution of (1.4) at (λ_c, μ_c) . Thus $(\lambda_c, \mu_c) \in \mathcal{P}$ and this completes the proof. $\square$

Lemma 3.6. Assume that (H_1) – (H_6) hold. Then there exists $s^* \in [\mu_*, \mu_u]$ such that (1.4) has a positive solution at $(0, s)$ for all $0 < s \leq s^*$, and no solution at $(0, s)$ for all $s > s^*$. Similarly, there exists $r^* \in [\lambda_*, \lambda_u]$ such that (1.4) has a positive solution at $(r, 0)$ for all $0 < r \leq r^*$, and no solution at $(r, 0)$ for all $r > r^*$, where (λ_u, μ_u) is upper bound of $(\mathcal{P}, \leq)$.

Proof. We know by Lemma 3.3 that (1.4) has a positive solution at $(0, s)$ for all $0 < s \leq \mu_*$. Thus $\{(0, s) : s > 0\} \cap \mathcal{P}$ is a nonempty chain in $\mathcal{P}$ and by Lemma 3.5, it has a unique supremum of the form $(0, s^*)$ in $\mathcal{P}$. The proof of the second part is similar. $\square$

Lemma 3.7. Assume that (H_1) – (H_6) hold. Then there exists a continuous curve Γ separating $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (1.4) has at least one positive solution for $(\lambda, \mu) \in \mathcal{O}_1 \cup \Gamma$ and no solution for $(\lambda, \mu) \in \mathcal{O}_2$.

Proof. We first construct the curve Γ on $\mathbb{R}_+^2 \setminus \{(0, 0)\}$. Define

$$L_x = \{(r, s) \in \mathbb{R}_+^2 \setminus \{(0, 0)\} : s = r - x\}, \quad x \in \mathbb{R}. \quad (3.15)$$

At $x = -s^*$, we know by Lemma 3.6, $(0, s^*) \in L_{-s^*} \cap \mathcal{P}$. Thus $L_{-s^*} \cap \mathcal{P}$ is a nonempty chain in $\mathcal{P}$ and Lemma 3.5 implies that the chain has a unique supremum. We show $\sup\{L_{-s^*} \cap \mathcal{P}\} = (0, s^*)$. Indeed, otherwise, we may choose $(\tilde{r}, \tilde{s}) (\neq (0, s^*)) \in L_{-s^*}$ such that $(0, s^*) < (\tilde{r}, \tilde{s})$ and (1.4) has a solution at $(\tilde{r}, \tilde{s})$. Thus by Lemma 3.2, (1.4) has a solution at $(0, \tilde{s})$ and this contradicts Lemma 3.6. Similarly, we get $\sup\{L_{r^*} \cap \mathcal{P}\} = (r^*, 0)$.

For $-s^* < x < r^*$, we know by Lemmas 3.3, 3.5, and 3.6 that $L_x \cap \mathcal{P}$ is a nonempty chain in $\mathcal{P}$ and thus the chain also has a unique supremum.

We notice that $L_x \cap \mathcal{P} = \emptyset$, for $x < -s^*$ or $x > r^*$. Now for $x \in [-s^*, r^*]$, let us define

$$\Gamma(x) = \sup(L_x \cap \mathcal{P}). \quad (3.16)$$

Then $\Gamma : [-s^*, r^*] \rightarrow \mathbf{R}_+^2 \setminus \{(0,0)\}$ is well defined and $\Gamma(-s^*) = (0, s^*)$ and $\Gamma(r^*) = (r^*, 0)$. Similar to [10, Theorem 3.1], it is easy to prove that Γ is continuous on $[-s^*, r^*]$.

Consequently, the curve Γ separates $\mathbf{R}_+^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$, where $\mathcal{O}_1$ is bounded and $\mathcal{O}_2$ is unbounded. It is obvious that (1.4) has a positive solution at $\Gamma(x)$ for all $x \in [-s^*, r^*]$. If $(\lambda, \mu) \in \mathcal{O}_1$ and so if $(\lambda, \mu) \in L_{x_0}$, then $x_0 \in [-s^*, r^*]$ and $(\lambda, \mu) < \Gamma(x_0)$. Thus by Lemma 3.2, $(\lambda, \mu) \in \mathcal{P}$. On the other hand, if $(\lambda, \mu) \in \mathcal{O}_2$ and if $(\lambda, \mu) \in L_{x_0}$, then either $\Gamma(x_0)$ is not defined when $x_0 \in [-s^*, r^*]$ or $(\lambda, \mu) > \Gamma(x_0)$ when $x_0 \notin [-s^*, r^*]$. We get $(\lambda, \mu) \notin \mathcal{P}$ for both cases and the proof is done. $\square$

Now, we show the existence of the second positive solution for $(\lambda, \mu) \in \mathcal{O}_1$. Let $(\lambda, \mu) \in \mathcal{O}_1$, then we may choose $x_0 \in [-s^*, r^*]$ such that $(\lambda, \mu) \in L_{x_0}$. We know by Lemma 3.7 that (1.4) has a positive solution at $\Gamma(x_0)$, so let (u^*, v^*) be the solution at $\Gamma(x_0)$ and let us denote $\Gamma(x_0) = (\lambda^*, \mu^*)$. Then obviously $(\lambda, \mu) < (\lambda^*, \mu^*)$ and we have the following theorem.

Lemma 3.8. *Let $(\lambda, \mu) \in \mathcal{O}_1$. Then there exists $\varepsilon_0 > 0$ and for all $x \in [0, 1]$, $0 < \varepsilon \leq \varepsilon_0$ such that*

$$\begin{aligned} u_\varepsilon^*(x) &> \lambda \int_0^1 k_1(x, y) f_1(y, u_\varepsilon^*(y), v_\varepsilon^*(y)) dy, \\ v_\varepsilon^*(x) &> \mu \int_0^1 k_2(x, y) f_2(y, u_\varepsilon^*(y), v_\varepsilon^*(y)) dy, \end{aligned} \quad (3.17)$$

where $u_\varepsilon^*(x) = u^*(x) + \varepsilon$ and $v_\varepsilon^*(x) = v^*(x) + \varepsilon$.

Proof. From (H_4) , there exists constant $M > 0$ such that

$$\min_{x \in [0, 1]} f_i(x, u^*(x), v^*(x)) \geq M > 0, \quad i = 1, 2. \quad (3.18)$$

Then by the uniform continuity of f_i on a compact set, there exists $\varepsilon_0 > 0$ small enough such that for all $x \in [0, 1]$, $0 < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} |f_1(x, u_\varepsilon^*(x), v_\varepsilon^*(x)) - f_1(x, u^*(x), v^*(x))| &< M \left(\frac{\lambda^*}{\lambda} - 1 \right), \\ |f_2(x, u_\varepsilon^*(x), v_\varepsilon^*(x)) - f_2(x, u^*(x), v^*(x))| &< M \left(\frac{\mu^*}{\mu} - 1 \right), \end{aligned} \quad (3.19)$$

$$\lambda f_1(x, u_\varepsilon^*(x), v_\varepsilon^*(x)) - \lambda^* f_1(x, u^*(x), v^*(x)) < (\lambda^* - \lambda)(M - f(x, u^*(x), v^*(x))) \leq 0,$$

$$\mu f_2(x, u_\varepsilon^*(x), v_\varepsilon^*(x)) - \mu^* f_2(x, u^*(x), v^*(x)) < (\mu^* - \mu)(M - f_2(x, u^*(x), v^*(x))) \leq 0.$$

Furthermore, we have for all $x \in [0, 1]$

$$u_\varepsilon^*(x) > \lambda^* \int_0^1 k_1(x, y) f_1(y, u^*(y), v^*(y)) dy > \lambda \int_0^1 k_1(x, y) f_1(y, u_\varepsilon^*(y), v_\varepsilon^*(y)) dy. \quad (3.20)$$

The inequalities for v_ε^* can be shown similarly and the proof is done. $\square$

We now state and prove our main result in this section.

Theorem 3.9. *Assume (H_1) – (H_6) . Then there exists a continuous curve Γ separating $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (1.4) has at least two positive solution on $\mathcal{O}_1$, at least one positive solution on Γ and no solution on $\mathcal{O}_2$.*

Proof. By Lemma 3.7, it is enough to show the existence of the second positive solution of (1.4) for $(\lambda, \mu) \in \mathcal{O}_1$. Let $(\lambda, \mu) \in \mathcal{O}_1$. Let $(\lambda, \mu) \in L_{x_0}$ and (u^*, v^*) be a positive solution of (1.4) at $\Gamma(x_0) =: (\lambda^*, \mu^*)$ and let $u_\varepsilon^*(x) = u^* + \varepsilon$, $v_\varepsilon^*(x) = v^* + \varepsilon$, where ε is given in Lemma 3.8. Let

$$\Omega = \{(u, v) \in E : -\varepsilon < u(x) < u_\varepsilon^*(x), -\varepsilon < v(x) < v_\varepsilon^*(x), x \in [0, 1]\}, \quad (3.21)$$

then Ω is bounded open in E , $0 \in \Omega$ and $T : P \cap \overline{\Omega} \rightarrow P$ be a completely continuous mapping, since it is completely continuous. Let $(u, v) \in P \cap \partial\Omega$. Then there exists $x_o \in [0, 1]$ such that either $u(x_o) = u_\varepsilon^*(x_o)$ or $v(x_o) = v_\varepsilon^*(x_o)$. Suppose that $u(x_o) = u_\varepsilon^*(x_o)$. Then by (H_4) and Lemma 3.8,

$$\begin{aligned} T_\lambda(u, v)(x_o) &= \lambda \int_0^1 k_1(x_o, y) f_1(y, u(y), v(y)) dy \\ &\leq \lambda \int_0^1 k_1(x_o, y) f_1(y, u_\varepsilon^*(y), v_\varepsilon^*(y)) dy \\ &< u_\varepsilon^*(x_o) = u(x_o) \leq \varpi u(x_o), \end{aligned} \quad (3.22)$$

for all $\varpi \geq 1$. Similarly, for the case $v(x_o) = v_\varepsilon^*(x_o)$, we can get $T_\mu(u, v)(x_o) < \varpi v(x_o)$ for all $\varpi \geq 1$. Thus $T(u, v) = (T_\lambda(u, v), T_\mu(u, v)) \neq \varpi(u, v)$, for all $(u, v) \in P \cap \overline{\Omega}$ and all $\varpi \geq 1$ and by Lemma 2.2,

$$i(T, P \cap \Omega, P) = 1. \quad (3.23)$$

Modifying (3.2) for $\lambda^* = \lambda$, that is, there exists $R_{f_1} > 0$ such that

$$f_1(x, u, v) \geq K_{f_1}(u + v), \quad \forall u + v \geq R_{f_1}, \quad (3.24)$$

where K_{f_1} satisfies

$$\frac{\lambda \sigma^2 K_{f_1}}{2} \int_\alpha^\beta k_1(y, y) dy > 1. \quad (3.25)$$

Let $R_0 = \max\{C_\Sigma, (1/\sigma)R_{f_1}, \|(u_\varepsilon^*, v_\varepsilon^*)\|\}$, where C_Σ is given in Lemma 3.1 with Σ a compact rectangle in $\mathbb{R}_+^2 \setminus \{(0,0)\}$ containing (λ, μ) . Let $P_{R_0} = \{(u, v) \in P, \|(u, v)\| < R_0\}$. Then $(u, v) \neq T(u, v)$ for $(u, v) \in \partial P_{R_0}$, by Lemma 3.1. Furthermore, if $(u, v) \in P_{R_0}$, then

$$u(t) + v(t) \geq \min_{x \in [\alpha, \beta]} (u(x) + v(x)) \geq \sigma \|(u, v)\| \geq R_{f_1}. \quad (3.26)$$

Thus by (3.24), $f_1(x, u(x), v(x)) \geq K_{f_1}(u(x) + v(x))$, for all $x \in [\alpha, \beta]$ and

$$\begin{aligned} T_\lambda(u, v)(x) &= \lambda \int_0^1 k_1(x, y) f_1(y, u(y), v(y)) dy \\ &\geq \lambda \sigma \int_\alpha^\beta k_1(y, y) f_1(y, u(y), v(y)) dy \\ &\geq \lambda \sigma K_{f_1} \int_\alpha^\beta k_1(y, y) (u(y) + v(y)) dy \\ &\geq \frac{\lambda \sigma^2 K_{f_1}}{2} \int_\alpha^\beta k_1(y, y) \|(u, v)\| ds > \|(u, v)\|. \end{aligned} \quad (3.27)$$

Therefore $\|T(u, v)\| \geq \|T_\lambda(u, v)\| > \|(u, v)\|$ and by Lemma 2.3,

$$i(T, P_{R_0}, P) = 0. \quad (3.28)$$

Consequently by the additivity of the fixed-point index,

$$0 = i(T, P_{R_0}, P) = i(T, P \cap \Omega, P) + i\left(T, P_{R_0} \setminus \overline{P \cap \Omega}, P\right). \quad (3.29)$$

Since $i(T, P \cap \Omega, P) = 1$, $i(T, P_{R_0} \setminus \overline{P \cap \Omega}, P) = -1$ and thus T has a fixed point on $P \cap \Omega$ and another on $P_{R_0} \setminus \overline{P \cap \Omega}$, and this completes the proof. $\square$

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