Research Article

Inclusion Relationships for Certain Subclasses of Meromorphic Functions Defined by Using the Extended Multiplier Transformations

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Let Σ denote the class of analytic functions in the punctured unit disc $U^* = \{z : 0 < |z| < 1\}$. Set $\varphi_{\lambda, \ell}^m(z) = 1/z + \sum_{k=0}^{\infty} [\ell + \lambda(k+1)/\ell]^m z^k$ ($m \in \mathbb{N}_0; \ell > 0; \lambda \geq 0; z \in U^*$), and define $\varphi_{\lambda, \ell}^{m, \mu}$ in terms of the Hadamard product by $\varphi_{\lambda, \ell}^m(z) * \varphi_{\lambda, \ell}^{m, \mu}(z) = 1/z(1-z)^\mu$ ($\mu > 0; z \in U^*$). In this paper, we introduce several new subclasses of analytic functions defined by means of the operator $I_\mu^m(\lambda, \ell)f(z) = \varphi_{\lambda, \ell}^{m, \mu}(z) * f(z)$ ($f \in \Sigma; m \in \mathbb{N}_0; \ell > 0; \lambda \geq 0; \mu > 0$). Inclusion properties of these classes and some applications involving integral operator are also considered.

1. Introduction

Let Σ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the punctured open unit disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. We denote by $\Sigma S(\eta)$, $\Sigma K(\eta)$, and $\Sigma C(\eta, \beta)$ ($0 \leq \eta, \beta < 1$) the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order η in U , convex of order η in U , and close-to-convex of order η and type β in U (see [1–3]).

Let M be the class of all function φ which are analytic and univalent in U and for which $\varphi(U)$ is convex with

$$\varphi(0) = 1, \operatorname{Re}\{\varphi(z)\} > 0 \quad (z \in U). \quad (1.2)$$

For two functions f and g analytic in U , we say that f is subordinate to g and write $f < g$ in U or $f(z) < g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = g(w(z))$. It is known that

$$f(z) < g(z) \implies f(0) = g(0), \quad f(U) \subset g(U). \quad (1.3)$$

Furthermore, if the function g is univalent in U (see, [4, page 4]),

$$f(z) < g(z) \iff f(0) = g(0), \quad f(U) \subset g(U). \quad (1.4)$$

Making use of the principle of subordination between analytic functions, we define the subclasses $\sum S(\eta; \varphi)$, $\sum K(\eta, \varphi)$, and $\sum C(\eta, \beta; \varphi, \psi)$ of the class $\sum$ for $0 \leq \eta, \beta < 1$ and $\varphi, \psi \in M$, which are defined by

$$\begin{aligned} \sum S(\eta, \varphi) &= \left\{ f : f \in \sum, \frac{1}{1-\eta} \left(\frac{-zf'(z)}{f(z)} - \eta \right) < \varphi(z) \ (z \in U) \right\}, \\ \sum K(\eta, \varphi) &= \left\{ f : f \in \sum, \frac{1}{1-\eta} \left(-\left[1 + \frac{zf''(z)}{f'(z)} \right] - \eta \right) < \varphi(z) \ (z \in U) \right\}, \\ \sum C(\eta, \beta; \varphi, \psi) &= \left\{ f : f \in \sum, \exists g \in S(\eta; \varphi) \text{ s.t. } \frac{1}{1-\beta} \left(-\frac{zf'(z)}{g(z)} - \beta \right) < \varphi(z) \ (z \in U) \right\}, \end{aligned} \quad (1.5)$$

respectively. For special choices for the parameters η and β as well as for special choices for the function φ and ψ , we will obtain various subclasses of meromorphic function of the above classes (see [5–7]).

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ($\mathbb{N} = \{1, 2, \dots\}$), we define the multiplier transformation $J^m(\lambda, \ell)$ for functions $f \in \sum$ (see [8, 9] with $p = 1$) by

$$J^m(\lambda, \ell)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\ell + \lambda(k+1)}{\ell} \right)^m a_k z^k \quad (\ell > 0; \lambda \geq 0; z \in U^*). \quad (1.6)$$

Obviously, we have

$$J^{m_1}(\lambda, \ell)(J^{m_2}(\lambda, \ell)f(z)) = J^{m_2}(\lambda, \ell)(J^{m_1}(\lambda, \ell)f(z)) = J^{m_1+m_2}(\lambda, \ell)f(z), \quad (1.7)$$

for all integers m_1 and m_2 .

We note that

- (i) $J^m(1, \ell)f(z) = I(m, \ell)f(z)$ (see [10, 11]);
- (ii) $J^m(\lambda, 1)f(z) = D_\lambda^m f(z)$ (see [12]);
- (iii) $J^m(1, 1)f(z) = I^m f(z)$ (see [13]).

Setting

$$\varphi_{\lambda,\ell}^m(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\ell + \lambda(k+1)}{\ell} \right)^m z^k \quad (m \in \mathbb{N}_0; \ell > 0; \lambda \geq 0; z \in U^*), \quad (1.8)$$

we define a new function $\varphi_{\lambda,\ell}^{m,\mu}(z)$ in terms of the Hadamard product (or convolution) by

$$\varphi_{\lambda,\ell}^m(z) * \varphi_{\lambda,\ell}^{m,\mu}(z) = \frac{1}{z(1-z)^\mu} \quad (\mu > 0; z \in U^*). \quad (1.9)$$

Essentially Choi et al. [14] motivated the Choi-Saigo-Srivastava operator for analytic functions, which includes an integral operator considered earlier by Noor [15] and others [16–18]; we now introduce the operator $I_\mu^m(\lambda, \ell) : \Sigma \rightarrow \Sigma$, which is defined here by;

$$I_\mu^m(\lambda, \ell)f(z) = \varphi_{\lambda,\ell}^{m,\mu}(z) * f(z) \quad (f \in \Sigma; m \in \mathbb{N}_0; \ell > 0; \lambda \geq 0; \mu > 0). \quad (1.10)$$

We note that

- (i) $I_2^0(1, 1)f(z) = zf'(z) + 2f(z)$ and $I_2^1(1, 1)f(z) = f(z)$;
- (ii) $I_\mu^m(1, \ell)f(z) = I_{\ell,\mu}^m f(z)$ (see [11]).

It is easily verified from the definition of the operator $I_\mu^m(\lambda, \ell)$ that

$$\lambda z \left(I_\mu^{m+1}(\lambda, \ell)f(z) \right)' = \ell I_\mu^m(\lambda, \ell)f(z) - (\lambda + \ell) I_\mu^{m+1}(\lambda, \ell)f(z) \quad (\lambda > 0), \quad (1.11)$$

$$z \left(I_\mu^m(\lambda, \ell)f(z) \right)' = \mu I_{\mu+1}^m(\lambda, \ell)f(z) - (\mu + 1) I_\mu^m(\lambda, \ell)f(z). \quad (1.12)$$

Next, by using the operator $I_\mu^m(\lambda, \ell)$ defined by (1.10), we introduce the following subclasses of meromorphic functions:

$$\begin{aligned} \Sigma S_{\lambda,\ell}^{m,\mu}(\eta; \varphi) &= \left\{ f : f \in \Sigma \text{ and } I_\mu^m(\lambda, \ell)f(z) \in \Sigma S(\eta; \varphi) \right\} \\ &\quad (\varphi \in M; \lambda, \ell, \mu > 0; m \in \mathbb{N}_0; 0 \leq \eta < 1), \\ \Sigma K_{\lambda,\ell}^{m,\mu}(\eta; \varphi) &= \left\{ f : f \in \Sigma \text{ and } I_\mu^m(\lambda, \ell)f(z) \in \Sigma K(\eta; \varphi) \right\} \\ &\quad (\varphi \in M; \lambda, \ell, \mu > 0; m \in \mathbb{N}_0; 0 \leq \eta < 1), \\ \Sigma C_{\lambda,\ell}^{m,\mu}(\eta, \beta; \varphi, \psi) &= \left\{ f : f \in \Sigma \text{ and } I_\mu^m(\lambda, \ell)f(z) \in \Sigma C(\eta, \beta; \varphi, \psi) \right\} \\ &\quad (\varphi, \psi \in M; \lambda, \ell, \mu > 0; m \in \mathbb{N}_0; 0 \leq \eta, \beta < 1). \end{aligned} \quad (1.13)$$

We also note that

$$f(z) \in \sum K_{\lambda, \ell}^{m, \mu}(\eta; \varphi) \iff -zf'(z) \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi). \quad (1.14)$$

In particular, we set

$$\begin{aligned} \sum S_{\lambda, \ell}^{m, \mu}\left(\eta; \frac{1 + Az}{1 + Bz}\right) &= \sum S_{\lambda, \ell}^{m, \mu}(\eta; A, B) \quad (-1 < B < A \leq 1), \\ \sum K_{\lambda, \ell}^{m, \mu}\left(\eta; \frac{1 + Az}{1 + Bz}\right) &= \sum K_{\lambda, \ell}^{m, \mu}(\eta; A, B) \quad (-1 < B < A \leq 1). \end{aligned} \quad (1.15)$$

The main object of this paper is to investigate several inclusion properties of the classes mentioned above. Some applications involving integral operator are also considered.

2. Inclusion Properties Involving the Operator $I_{\mu}^m(\lambda, \ell)$

The following lemmas will be required in our investigation.

Lemma 2.1 (see [19]). *Let φ be convex univalent in U with $\varphi(0) = 1$ and $\operatorname{Re}\{\beta\varphi(z) + v\} > 0$ ($\beta, v \in \mathbb{C}$). If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + v} < \varphi(z) \quad (2.1)$$

implies that

$$p(z) < \varphi(z). \quad (2.2)$$

Lemma 2.2 (see [20]). *Let φ be convex univalent in U and let w be analytic in U with $\operatorname{Re}\{w(z)\} \geq 0$. If $p(z)$ is analytic in U and $p(0) = \varphi(0)$, then*

$$p(z) + w(z)zp'(z) < p(z), \quad (2.3)$$

implies that

$$p(z) < \varphi(z). \quad (2.4)$$

At first, with the help of Lemma 2.1, we prove the following theorem.

Theorem 2.3. *Let $\varphi \in M$ with*

$$\max_{z \in U}(\operatorname{Re}\{\varphi(z)\}) < \min\left(\frac{\mu + 1 - \eta}{1 - \eta}, \frac{(\ell/\lambda) + 1 - \eta}{1 - \eta}\right) \quad (\lambda, \mu, \ell > 0; 0 \leq \eta < 1); \quad (2.5)$$

then

$$\sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) \subset \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi) \subset \sum S_{\lambda, \ell}^{m+1, \mu}(\eta; \varphi). \quad (2.6)$$

Proof. We begin by showing the first inclusion relationship:

$$\sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) \subset \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi), \quad (2.7)$$

which is asserted by Theorem 2.3. Let $f \in \sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi)$ and set

$$p(z) = \frac{1}{1-\eta} \left(-\frac{z \left(I_{\mu}^m(\lambda, \ell) f(z) \right)'}{I_{\mu}^m(\lambda, \ell) f(z)} - \eta \right), \quad (2.8)$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. Then, by applying (1.11) in (2.8), we obtain

$$\mu \frac{I_{\mu+1}^m(\lambda, \ell) f(z)}{I_{\mu}^m(\lambda, \ell) f(z)} = -(1-\eta)p(z) + (\mu+1-\eta). \quad (2.9)$$

Differentiating (2.9) logarithmically with respect to z and multiplying the resulting equation by z , we have

$$\frac{1}{1-\eta} \left(-\frac{z \left(I_{\mu+1}^m(\lambda, \ell) f(z) \right)'}{I_{\mu+1}^m(\lambda, \ell) f(z)} - \eta \right) = \frac{zp'(z)}{-(1-\eta)p(z) + \mu+1-\eta} + p(z) \quad (z \in U). \quad (2.10)$$

Since

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{\mu+1-\eta}{1-\eta} \quad (\mu > 0; 0 \leq \eta < 1; z \in U), \quad (2.11)$$

we see that

$$\operatorname{Re}\{\mu+1-\eta - (1-\eta)\varphi(z)\} > 0 \quad (z \in U). \quad (2.12)$$

Applying Lemma 2.1 to (2.10), it follows that $p < \varphi$ in U , that is,

$$f \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi). \quad (2.13)$$

For the second inclusion relationship asserted by Theorem 2.3, using arguments similar to those detailed above with (1.11), we obtain

$$\sum S_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi) \subset \sum S_{\lambda, \ell}^{m+1, \mu}(\eta, \beta; \varphi). \quad (2.14)$$

We thus complete the proof of Theorem 2.3. $\square$

Theorem 2.4. *Let $\varphi \in M$ with*

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \min\left(\frac{\mu + 1 - \eta}{1 - \eta}, \frac{(\ell/\lambda) + 1 - \eta}{1 - \eta}\right) \quad (\lambda, \mu, \ell > 0; 0 \leq \eta < 1); \quad (2.15)$$

then

$$\sum K_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) \subset \sum K_{\lambda, \ell}^{m, \mu}(\eta; \varphi) \subset \sum K_{\lambda, \ell}^{m+1, \mu}(\eta; \varphi). \quad (2.16)$$

Proof. Applying (1.14) and Theorem 2.3, we observe that

$$\begin{aligned} f(z) \in \sum K_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) &\iff I_{\mu+1}^m(\lambda, \ell)f(z) \in \sum K(\eta; \varphi) \\ &\iff -z\left(I_{\mu+1}^m(\lambda, \ell)f(z)\right)' \in \sum S(\eta; \varphi) \\ &\iff I_{\mu+1}^m(\lambda, \ell)(-zf'(z)) \in \sum S(\eta; \varphi) \\ &\iff -zf'(z) \in \sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) \\ &\implies -zf'(z) \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi) \\ &\iff I_{\mu}^m(\lambda, \ell)(-zf'(z)) \in \sum S(\eta; \varphi) \\ &\iff -z\left(I_{\mu}^m(\lambda, \ell)f(z)\right)' \in \sum S(\eta; \varphi) \\ &\iff I_{\mu}^m(\lambda, \ell)f(z) \in \sum K(\eta; \varphi) \\ &\iff f(z) \in \sum K_{\lambda, \ell}^{m, \mu}(\eta; \varphi), \end{aligned}$$

$$\begin{aligned}
f(z) \in \sum K_{\lambda, \ell}^{m, \mu}(\eta; \varphi) &\iff -zf'(z) \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi) \\
&\implies -zf'(z) \in S_{\lambda, \ell}^{m+1, \mu}(\eta; \varphi) \\
&\iff -z\left(I_{\mu}^{m+1}(\lambda, \ell)f(z)\right)' \in \sum S(\eta; \varphi) \\
&\iff I_{\mu}^{m+1}(\lambda, \ell)f(z) \in \sum K(\eta; \varphi) \\
&\iff f(z) \in \sum K_{\lambda, \ell}^{m+1, \mu}(\eta; \varphi),
\end{aligned} \tag{2.17}$$

which evidently prove Theorem 2.4. $\square$

By setting

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 < B < A \leq 1; z \in U) \tag{2.18}$$

in Theorems 2.3 and 2.4, we deduce the following corollary.

Corollary 2.5. *Suppose that*

$$\frac{1 + A}{1 + B} < \min\left(\frac{\mu + 1 - \eta}{1 - \eta}, \frac{(\ell/\lambda) + 1 - \eta}{1 - \eta}\right) \quad (\lambda, \mu, \ell > 0; 0 \leq \eta < 1; -1 < B < A \leq 1). \tag{2.19}$$

Then, for the function classes defined by (1.15),

$$\begin{aligned}
\sum S_{\lambda, \ell}^{m, \mu+1}(\eta; A, B) &\subset \sum S_{\lambda, \ell}^{m, \mu}(\eta; A, B) \subset \sum S_{\lambda, \ell}^{m+1, \mu}(\eta; A, B), \\
\sum K_{\lambda, \ell}^{m, \mu+1}(\eta; A, B) &\subset \sum K_{\lambda, \ell}^{m, \mu}(\eta; A, B) \subset \sum K_{\lambda, \ell}^{m+1, \mu}(\eta; A, B).
\end{aligned} \tag{2.20}$$

Next by using Lemma 2.2, one obtains the following inclusion relationships for the class $\sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi)$.

Theorem 2.6. *Let $\varphi, \psi \in M$ with*

$$\max_{z \in U}(\operatorname{Re}\{\varphi(z)\}) < \min\left(\frac{\mu + 1 - \eta}{1 - \eta}, \frac{(\ell/\lambda) + 1 - \eta}{1 - \eta}\right) \quad (\lambda, \mu, \ell > 0; 0 \leq \eta < 1). \tag{2.21}$$

Then

$$\sum C_{\lambda, \ell}^{m, \mu+1}(\eta, \beta; \varphi, \psi) \subset \sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi) \subset \sum C_{\lambda, \ell}^{m+1, \mu}(\eta, \beta; \varphi, \psi). \quad (2.22)$$

Proof. We begin by proving that

$$\sum C_{\lambda, \ell}^{m, \mu+1}(\eta, \beta; \varphi, \psi) \subset \sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi), \quad (2.23)$$

which is the first inclusion relationship asserted by Theorem 2.6. Let

$$f \in \sum C_{\lambda, \ell}^{m, \mu+1}(\eta, \beta; \varphi, \psi). \quad (2.24)$$

Then, in view of the definition of the function class $\sum C_{\lambda, \ell}^{m, \mu+1}(\eta, \beta; \varphi, \psi)$, there exists a function $k(z) \in \sum S(\eta; \varphi)$ such that

$$\frac{1}{1-\beta} \left(-\frac{z \left(I_{\mu+1}^m(\lambda, \ell) f(z) \right)'}{k(z)} - \beta \right) < \psi(z). \quad (2.25)$$

Choose the function $g(z)$ such that $I_{\mu+1}^m(\lambda, \ell) g(z) = k(z)$. Then $g(z) \in \sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi)$ and

$$\frac{1}{1-\beta} \left(-\frac{z \left(I_{\mu+1}^m(\lambda, \ell) f(z) \right)'}{I_{\mu+1}^m(\lambda, \ell) g(z)} - \beta \right) < \psi(z). \quad (2.26)$$

Now let

$$p(z) = \frac{1}{1-\beta} \left(-\frac{z \left(I_{\mu+1}^m(\lambda, \ell) f(z) \right)'}{I_{\mu+1}^m(\lambda, \ell) g(z)} - \beta \right), \quad (2.27)$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. Using (1.12), we find that

$$\begin{aligned}
& \frac{1}{1-\beta} \left(-\frac{z(I_{\mu+1}^m(\lambda, \ell)f(z))'}{I_{\mu+1}^m(\lambda, \ell)g(z)} - \beta \right) \\
&= \frac{1}{1-\beta} \left(\frac{I_{\mu+1}^m(\lambda, \ell)(-zf'(z))}{I_{\mu+1}^m(\lambda, \ell)g(z)} - \beta \right) \\
&= \frac{1}{1-\beta} \left(\frac{z(I_{\mu}^m(\lambda, \ell)(-zf'(z)))' + (\mu+1)I_{\mu}^m(\lambda, \ell)(-zf'(z))}{z(I_{\mu}^m(\lambda, \ell)g(z))' + (\mu+1)I_{\mu}^m(\lambda, \ell)g(z)} - \beta \right) \\
&= \frac{1}{1-\beta} \\
&\quad \times \left(\frac{\left(z(I_{\mu}^m(\lambda, \ell)(-zf'(z)))' / I_{\mu}^m(\lambda, \ell)g(z) \right) + (\mu+1) \left(I_{\mu}^m(\lambda, \ell)(-zf'(z)) / I_{\mu}^m(\lambda, \ell)g(z) \right)}{\left(z(I_{\mu+1}^m(\lambda, \ell)g(z))' / I_{\mu}^m(\lambda, \ell)g(z) \right) + (\mu+1)} - \beta \right). \tag{2.28}
\end{aligned}$$

Since

$$g \in \sum S_{\lambda, \ell}^{m, \mu+1}(\eta; \varphi) \subset \sum S_{\lambda, \ell}^{m, \mu}(\eta; \varphi), \tag{2.29}$$

by Theorem 2.3, then we set

$$q(z) = \frac{1}{1-\eta} \left(-\frac{z(I_{\mu}^m(\lambda, \ell)g(z))'}{I_{\mu}^m(\lambda, \ell)g(z)} - \eta \right), \tag{2.30}$$

where $q < \varphi$ in U with the assumption that $\varphi \in M$. Then by (2.27) and (2.28), we observe that

$$I_{\mu}^m(\lambda, \ell)(-zf'(z)) = (1-\beta)p(z)I_{\mu}^m(\lambda, \ell)g(z) + \beta I_{\mu}^m(\lambda, \ell)g(z), \tag{2.31}$$

$$\begin{aligned}
& \frac{1}{1-\beta} \left(-\frac{z(I_{\mu+1}^m(\lambda, \ell)f(z))'}{I_{\mu+1}^m(\lambda, \ell)g(z)} - \beta \right) \\
&= \frac{1}{1-\beta} \left(\frac{\left(z(I_{\mu}^m(\lambda, \ell)(-zf'(z)))' / I_{\mu}^m(\lambda, \ell)g(z) \right) + (\mu+1)((1-\beta)p(z) + \beta)}{\mu+1-\eta-(1-\eta)q(z)} - \beta \right). \tag{2.32}
\end{aligned}$$

Differentiating both sides of (2.31) with respect to z , multiplying by z and dividing by $I_\mu^m(\lambda, \ell)g(z)$, we obtain

$$\frac{z(I_\mu^m(\lambda, \ell)(-zf'(z)))'}{I_\mu^m(\lambda, \ell)g(z)} = (1 - \beta)zp'(z) - [(1 - \beta)p(z) + \beta][(1 - \eta)q(z) + \eta]. \quad (2.33)$$

Now making use of (2.26), (2.32), and (2.33), we get

$$\frac{1}{1 - \beta} \left(-\frac{z(I_\mu^m(\lambda, \ell)f(z))'}{I_\mu^m(\lambda, \ell)g(z)} - \beta \right) = p(z) + \frac{zp'(z)}{(\mu + 1 - \eta) - (1 - \eta)q(z)} \prec \psi(z). \quad (2.34)$$

Since $\mu > 0$ and $q \prec \varphi$ in U with

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{\mu + 1 - \eta}{1 - \eta}, \quad (2.35)$$

we have

$$\operatorname{Re}\{\mu + 1 - \eta - (1 - \eta)q(z)\} > 0 \quad (z \in U). \quad (2.36)$$

Hence, by taking

$$w(z) = \frac{1}{(\mu + 1 - \eta) - (1 - \eta)q(z)} \quad (2.37)$$

in (2.34), and then applying Lemma 2.2, we can show that $p \prec \psi$ in U , so that

$$f \in \sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi). \quad (2.38)$$

For the second inclusion relationship asserted by Theorem 2.6, using arguments similar to those detailed above with (1.11), we obtain

$$\sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi) \subset \sum C_{\lambda, \ell}^{m+1, \mu}(\eta, \beta; \varphi, \psi). \quad (2.39)$$

We thus complete the proof of Theorem 2.6. $\square$

3. Inclusion Properties Involving the Integral Operator F_c

In this section, we consider the integral operator F_c (see, [4, page 11]) defined by

$$F_c(f) = F_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt \quad \left(f \in \Sigma; c > 0 \right). \quad (3.1)$$

From the definition (3.1), it is easily verified that

$$z \left(I_\mu^m(\lambda, \ell) F_c(f)(z) \right)' = c I_\mu^m(\lambda, \ell) f(z) - (c+1) I_\mu^m(\lambda, \ell) F_c(f)(z). \quad (3.2)$$

By using (3.2) we can prove the following theorems (see Cho et al. [11]).

Theorem 3.1. *Let $\varphi \in M$ with*

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{c+1-\eta}{1-\eta} \quad (c > 0; 0 \leq \eta < 1). \quad (3.3)$$

If $f \in \Sigma S_{\lambda, \ell}^{m, \mu}(\eta; \varphi)$, then

$$F_c(f) \in \Sigma S_{\lambda, \ell}^{m, \mu}(\eta; \varphi). \quad (3.4)$$

Theorem 3.2. *Let $\varphi \in M$ with*

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{c+1-\eta}{1-\eta} \quad (c > 0; 0 \leq \eta < 1). \quad (3.5)$$

If $f \in \Sigma K_{\lambda, \ell}^{m, \mu}(\eta; \varphi)$, then

$$F_c(f) \in \Sigma K_{\lambda, \ell}^{m, \mu}(\eta; \varphi). \quad (3.6)$$

From Theorems 3.1 and 3.2, we can easily deduce the following.

Corollary 3.3. *Suppose that*

$$\frac{1+A}{1+B} < \frac{c+1-\eta}{1-\eta} \quad (c > 0; -1 < B < A \leq 1; 0 \leq \eta < 1). \quad (3.7)$$

Then for the function classes defined by (1.15), the following inclusion relationships hold true:

$$\begin{aligned} f \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; A, B) &\implies F_c(f) \in \sum S_{\lambda, \ell}^{m, \mu}(\eta; A, B), \\ f \in \sum K_{\lambda, \ell}^{m, \mu}(\eta; A, B) &\implies F_c(f) \in \sum K_{\lambda, \ell}^{m, \mu}(\eta; A, B). \end{aligned} \quad (3.8)$$

Theorem 3.4. Let $\varphi, \psi \in M$ with

$$\max_{z \in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{c+1-\eta}{1-\eta} \quad (c > 0; 0 \leq \eta < 1). \quad (3.9)$$

If $f \in \sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi)$, then

$$F_c(f) \in \sum C_{\lambda, \ell}^{m, \mu}(\eta, \beta; \varphi, \psi). \quad (3.10)$$

Remark 3.5. Putting $\lambda = 1$ in all the above results, we will obtain the results obtained by Cho et al. [11].

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