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Research Article

Quadruple Fixed Point Theorems for Weak ϕ-Contractions

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The notion of coupled fixed point is introduced in by Gnana Bhaskar and Lakshmikantham (2006). Very recently, the concept of tripled fixed point is introduced by Berinde and Borcut (2011). In this paper, quadruple fixed point is introduced, and some new fixed point theorems are obtained.

1. Introduction and Preliminaries

Very recently, Berinde and Borcut [1] introduced the concept of triple fixed point. Their contributions are inspired from the remarkable paper of Gnana Bhaskar and Lakshmikantham [2] in which they introduced the notion of coupled fixed point and proved some fixed point theorems under certain condition. Later, Lakshmikantham and Ljubomir Ćirić in [3] extended these results by defining of g-monotone property. Many authors focused on coupled fixed point theory and proved remarkable results (see, e.g., [4–9]).

Here we recall the basic definitions and results from which triple and quadruple fixed point [10] notions are inspired. Let (X, d) be a metric space and $X^2 := X \times X$. Then the mapping $\rho := X^2 \times X^2 :\to E$ such that $\rho((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + d(y_1, y_2)$ forms a metric on X^2. A sequence $(\{x_n\}, \{y_n\}) \in X^2$ is said to be a double sequence of X.

Definition 1.1 (see [2]). Let $(X, \leq)$ be partially ordered set and $F : X \times X \to X$. F is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$\begin{aligned} x_1 \leq x_2 &\Longrightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ y_1 \leq y_2 &\Longrightarrow F(x, y_2) \leq F(x, y_1), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{1.1}$$

Definition 1.2 (see [2]). An element $(x, y) \in X \times X$ is said to be a couple fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{1.2}$$

Throughout this paper, let $(X, \leq)$ be partially ordered set and d a metric on X such that (X, d) is a complete metric space. Further, the product spaces $X \times X$ satisfy the following:

$$(u, v) \leq (x, y) \Longleftrightarrow u \leq x, y \leq v; \quad \forall (x, y), (u, v) \in X \times X. \tag{1.3}$$

The following two results of Gnana Bhaskar and Lakshmikantham in [2] were extended to class of cone metric spaces in [7].

Theorem 1.3. *Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall u \leq x, y \leq v. \tag{1.4}$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then, there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 1.4. *Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that X has the following properties:*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all n;*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \leq y_n$, for all n.*

Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall u \leq x, y \leq v. \tag{1.5}$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then, there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Inspired by Definition 1.1, Berinde and Borcut [1] introduced the following partial order on the product space $X^3 = X \times X \times X$:

$$(u, v, w) \leq (x, y, z) \quad \text{iff } x \geq u, y \leq v, z \geq w, \tag{1.6}$$

where $(u, v, w), (x, y, z) \in X^3$. Regarding this partial order, we state the definition of the following mapping.

Definition 1.5 (see [1]). Let $(X, \leq)$ be partially ordered set and $F : X^3 \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x and y, and it is monotone nonincreasing in y, that is, for any $x, y, z \in X$

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ &y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ &z_1, z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \tag{1.7}$$

Definition 1.6 (see [1]). An element $(x, y, z, w)X^3$ is called a quadruple fixed point of F: $X^3 \to X$ if

$$F(x, y, z) = x, \qquad F(y, z, y) = y, \qquad F(z, y, x) = z. \tag{1.8}$$

For a metric space (X, d), the function $\rho : X^3 \to [0, \infty)$, given by

$$\rho((x, y, z), (u, v, w)) := d(x, u) + d(y, v) + d(z, w), \tag{1.9}$$

forms a metric space on X^3; that is, (X^3, ρ) is a metric induced by (X, d).

Theorem 1.7. *Let $(X, \leq)$ be partially ordered set, and let (X, d) be a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w) \tag{1.10}$$

for all $x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 \leq F(x_0, y_0, z_0), \tag{1.11}$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, z, y) = y, \qquad F(z, y, x) = z. \tag{1.12}$$

The aim of this paper is to introduce the concept of quadruple fixed point and prove the related fixed point theorems.

2. Quadruple Fixed Point Theorems

Let $(X, \leq)$ be partially ordered set and (X, d) a complete metric space. We consider the following partial order on the product space $X^4 = X \times X \times X \times X$:

$$(u, v, r, t) \leq (x, y, z, w) \quad \text{iff } x \geq u, y \leq v, z \geq r, t \leq w, \tag{2.1}$$

where $(u,v,r,t),(x,y,z,w) \in X^4$. Regarding this partial order, we state the definition of the following mapping.

Definition 2.1. Let $(X,\leq)$ be partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone nondecreasing in x and z, and it is monotone nonincreasing in y and w, that is, for any $x,y,z,w \in X$

$$\begin{aligned} x_1,x_2 \in X, &\quad x_1 \leq x_2 \Longrightarrow F(x_1,y,z,w) \leq F(x_2,y,z,w), \\ y_1,y_2 \in X, &\quad y_1 \leq y_2 \Longrightarrow F(x,y_1,z,w) \geq F(x,y_2,z,w), \\ z_1,z_2 \in X, &\quad z_1 \leq z_2 \Longrightarrow F(x,y,z_1,w) \leq F(x,y,z_2,w), \\ w_1,w_2 \in X, &\quad w_1 \leq w_2 \Longrightarrow F(x,y,z,w_1) \geq F(x,y,z,w_2). \end{aligned} \tag{2.2}$$

Definition 2.2. An element $(x,y,z) \in X^4$ is called a triple fixed point of $F : X^4 \to X$ if

$$F(x,y,z,w) = x, \qquad F(x,w,z,y) = y, \qquad F(z,y,x,w) = z, \qquad F(z,w,x,y) = w. \tag{2.3}$$

For a metric space (X,d), the function $\rho : X^4 \to [0,\infty)$, given by

$$\rho((x,y,z,w),(u,v,r,t)) := d(x,u) + d(y,v) + d(z,r) + d(w,t), \tag{2.4}$$

forms a metric space on X^4; that is, (X^4,ρ) is a metric induced by (X,d). Let Φ denote the all functions $\phi : [0,\infty) \to [0,\infty)$ which satisfies that $\lim_{t\to r}\phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+}\phi(t) = 0$.

The aim of this paper is to prove the following theorem.

Theorem 2.3. *Let $(X,\leq)$ be partially ordered set and (X,d) a complete metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on X. Assume that for all $x \geq u, y \leq v, z \geq w$,*

$$\begin{aligned} d(F(x,y,z,w),F(u,v,r,t)) \leq{}& \frac{1}{4}[d(x,u) + d(y,v) + d(z,r) + d(w,t)] \\ &- \phi\left(\frac{1}{4}[d(x,u) + d(y,v) + d(z,r) + d(w,t)]\right), \end{aligned} \tag{2.5}$$

where $\phi \in \Phi$. Suppose there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$\begin{aligned} x_0 &\leq F(x_0,y_0,z_0,w_0), \qquad y_0 \geq F(x_0,w_0,z_0,y_0), \\ z_0 &\leq F(z_0,y_0,x_0,w_0), \qquad w_0 \geq F(z_0,w_0,x_0,y_0). \end{aligned} \tag{2.6}$$

Suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if nondecreasing sequence $x_n \to x$ (resp., $z_n \to z$), then $x_n \leq x$ (resp., $z_n \leq z$) for all n,*

(ii) *if nonincreasing sequence $y_n \to y$ (resp., $w_n \to w$), then $y_n \geq y$ (resp., $w_n \geq w$) for all n.*

Then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x,y,z,w) &= x, \qquad F(x,w,z,y) = y, \\ F(z,y,x,w) &= z, \qquad F(z,w,x,y) = w. \end{aligned} \tag{2.7}$$

Proof. We construct a sequence $\{(x_n, y_n, z_n, w_n)\}$ in the following way. Set

$$\begin{aligned} x_1 &= F(x_0, y_0, z_0, w_0) \geq x_0, \\ y_1 &= F(x_0, w_0, z_0, y_0) \leq y_0, \\ z_1 &= F(z_0, y_0, x_0, w_0) \geq z_0, \\ w_1 &= F(z_0, w_0, x_0, y_0) \leq w_0, \end{aligned} \tag{2.8}$$

and by the mixed monotone property of F, for $n \geq 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \geq x_{n-1} \geq \cdots \geq x_0, \\ y_n &= F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) \leq y_{n-1} \leq \cdots \leq y_0, \\ z_n &= F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \geq z_{n-1} \geq \cdots \geq z_0, \\ w_n &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \leq w_{n-1} \leq \cdots \leq w_0. \end{aligned} \tag{2.9}$$

Due to (2.5) and (2.9), we have

$$\begin{aligned} d(x_1, x_2) &= d(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)) \\ &\leq \frac{1}{4}\left[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1) + d(w_0, w_1)\right] \\ &\quad - \phi\left(\frac{1}{4}\left[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1) + d(w_0, w_1)\right]\right) \\ &\leq \frac{1}{4}\left[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1) + d(w_0, w_1)\right], \end{aligned} \tag{2.10}$$

$$\begin{aligned}
d(y_1,y_2) &= d(F(x_0,w_0,z_0,y_0),F(x_1,w_1,z_1,y_1)) \\
&\le \frac{1}{4}\left[d(x_0,x_1)+d(w_0,w_1)+d(z_0,z_1)+d(y_0,y_1)\right] \\
&\quad -\phi\left(\frac{1}{4}\left[d(x_0,x_1)+d(w_0,w_1)+d(z_0,z_1)+d(y_0,y_1)\right]\right) \\
&\le \frac{1}{4}\left[d(x_0,x_1)+d(w_0,w_1)+d(z_0,z_1)+d(y_0,y_1)\right],
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
d(z_1,z_2) &= d(F(z_0,y_0,x_0,w_0),F(z_1,y_1,x_1,w_1)) \\
&\le \frac{1}{4}\left[d(z_0,z_1)+d(y_0,y_1)+d(x_0,x_1)+d(w_0,w_1)\right] \\
&\quad -\phi\left(\frac{1}{4}\left[d(z_0,z_1)+d(y_0,y_1)+d(x_0,x_1)+d(w_0,w_1)\right]\right) \\
&\le \frac{1}{4}\left[d(z_0,z_1)+d(y_0,y_1)+d(x_0,x_1)+d(w_0,w_1)\right],
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
d(w_1,w_2) &= d(F(z_0,w_0,x_0,y_0),F(z_1,w_1,x_1,y_1)) \\
&\le \frac{1}{4}\left[d(z_0,z_1)+d(w_0,w_1)+d(x_0,x_1)+d(y_0,y_1)\right] \\
&\quad -\phi\left(\frac{1}{4}\left[d(z_0,z_1)+d(w_0,w_1)+d(x_0,x_1)+d(y_0,y_1)\right]\right) \\
&\le \frac{1}{4}\left[d(z_0,z_1)+d(w_0,w_1)+d(x_0,x_1)+d(y_0,y_1)\right].
\end{aligned} \tag{2.13}$$

Regarding (2.5) together with (2.14) we have

$$\begin{aligned}
d(x_2,x_3) &= d(F(x_1,y_1,z_1,w_1),F(x_2,y_2,z_2,w_2)) \\
&\le \frac{1}{4}\left[d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+\mathrm{d}(w_1,w_2)\right] \\
&\quad -\phi\left(\frac{1}{4}\left[d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)\right]\right) \\
&\le \frac{1}{4}\left[d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)\right], \\
d(y_2,y_3) &= d(F(x_1,w_1,z_1,y_1),F(x_2,w_2,z_2,y_2)) \\
&\le \frac{1}{4}\left[d(x_1,x_2)+d(w_1,w_2)+d(z_1,z_2)+d(y_1,y_2)\right] \\
&\quad -\phi\left(\frac{1}{4}\left[d(x_1,x_2)+d(w_1,w_2)+d(z_1,z_2)+d(y_1,y_2)\right]\right) \\
&\le \frac{1}{4}\left[d(x_1,x_2)+d(w_1,w_2)+d(z_1,z_2)+d(y_1,y_2)\right],
\end{aligned}$$

$$
\begin{aligned}
d(z_2,z_3) &= d\big(F(z_1,y_1,x_1,w_1),F(z_2,y_2,x_2,w_2)\big)\\
&\le \frac{1}{4}\big[d(z_1,z_2)+d(y_1,y_2)+d(x_1,x_2)+d(w_1,w_2)\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(z_1,z_2)+d(y_1,y_2)+d(x_1,x_2)+d(w_1,w_2)\big]\right)\\
&\le \frac{1}{4}\big[d(z_1,z_2)+d(y_1,y_2)+d(x_1,x_2)+d(w_1,w_2)\big],\\
d(w_2,w_3) &= d\big(F(z_1,w_1,x_1,y_2),F(z_2,w_2,x_2,y_2)\big)\\
&\le \frac{1}{4}\big[d(z_1,z_2)+d(w_1,w_2)+d(x_1,x_2)+d(y_1,y_2)\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(z_1,z_2)+d(w_1,w_2)+d(x_1,x_2)+d(y_1,y_2)\big]\right)\\
&\le \frac{1}{4}\big[d(z_1,z_2)+d(w_1,w_2)+d(x_1,x_2)+d(y_1,y_2)\big].
\end{aligned}
\tag{2.14}
$$

Inductively we have

$$
\begin{aligned}
d(x_{n+1},x_{n+2}) &= d\big(F(x_n,y_n,z_n,w_n),F(x_{n+1},y_{n+1},z_{n+1},w_{n+1})\big)\\
&\le \frac{1}{4}\big[d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{\mathrm{n}+1})+d(w_n,w_{n+1})\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1})+d(w_n,w_{n+1})\big]\right)\\
&\le \frac{1}{4}\big[d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1})+d(w_n,w_{n+1})\big],\\
d(y_{n+1},y_{n+2}) &= d\big(F(x_n,w_n,z_n,y_n),F(x_{n+1},w_{n+1},z_{n+1},y_{n+1})\big)\\
&\le \frac{1}{4}\big[d(x_n,x_{n+1})+d(w_n,w_{n+1})+d(z_n,z_{n+1})+d(y_n,y_{n+1})\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(x_n,x_{n+1})+d(w_n,w_{n+1})+d(z_n,z_{n+1})+d(y_n,y_{n+1})\big]\right)\\
&\le \frac{1}{4}\big[d(x_n,x_{n+1})+d(w_n,w_{n+1})\ \ +d(z_n,z_{n+1})+d(y_n,y_{n+1})\big],\\
d(z_{n+1},z_{n+2}) &= d\big(F(z_n,y_n,x_n,w_n),F(z_{n+1},y_{n+1},x_{n+1},w_{n+1})\big)\\
&\le \frac{1}{4}\big[d(z_n,z_{n+1})+d(y_n,y_{n+1})+d(x_n,x_{n+1})+d(w_n,w_{n+1})\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(z_n,z_{n+1})+d(y_n,y_{n+1})+d(x_n,x_{n+1})+d(w_n,w_{n+1})\big]\right)\\
&\le \frac{1}{4}\big[d(z_n,z_{n+1})+d(y_n,y_{n+1})+d(x_n,x_{n+1})+d(w_n,w_{n+1})\big],
\end{aligned}
$$

$$\begin{aligned} d(w_{n+1}, w_{n+2}) &= d\big(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})\big) \\ &\le \frac{1}{4}\big[d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1})\big] \\ &\quad - \phi\left(\frac{1}{4}\big[d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1})\big]\right) \\ &\le \frac{1}{4}\big[d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1})\big]. \end{aligned} \tag{2.15}$$

Set $\delta_n = d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1}) + d(w_n, w_{n-1})$. Due to (2.15), we conclude that $\{\delta_n\}$ is a nonincreasing sequence. Since it is bounded below, there is some $\delta \ge 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{2.16}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$.

Again by (2.15) and (2.9) together with (2.5), we have

$$\delta_n \le \delta_n - 4\phi\left(\frac{1}{4}[\delta_n]\right). \tag{2.17}$$

Letting $n \to \infty$ in (2.17) and having in mind that we suppose $\lim_{t\to r}\phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+}\phi(t) = 0$, we have

$$\delta \le \delta - 4\phi\left(\frac{1}{4}\delta\right), \tag{2.18}$$

which is a contradiction. Thus, $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}\big[d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1}) + d(w_n, w_{n-1})\big] = 0. \tag{2.19}$$

Now, we shall prove that $\{x_n\}$,$\{y_n\}$,$\{z_n\}$, and $\{w_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$,$\{y_n\}$,$\{z_n\}$, and $\{w_n\}$ is not Cauchy. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{n(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{n(k)}\}$ of $\{y_n\}$ and $\{z_{n(k)}\}$, $\{z_{n(k)}\}$ of $\{z_n\}$ and $\{w_{n(k)}\}$, $\{w_{n(k)}\}$ of $\{w_n\}$ with $n(k) > m(k) \ge k$ such that

$$d\big(x_{n(k)}, x_{m(k)}\big) + d\big(y_{n(k)}, y_{m(k)}\big) + d\big(z_{n(k)}, z_{m(k)}\big) + d\big(w_{n(k)}, w_{m(k)}\big) \ge \varepsilon. \tag{2.20}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.20) and $n(k) > m(k) \ge k$. Thus,

$$d\big(x_{n(k)-1}, x_{m(k)}\big) + d\big(y_{n(k)-1}, y_{m(k)}\big) + d\big(z_{n(k)-1}, z_{m(k)}\big) + d\big(w_{n(k)-1}, w_{m(k)}\big) < \varepsilon. \tag{2.21}$$

By using triangle inequality and having (2.20), (2.21) in mind

$$\begin{aligned}
\varepsilon \le t_k &= d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) \\
&\le d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{n(k)-1}\right) + d\left(y_{n(k)-1}, y_{m(k)}\right) \\
&\quad + d\left(z_{n(k)}, z_{n(k)-1}\right) + d\left(z_{n(k)-1}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{n(k)-1}\right) + d\left(w_{n(k)-1}, w_{m(k)}\right) \\
&< d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(y_{n(k)}, y_{n(k)-1}\right) + d\left(z_{n(k)}, z_{n(k)-1}\right) + d\left(w_{n(k)}, w_{n(k)-1}\right) + \varepsilon.
\end{aligned} \tag{2.22}$$

Letting $k \to \infty$ in (2.22) and using (2.16)

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) = \varepsilon. \tag{2.23}$$

Again by triangle inequality,

$$\begin{aligned}
t_k &= d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) \\
&\le d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{m(k)}\right) \\
&\quad + d\left(y_{n(k)}, y_{n(k)+1}\right) + d\left(y_{n(k)+1}, y_{m(k)+1}\right) + d\left(y_{m(k)+1}, y_{m(k)}\right) \\
&\quad + d\left(z_{n(k)}, z_{n(k)+1}\right) + d\left(z_{n(k)+1}, z_{m(k)+1}\right) + d\left(z_{m(k)+1}, z_{m(k)}\right) \\
&\quad + d\left(w_{n(k)}, w_{n(k)+1}\right) + d\left(w_{n(k)+1}, w_{m(k)+1}\right) + d\left(w_{m(k)+1}, w_{m(k)}\right) \\
&\le \delta_{n(k)+1} + \delta_{m(k)+1} + d\left(x_{n(k)+1}, x_{m(k)+1}\right) + d\left(y_{n(k)+1}, y_{m(k)+1}\right) \\
&\quad + d\left(z_{n(k)+1}, z_{m(k)+1}\right) + d\left(w_{n(k)+1}, w_{m(k)+1}\right).
\end{aligned} \tag{2.24}$$

Since $n(k) > m(k)$, then

$$\begin{aligned}
x_{n(k)} &\ge x_{m(k)}, \qquad & y_{n(k)} &\le y_{m(k)}, \\
z_{n(k)} &\ge z_{m(k)}, \qquad & w_{n(k)} &\le w_{m(k)}.
\end{aligned} \tag{2.25}$$

Hence from (2.25), (2.9) and (2.5), we have

$$\begin{aligned}
&d\left(x_{n(k)+1}, x_{m(k)+1}\right) \\
&\quad = d\left(F\left(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}\right)\right) \\
&\quad \le \frac{1}{4}\left[d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right)\right] \\
&\qquad - \phi\left(\frac{1}{4}\left[d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right)\right]\right),
\end{aligned}$$

$$
\begin{aligned}
&d\left(y_{n(k)+1}, y_{m(k)+1}\right) \\
&\quad = d\left(F\left(x_{n(k)}, w_{n(k)}, z_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, w_{m(k)}, z_{m(k)}, y_{m(k)}\right)\right) \\
&\quad \le \frac{1}{4}\left[d\left(x_{n(k)}, x_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right)\right] \\
&\qquad - \phi\left(\frac{1}{4}\left[d\left(x_{n(k)}, x_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) + d\left(z_{n(k)}, z_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right)\right]\right), \\
&d\left(z_{n(k)+1}, z_{m(k)+1}\right) \\
&\quad = d\left(F\left(z_{n(k)}, y_{n(k)}, x_{n(k)}, w_{n(k)}\right), F\left(z_{m(k)}, y_{m(k)}, x_{m(k)}, w_{m(k)}\right)\right) \\
&\quad \le \frac{1}{4}\left[d\left(z_{n(k)}, z_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right)\right] \\
&\qquad - \phi\left(\frac{1}{4}\left[d\left(z_{n(k)}, z_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right)\right]\right), \\
&d\left(w_{n(k)+1}, w_{m(k)+1}\right) \\
&\quad = d\left(F\left(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}\right), F\left(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}\right)\right) \\
&\quad \le \frac{1}{4}\left[d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right)\right] \\
&\qquad - \phi\left(\frac{1}{4}\left[d\left(z_{n(k)}, z_{m(k)}\right) + d\left(w_{n(k)}, w_{m(k)}\right) + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right)\right]\right).
\end{aligned}
\tag{2.26}
$$

Combining (2.24) with (2.26), we obtain that

$$
\begin{aligned}
t_k &\le \delta_{n(k)+1} + \delta_{m(k)+1} + d\left(x_{n(k)+1}, x_{m(k)+1}\right) + d\left(y_{n(k)+1}, y_{m(k)+1}\right) \\
&\quad + d\left(z_{n(k)+1}, z_{m(k)+1}\right) + d\left(w_{n(k)+1}, w_{m(k)+1}\right) \\
&\le \delta_{n(k)+1} + \delta_{m(k)+1} + t_k - 4\phi\left(\frac{1}{4}t_k\right).
\end{aligned}
\tag{2.27}
$$

Letting $k \to \infty$ and having in mind (2.19) we get a contradiction. This shows that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are Cauchy sequences. Since X is complete metric space, there exists $x, y, z, w \in X$ such that

$$
\begin{aligned}
&\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} y_n = y, \\
&\lim_{n\to\infty} z_n = z, \qquad \lim_{n\to\infty} w_n = w.
\end{aligned}
\tag{2.28}
$$

Suppose now the assumption (a) holds. Then by (2.9) and (2.28), we have

$$\begin{aligned} x &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \\ &= F\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} w_{n-1}\right) \\ &= F(x, y, z, w). \end{aligned} \tag{2.29}$$

Analogously, we also observe that

$$\begin{aligned} y &= \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) = F(x, w, z, y), \\ z &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) = F(z, y, x, w), \\ w &= \lim_{n\to\infty} w_n = \lim_{n\to\infty} F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) = F(z, w, x, y). \end{aligned} \tag{2.30}$$

Thus, we have

$$\begin{aligned} F(x, y, z, w) &= x, \qquad F(x, w, z, y) = y, \\ F(z, y, x, w) &= z, \qquad F(z, w, x, y) = w. \end{aligned} \tag{2.31}$$

Suppose now the assumption (b) holds. Since $\{x_n\}, \{z_n\}$ are nondecreasing and $x_n \to x$, $z_n \to z$, and also $\{y_n\}, \{w_n\}$ are nonincreasing and $y_n \to y$, $w_n \to w$, then by assumption (b) we have

$$x_n \geq x, \qquad y_n \leq y, \qquad z_n \geq z, \qquad w_n \leq w \tag{2.32}$$

for all n. Consider now

$$\begin{aligned} d(x, F(x, y, z, w)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y, z, w)) \\ &= d(x, x_{n+1}) + d(F(x_n, y_n, z_n, w_n), F(x, y, z, w)) \\ &\leq \frac{1}{4}\left[d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)\right] \\ &\quad - \phi\left(\frac{1}{4}\left[d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)\right]\right). \end{aligned} \tag{2.33}$$

Taking $n \to \infty$ in (2.33) and using (2.28), we get that $d(x, F(x, y, z, w)) = 0$. Thus, $x = F(x, y, z, w)$. Analogously, we get that

$$F(x, w, z, y) = y, \qquad F(z, y, x, w) = z, \qquad F(z, w, x, y) = w. \tag{2.34}$$

Thus, we proved that F has a quadruple fixed point. □

Corollary 2.4. *Let $(X, \leq)$ be partially ordered set and (X, d) a complete metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that*

$$d(F(x, y, z, w), F(u, v, r, t)) \leq \frac{k}{4}\left[d(x, u) + d(y, v) + d(z, r) + d(w, t)\right] \tag{2.35}$$

for all $x \geq u$, $y \leq v$, $z \geq w$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 &\leq F(x_0, y_0, z_0, w_0), \qquad & y_0 &\geq F(x_0, w_0, z_0, y_0), \\ z_0 &\leq F(z_0, y_0, x_0, w_0), \qquad & w_0 &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \tag{2.36}$$

Suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if nondecreasing sequence $x_n \to x$ (resp., $z_n \to z$), then $x_n \leq x$ (resp., $z_n \leq z$) for all n,*

(ii) *if nonincreasing sequence $y_n \to y$ (resp., $w_n \to w$), then $y_n \geq y$ (resp., $w_n \geq w$) for all n,*

then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x, y, z, w) &= x, \qquad & F(x, w, z, y) &= y, \\ F(z, y, x, w) &= z, \qquad & F(z, w, x, y) &= w. \end{aligned} \tag{2.37}$$

Proof. It is sufficient to take $\phi(t) = ((1 - k)/2)t$ in previous theorem. □

3. Uniqueness of Quadruple Fixed Point

In this section we shall prove the uniqueness of quadruple fixed point. For a product X^4 of a partial ordered set $(X, \leq)$ we define a partial ordering in the following way. For all $(x, y, z, t), (u, v, r, t) \in X \times X \times X \times X$

$$(x, y, z, w) \leq (u, v, r, t) \Longleftrightarrow x \leq u,\ y \geq v,\ z \leq r,\ w \geq r. \tag{3.1}$$

We say that (x, y, z, w) is equal (u, v, r, t) if and only if $x = u, y = v, z = r$, and $w = t$.

Theorem 3.1. *In addition to hypothesis of Theorem 2.3, suppose that for all $(x, y, z, t), (u, v, r, t) \in X \times X \times X \times X$; there exists $(a, b, c, d) \in X \times X \times X \times X$ that is comparable to (x, y, z, t) and (u, v, r, t); then F has a unique quadruple fixed point.*

Proof. The set of quadruple fixed point of F is not empty due to Theorem 2.3. Assume, now, that (x, y, z, t) and (u, v, r, t) are the quadruple fixed point of F, that is,

$$\begin{aligned} F(x,y,z,w) &= x, & F(u,v,r,t) &= u, \\ F(x,w,z,y) &= y, & F(u,t,r,v) &= v, \\ F(z,y,x,w) &= z, & F(r,v,u,t) &= r, \\ F(z,w,x,y) &= w, & F(r,t,u,v) &= t. \end{aligned} \tag{3.2}$$

We shall show that (x, y, z, w) and (u, v, r, t) are equal. By assumption, there exists $(a, b, c, d) \in X \times X \times X \times X$ that is comparable to (x, y, z, t) and (u, v, r, t). Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$ such that

$$\begin{aligned} a = a_0, \quad b = b_0, \quad c = c_0, \quad d = d_0, \\ a_n &= F(a_{n-1}, b_{n-1}, z_{n-1}, d_{n-1}), \\ b_n &= F(a_{n-1}, d_{n-1}, c_{n-1}, b_{n-1}), \\ c_n &= F(c_{n-1}, b_{n-1}, a_{n-1}, d_{n-1}), \\ d_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}) \end{aligned} \tag{3.3}$$

for all n. Since (x, y, z, w) is comparable with (a, b, c, d), we may assume that $(x, y, z, w) \geq (a, b, c, d) = (a_0, b_0, c_0, d_0)$. Recursively, we get that

$$(x, y, z, w) \geq (a_n, b_n, c_n, d_n) \quad \forall n. \tag{3.4}$$

By (3.4) and (2.5), we have

$$\begin{aligned} d(x, a_{n+1}) &= d(F(x,y,z,w), F(a_n, b_n, c_n, d_n)) \\ &\leq \frac{1}{4}[d(x, a_n) + d(y, b_n) + d(z, c_n) + d(w, d_n)] \\ &\quad - \phi\left(\frac{1}{4}[d(x, a_n) + d(y, b_n) + d(z, c_n) + d(w, d_n)]\right), \\ d(b_{n+1}, y) &= d(F(a_n, d_n, c_n, b_n)\ , F(x, w, z, y)) \\ &\leq \frac{1}{4}[d(a_n, x) + d(d_n, w) + d(c_n, z) + d(b_n, y)] \\ &\quad - \phi\left(\frac{1}{4}[d(a_n, x) + d(d_n, w) + d(c_n, z) + d(b_n, y)]\right), \end{aligned}$$

$$\begin{aligned}
d(z,c_{n+1}) &= d\big(F(z,y,x,w),F(c_n,b_n,a_n,d_n)\big)\\
&\le \frac{1}{4}\big[d(z,c_n)+d(y,b_n)+d(x,a_n)+d(w,d_n)\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(z,c_n)+d(y,b_n)+d(x,a_n)+d(w,d_n)\big]\right),\\
d(d_{n+1},w) &= d\big(F(c_n,d_n,a_n,b_n),F(z,w,x,y)\big)\\
&\le \frac{1}{4}\big[d(c_n,z)+d(d_n,w)+d(a_n,x)+d(b_n,y)\big]\\
&\quad -\phi\left(\frac{1}{4}\big[d(c_n,z)+d(d_n,w)+d(a_n,x)+d(b_n,y)\big]\right).
\end{aligned} \tag{3.5}$$

Set $\gamma_n = d(x,a_n)+d(y,b_n)+d(z,c_n)+d(w,d_n)$. Then, due to (3.5), we have

$$\gamma_{n+1} \le \gamma_n, \quad \forall n. \tag{3.6}$$

□

Hence, the sequence $\{\gamma_n\}$ is decreasing and bounded below. Thus, there exists $\gamma \ge 0$ such that

$$\lim_{n\to\infty}\gamma_n = \gamma. \tag{3.7}$$

Now, we shall show that $\gamma = 0$. Suppose, to the contrary, that $\gamma > 0$. Again by (3.5), we have

$$\gamma_{n+1} \le \gamma_n - 4\left(\frac{1}{4}\phi(\gamma_n)\right), \quad \forall n. \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we obtain that

$$\gamma \le \gamma - 4\left(\frac{1}{4}\phi(\gamma)\right), \tag{3.9}$$

which is a contradiction. Therefore, $\gamma = 0$. That is,

$$\lim_{n\to\infty}\gamma_n = 0. \tag{3.10}$$

Consequently, we have

$$\begin{aligned}
&\lim_{n\to\infty} d(x,a_n) = 0, \qquad \lim_{n\to\infty} d(y,b_n) = 0,\\
&\lim_{n\to\infty} d(z,c_n) = 0, \qquad \lim_{n\to\infty} d(w,d_n) = 0.
\end{aligned} \tag{3.11}$$

Analogously, we show that

$$\begin{aligned} &\lim_{n\to\infty} d(u,a_n)=0, \qquad \lim_{n\to\infty} d(v,b_n)=0,\\ &\lim_{n\to\infty} d(r,c_n)=0, \qquad \lim_{n\to\infty} d(s,d_n)=0. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) yields that (x,y,z,w) and (u,v,r,t) are equal.

Example 3.2. Let $X=\mathbb{R}$ with the metric $d(x,y)=|x-y|$, for all $x,y\in X$ and the usual ordering.

Let $F: X^4 \to X$ be given by

$$F(x,y,z,w)=\frac{x-y+z-w}{16}, \quad \forall x,y,z,w\in X, \tag{3.13}$$

and let $\phi:[0,\infty)\to[0,\infty)$ be given by $\phi(t)=t/4$ for all $t\in[0,\infty)$.

It is easy to check that all the conditions of Theorem 2.3 are satisfied and $(0,0,0,0)$ is the unique quadruple fixed point of F.

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