Research Article

Weak and Almost Sure Convergence for Products of Sums of Associated Random Variables

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We study weak convergence of product of sums of stationary sequences of associated random variables to the log-normal law. The almost sure version of this result is also presented. The obtained theorems extend and generalize some of the results known so far for independent or associated random variables.

1. Introduction

In this paper, we consider sequences $(X_n)_{n \in \mathbb{N}}$ of random variables (r.v.) defined on some probability space $(\Omega, \mathfrak{F}, P)$, which are identically distributed but not necessarily independent. Let us assume that the r.v. are positive, square integrable and introduce the following notation $EX_1 = \mu$, $\text{Var}(X_1) = \sigma^2 > 0$ and denote by $\gamma = \sigma/\mu$ the coefficient of variation, further let $S_n = \sum_{k=1}^n X_k$. The first result concerning the asymptotic behavior of the products of partial sums of independent and identically distributed (i.i.d.) r.v. was obtained in 2002 by Rempała and Wesołowski (see [1]), who proved that

$$\left(\frac{\prod_{i=1}^n S_i}{n! \mu^n} \right)^{1/\gamma \sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}, \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

where $\mathcal{N}$ is a standard normal variable, and $\xrightarrow{d}$ stands for the convergence in distribution.

Recently, this result was extended in different ways, we refer the reader to [2, 3], where further references are given. In particular Gonchigdanzan and Rempała (see [2]) studied the almost sure version of (1.1) and proved that for a sequence of i.i.d. r.v., we have

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \mathbf{I} \left[\left(\frac{\prod_{j=1}^k S_j}{k! \mu^k} \right)^{1/(\gamma \sqrt{k})} \leq x \right] = F(x), \quad (1.2)$$

almost surely (a.s.), for all $x \geq 0$. Here and in the sequel, $\mathbf{I}(\cdot)$ is an indicator, and $F(x)$ is the distribution function of the log-normal random variable $\exp(\sqrt{2}\mathcal{N})$.

Li and Wang (see [3]) extended the convergence in (1.2) to the case of associated r.v. Let us recall the notion of association introduced by Esary et al. in [4]. $(X_n)_{n \in \mathbb{N}}$ is a sequence of associated r.v., if for every finite subcollection $X_{i_1}, \dots, X_{i_n}$ and any coordinatewise nondecreasing functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\text{Cov}(f(X_{i_1}, \dots, X_{i_n}), g(X_{i_1}, \dots, X_{i_n})) \geq 0, \quad (1.3)$$

whenever this covariance exists. We will also use the notion of positively quadrant-dependent r.v. according to Lehmann (see [5]). The random variables X, Y are said to be positively quadrant dependent (PQD) if

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y), \quad (1.4)$$

for all $x, y \in \mathbb{R}$. We refer the reader to the monograph [6] of Bulinski and Shashkin as the survey on association. Let us only mention that associated r.v. are pairwise PQD, independent r.v. are associated, and associated and uncorrelated r.v. are independent. Moreover, nondecreasing functions of independent r.v. and nonnegatively correlated Gaussian r.v. are associated. Associated and PQD r.v. are examples of the so-called positively dependent r.v. They are nonnegatively correlated, and in view of the properties, the covariance is often used as a measure of their dependence.

The motivation for the study of the convergence in (1.1) goes back to the paper of Arnold and Villaseñor (see [7]), who considered the limiting properties of sums of record values. The convergence in (1.1) may be also used, for example, to construct the distribution-free tests for the coefficient of variation γ . On the other hand, associated r.v. play very important role in mathematical physics and statistics, and there is a lot of papers devoted to limit theorems under this kind of dependence (see [6, 8, 9]).

The aim of our paper is to generalize the results of [1, 2] by proving (1.1) and (1.2) for strictly stationary sequences of associated r.v. We also relax the condition on the rate of decay of the covariance coefficients imposed in [3].

2. Main Results

Let us state the first of our main results. In the following theorem, we generalize the main result of [1]. We shall consider sequences of associated r.v. instead of independent r.v.

Theorem 2.1. Let $(X_n)_{n \in \mathbb{N}}$ be a strictly stationary sequence of positive and square-integrable associated r.v. Assume that

$$0 < \sigma_1^2 := \sigma^2 + 2 \sum_{k=1}^{\infty} \text{Cov}(X_1, X_{k+1}) < \infty. \quad (2.1)$$

Then,

$$\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/\gamma_1 \sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}, \quad \text{as } n \rightarrow \infty, \quad (2.2)$$

where $\gamma_1 = \sigma_1 / \mu$.

Remark 2.2. Let us note that i.i.d. sequences are trivial examples of stationary associated sequences satisfying (2.1). Therefore, the above theorem is a generalization of the main result in [1]. Furthermore, our proof simplifies the original one by employing the Marcinkiewicz-Zygmund-type theorem. As a nontrivial example of associated r.v. satisfying our assumptions, one can take a stationary sequence of Gaussian r.v. with covariance structure such that (2.1) holds. Other examples may be constructed, for example, as the moving averages of i.i.d. r.v. with positive coefficients.

In the next theorem, we present the almost sure version of the limiting result obtained in Theorem 2.1. Our results generalize the ones obtained in [2, 3].

Theorem 2.3. Assume that the conditions of Theorem 2.1 are satisfied, then,

$$\limsup_{n \rightarrow \infty} \sup_{x \geq 0} \left| \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I} \left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i} \right)^{1/\gamma_1 \sqrt{i}} \leq x \right] - F(x) \right| = 0 \quad \text{a.s.}, \quad (2.3)$$

where $F(x)$ denotes the distribution of the random variable $\exp(\sqrt{2}\mathcal{N})$.

Remark 2.4. Our Theorem 2.3 generalizes the result of [2], where i.i.d. r.v. were considered. It is also more general than the main result of [3], where a stronger and technical condition

$$\text{Cov}(X_1, X_{n+1}) = O\left(n^{-1}(\log n)^{-2-\varepsilon}\right), \quad \text{for some } \varepsilon > 0 \quad (2.4)$$

was imposed. Our assumption (2.1) on the summability of covariances is natural and was used previously in the literature (see, e.g., [9]). Let us also mention that the method used in the proof is different than in [2, 3].

3. Auxiliary Lemmas and Proofs of the Main Results

In order to prove Theorem 2.1, we need four lemmas. For the first one, let us introduce the following notation, which will be also used in the sequel. Let us denote by σ_n^2 the variance of the sum $\sum_{k=1}^n ((S_k - k\mu)/k)$, then,

$$\sigma_n^2 = \text{Var} \left(\sum_{k=1}^n b_{k,n} (X_k - \mu) \right) = \text{Var} \left(\sum_{k=1}^n b_{k,n} X_k \right), \quad (3.1)$$

where $b_{k,n} = \sum_{i=k}^n (1/i)$.

The first lemma describes the asymptotic behavior of the variance σ_n^2 .

Lemma 3.1. *Assume that the conditions of Theorem 2.1 are satisfied, then,*

$$\lim_{n \rightarrow \infty} \frac{\sigma_n^2}{2n} = \sigma_1^2 = \sigma^2 + 2 \sum_{k=1}^{\infty} \text{Cov}(X_1, X_{k+1}). \quad (3.2)$$

Proof. Let us observe that, by stationarity, we have

$$\frac{\sigma_n^2}{2n} = \frac{\sum_{k=1}^n b_{k,n}^2 \text{Var}(X_k)}{2n} + \frac{2 \sum_{k=1}^{n-1} f_{k,n} \text{Cov}(X_1, X_{k+1})}{2n}, \quad (3.3)$$

where

$$f_{k,n} = b_{1,n} b_{1+k,n} + b_{2,n} b_{2+k,n} + \cdots + b_{n-k,n} b_{n,n} = \sum_{j=1}^{n-k} b_{j,n} b_{j+k,n}. \quad (3.4)$$

For $k \in \mathbb{N}$, let us put $g_k = \sum_{i=k}^{\infty} \text{Cov}(X_1, X_{i+1})$, then $g_{k+1} - g_k = -\text{Cov}(X_1, X_{k+1})$. In order to calculate the second summand in (3.3), we shall apply the summation by parts formula

$$\sum_{k=1}^{n-1} f_{k,n} (g_{k+1} - g_k) = (g_n f_{n-1,n} - f_{1,n} g_1) - \sum_{k=1}^{n-2} g_{k+1} (f_{k+1,n} - f_{k,n}). \quad (3.5)$$

Thus, in our case,

$$\sum_{k=1}^{n-1} f_{k,n} \text{Cov}(X_1, X_{k+1}) = f_{1,n} g_1 - g_n f_{n-1,n} + \sum_{k=1}^{n-2} g_{k+1} (f_{k+1,n} - f_{k,n}). \quad (3.6)$$

Applying the identity

$$\sum_{j=1}^n b_{j,n}^2 = 2n - b_{1,n} \quad (3.7)$$

(see formula (4) in [1]), we get

$$\begin{aligned}
 f_{1,n} &= \sum_{j=1}^{n-1} b_{j,n} b_{j+1,n} = \sum_{j=1}^{n-1} b_{j,n} \left(b_{j,n} - \frac{1}{j} \right) \\
 &= \sum_{j=1}^{n-1} b_{j,n}^2 - \sum_{j=1}^{n-1} b_{j,n} \frac{1}{j} = 2n - b_{1,n} \\
 &\quad - \frac{1}{n^2} - \sum_{j=1}^{n-1} b_{j,n} \frac{1}{j}.
 \end{aligned} \tag{3.8}$$

Thus, taking into account that $f_{n-1,n} = b_{1,n} b_{n,n}$, the formula (3.3) takes the following form:

$$\begin{aligned}
 \frac{\sigma_n^2}{2n} &= \frac{\sigma^2 \sum_{k=1}^n b_{k,n}^2}{2n} + \frac{2 \left(f_{1,n} g_1 - g_n f_{n-1,n} + \sum_{k=1}^{n-2} g_{k+1} (f_{k+1,n} - f_{k,n}) \right)}{2n} \\
 &= \sigma^2 \left(\frac{2n - b_{1,n}}{2n} \right) + \frac{2n - b_{1,n} - (1/n^2) - \sum_{j=1}^{n-1} b_{j,n} (1/j)}{n} \sum_{k=1}^{\infty} \text{Cov}(X_1, X_{k+1}) \\
 &\quad - \frac{\sum_{i=n}^{\infty} \text{Cov}(X_1, X_{i+1}) b_{1,n} b_{n,n}}{n} - \frac{f_{1,n} - f_{n-1,n}}{n} \sum_{k=1}^{n-2} g_{k+1} c_{k,n-2},
 \end{aligned} \tag{3.9}$$

where $c_{k,n-2}$, for $k = 1, 2, \dots, n-2$ are defined as follows:

$$c_{k,n-2} = \frac{f_{k,n} - f_{k+1,n}}{\sum_{k=1}^{n-2} (f_{k,n} - f_{k+1,n})} = \frac{f_{k,n} - f_{k+1,n}}{f_{1,n} - f_{n-1,n}}. \tag{3.10}$$

On account that $f_{k,n} > f_{k+1,n}$, we have $c_{k,n-2} > 0$. Furthermore, $\sum_{k=1}^{n-2} c_{k,n-2} = 1$. We shall prove that $c_{k,n-2} \rightarrow 0$, as $n \rightarrow \infty$, for every $k \in \mathbb{N}$. At first, let us note that

$$\begin{aligned}
 0 &< f_{k,n} - f_{k+1,n} = \sum_{j=1}^{n-k} b_{j,n} b_{j+k,n} - \sum_{j=1}^{(n-k)-1} b_{j,n} b_{j+k+1,n} \\
 &= b_{n-k,n} b_{n,n} + \sum_{j=1}^{(n-k)-1} b_{j,n} \frac{1}{j+k} = b_{n-k,n} b_{n,n} + \sum_{j=1}^{(n-k)-1} \sum_{i=j}^n \frac{1}{i} \frac{1}{j+k} \\
 &< b_{1,n} b_{n,n} + \sum_{j=1}^n \sum_{i=1}^n \frac{1}{i} \frac{1}{j} = b_{1,n} b_{n,n} + (b_{1,n})^2 < 2(1 + \log n)^2,
 \end{aligned} \tag{3.11}$$

by the inequality $b_{1,n} - \log n < 1$. On the other hand, we have

$$\begin{aligned}
 f_{1,n} - f_{n-1,n} &= \sum_{j=1}^{n-1} b_{j,n} b_{j+1,n} - b_{1,n} b_{n,n} = \sum_{j=1}^{n-1} b_{j,n} \left(b_{j,n} - \frac{1}{j} \right) - b_{1,n} b_{n,n} \\
 &= \sum_{j=1}^n b_{j,n}^2 - b_{n,n}^2 - \sum_{j=1}^{n-1} b_{j,n} \frac{1}{j} - b_{1,n} b_{n,n} \\
 &= 2n - b_{1,n} - \frac{1}{n^2} - \sum_{j=1}^{n-1} b_{j,n} \frac{1}{j} - b_{1,n} \frac{1}{n} > 2n - (2 + \log n)^2 > n,
 \end{aligned} \tag{3.12}$$

for sufficiently large n ($n \geq 29$). Therefore, from (3.11) and (3.12), we get

$$c_{k,n-2} \leq \frac{2(1 + \log n)^2}{n} \longrightarrow 0, \tag{3.13}$$

for each $k \in \mathbb{N}$. From the Toeplitz theorem on the transformation of sequences into sequences (Problem 2.3.1 in [10]), we obtain

$$\sum_{k=1}^{n-2} g_{k+1} c_{k,n-2} \longrightarrow 0, \tag{3.14}$$

since $g_{k+1} \rightarrow 0$. It is easy to see that $0 < f_{1,n} - f_{n-1,n} < \sum_{j=1}^n b_{j,n}^2 < 2n$.

Thus, we see that from (3.9) the conclusion follows. $\square$

In the following lemma let us recall Theorem 2.3 from [11], which will be one of the tools needed in the sequel.

Lemma 3.2. *Let $\{a_{n,k}; n \in \mathbb{N}, 1 \leq k \leq n\}$ be a triangular array of real numbers satisfying*

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^n a_{n,k}^2 < \infty, \quad \lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} |a_{n,k}| = 0. \tag{3.15}$$

Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of centered, square-integrable associated r.v. such that the family $\{Y_n^2; n \in \mathbb{N}\}$ is uniformly integrable and

$$\text{Var} \left(\sum_{k=1}^n a_{n,k} Y_k \right) = 1. \tag{3.16}$$

If furthermore $\sum_{j: |k-j| \geq n} \text{Cov}(Y_k, Y_j) \rightarrow 0$, uniformly in $k \in \mathbb{N}$, as $n \rightarrow \infty$, then $\sum_{k=1}^n a_{n,k} Y_k \xrightarrow{d} \mathcal{N}$, as $n \rightarrow \infty$.

In the next lemma, we state the central limit theorem for sums of associated r.v. The proof particularly relies on Lemmas 3.1 and 3.2.

Lemma 3.3. Assume that the conditions of Theorem 2.1 are satisfied, then

$$\frac{1}{\sqrt{2n\sigma_1}} \sum_{k=1}^n \frac{S_k - k\mu}{k} = \frac{1}{\sqrt{2n\sigma_1}} \sum_{k=1}^n b_{k,n}(X_k - \mu) \xrightarrow{d} \mathcal{N}, \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

Proof. We apply Lemma 3.2 to the sequence $Y_k = X_k - \mu$ with $a_{n,k} = b_{k,n}/\sigma_n$. Let us check the assumptions. Obviously the family $\{Y_n^2; n \in \mathbb{N}\}$ is uniformly integrable. It is easy to see that by Lemma 3.1

$$\sum_{k=1}^n a_{n,k}^2 = \frac{\sum_{k=1}^n b_{k,n}^2}{\sigma_n^2} = \frac{2n - b_{1,n}}{2n} \frac{2n}{\sigma_n^2} \rightarrow \frac{1}{\sigma_1^2}, \quad \text{as } n \rightarrow \infty, \quad (3.18)$$

thus, $\sup_{n \in \mathbb{N}} \sum_{k=1}^n a_{n,k}^2 < \infty$. Further we have $\max_{1 \leq k \leq n} |a_{nk}| = b_{1,n}/\sigma_n \leq (1 + \log n)/c\sqrt{n} \rightarrow 0$.

Obviously $\text{Var}(\sum_{k=1}^n a_{n,k} Y_k) = 1$, thus,

$$\begin{aligned} \frac{1}{\sigma_n} \sum_{k=1}^n b_{k,n}(X_k - \mu) &\xrightarrow{d} \mathcal{N}, \\ \frac{1}{\sqrt{2n\sigma_1}} \sum_{k=1}^n b_{k,n}(X_k - \mu) &= \frac{1}{\sqrt{2n\sigma_1}} \sigma_n \frac{1}{\sigma_n} \sum_{k=1}^n b_{k,n}(X_k - \mu) \xrightarrow{d} \mathcal{N}, \end{aligned} \quad (3.19)$$

since we have $\sqrt{\sigma_n^2/2n\sigma_1^2} \rightarrow 1$, as $n \rightarrow \infty$. □

We shall also need the following strong law of large numbers of the Marcinkiewicz-Zygmund type.

Lemma 3.4. Let $(X_n)_{n \in \mathbb{N}}$ be a strictly stationary sequence of square-integrable associated r.v. If $\sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty$, then for every $p \in (0, 2)$, one has

$$\frac{1}{n^{1/p}} \sum_{k=1}^n (X_k - EX_k) \rightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty. \quad (3.20)$$

Proof. We directly apply Theorem 3.3 in [12]. This result states that if $(b_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence of positive real numbers and $(X_n)_{n \in \mathbb{N}}$ is a sequence of associated r.v. with

$$\sum_{j=1}^{\infty} \frac{\text{Var}(X_j)}{b_j^2} + \sum_{j \neq k}^{\infty} \frac{\text{Cov}(X_j, X_k)}{b_j b_k} < \infty, \quad (3.21)$$

then, $b_n^{-1} \sum_{j=1}^n (X_j - EX_j) \rightarrow 0$ almost surely, as $n \rightarrow \infty$. □

Proof of Theorem 2.1. After taking the logarithm it suffices to show that

$$L := \frac{1}{\gamma_1 \sqrt{2n}} \sum_{k=1}^n \log \left(\frac{S_k}{k\mu} \right) \xrightarrow{d} \mathcal{N}. \quad (3.22)$$

We shall use the expansion of the logarithm $\log(1+x) = x - (x^2/2(1+\theta x)^2)$, where $\theta \in (0,1)$ depends on $x > -1$. We have

$$L = \frac{1}{\gamma_1 \sqrt{2n}} \sum_{k=1}^n \frac{S_k - k\mu}{k\mu} - \frac{1}{\gamma_1 \sqrt{2n}} \sum_{k=1}^n \frac{((S_k - k\mu)/k\mu)^2}{2(1 + \theta_k((S_k - k\mu)/k\mu))^2}, \quad (3.23)$$

where θ_k are random variables taking values in $(0,1)$. The first term in (3.23) converges weakly to the standard normal law by Lemma 3.3. We shall prove that the second one converges almost surely to 0. By Lemma 3.4 with $p = 4/3$, we get $(S_k - k\mu)/k^{3/4} \rightarrow 0$, almost surely as $k \rightarrow \infty$, consequently $(S_k - k\mu)/k \rightarrow 0$ almost surely, as $k \rightarrow \infty$. Therefore,

$$\frac{((S_k - k\mu)/k^{3/4})^2}{(1 + \theta_k((S_k - k\mu)/k\mu))^2} \rightarrow 0, \quad \text{almost surely, as } k \rightarrow \infty. \quad (3.24)$$

The weighted average with weights $1/\sqrt{k}$ is also convergent to 0, accordingly

$$\begin{aligned} & \frac{1}{\gamma_1 \sqrt{2n}} \sum_{k=1}^n \frac{((S_k - k\mu)/k\mu)^2}{(1 + \theta_k((S_k - k\mu)/k\mu))^2} \\ &= \frac{1}{\gamma_1 \sqrt{2n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} \frac{((S_k - k\mu)/k^{3/4}\mu)^2}{(1 + \theta_k((S_k - k\mu)/k\mu))^2} \rightarrow 0, \end{aligned} \quad (3.25)$$

almost surely, as $n \rightarrow \infty$, and the proof is completed. $\square$

In the proof of Theorem 2.3, we shall use the following inequality concerning indicators. The proof is immediate.

Lemma 3.5. *For any random variables X, Y and real numbers $t \in \mathbb{R}$, $a > 0$, the following inequalities hold:*

$$\begin{aligned} \mathbf{I}[X + Y \leq t] &\leq \mathbf{I}[X \leq t + a] + \mathbf{I}[|Y| > a], \\ \mathbf{I}[X + Y \leq t] &\geq \mathbf{I}[X \leq t - a] - \mathbf{I}[|Y| > a]. \end{aligned} \quad (3.26)$$

The proof of Theorem 2.3 is based on Theorem 1 in [8], let us state it here as a lemma.

Lemma 3.6. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers, one puts $b_n = \sum_{k=1}^n a_k$, and assume that $a_n/b_n \rightarrow 0$, $b_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of pairwise PQD r.v. with distribution*

functions $F_n(x)$, respectively. Suppose that $F_n \xrightarrow{d} F$, where F is a continuous distribution function, and

$$\sum_{j=1}^{\infty} \sum_{i=1}^j \frac{a_i a_j \text{Cov}(Y_i, Y_j)}{b_j^2} < \infty. \quad (3.27)$$

Then,

$$\sup_{-\infty < x < \infty} \left| \frac{1}{b_n} \sum_{i=1}^n a_i \mathbf{I}[Y_i \leq x] - F(x) \right| \rightarrow 0 \quad \text{a.s., as } n \rightarrow \infty. \quad (3.28)$$

Proof of Theorem 2.3. Let $\varepsilon > 0$ be fixed. We put $C_k = S_k/k\mu$ and $\log x = t$. By using the expansion of the logarithm and Lemma 3.5, we get

$$\begin{aligned} \mathbf{I} \left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i} \right)^{1/\gamma_1 \sqrt{i}} \leq x \right] &= \mathbf{I} \left[\frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1) + R_i \leq t \right] \\ &\leq \mathbf{I} \left[\frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1) \leq t + \varepsilon \right] + \mathbf{I}[|R_i| > \varepsilon], \\ \mathbf{I} \left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i} \right)^{1/\gamma_1 \sqrt{i}} \leq x \right] &\geq \mathbf{I} \left[\frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1) \leq t - \varepsilon \right] - \mathbf{I}[|R_i| > \varepsilon], \end{aligned} \quad (3.29)$$

where $R_i = (-1/\gamma_1 \sqrt{i}) \sum_{k=1}^i ((C_k - 1)^2 / 2(1 + \theta_k(C_k - 1))^2)$. As in the proof of Theorem 2.1, we see that $R_i \rightarrow 0$ almost surely, as $i \rightarrow \infty$. Thus, for almost every $\omega \in \Omega$ there exists $i_0(\omega)$ such that for $i \geq i_0$ we have $|R_i(\omega)| < \varepsilon$, therefore,

$$\mathbf{I}[|R_i(\omega)| > \varepsilon] \rightarrow 0, \quad i \rightarrow \infty, \quad (3.30)$$

for almost every ω . Now, we shall apply Lemma 3.6 to the sequence

$$Y_i = \frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1), \quad (3.31)$$

and weights $a_i = 1/i$ and $b_n = \log n$. The r.v. Y_i are not only PQD but associated as well, as sums of the r.v. $C_k - 1$ which are associated. By Lemma 3.3, we have

$$\frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1) \xrightarrow{d} \sqrt{2} \mathcal{N}, \quad \text{as } i \rightarrow \infty. \quad (3.32)$$

We shall check the condition (3.27). Let us observe that

$$Y_i = \frac{1}{\gamma_1 \sqrt{i}} \sum_{k=1}^i (C_k - 1) = \frac{1}{\sigma_1 \sqrt{i}} \sum_{k=1}^i b_{k,i} (X_k - \mu), \quad (3.33)$$

whereas before $b_{k,i} = \sum_{v=k}^i (1/v)$.

For $1 \leq i < j$, we have the following decomposition:

$$\begin{aligned} \sum_{k=1}^j b_{k,j} (X_k - \mu) &= \sum_{k=1}^i (b_{k,i} + b_{i+1,j}) (X_k - \mu) + \sum_{k=i+1}^j b_{k,j} (X_k - \mu) \\ &= \sum_{k=1}^i b_{k,i} (X_k - \mu) + b_{i+1,j} \sum_{k=1}^i (X_k - \mu) + \sum_{k=i+1}^j b_{k,j} (X_k - \mu). \end{aligned} \quad (3.34)$$

Thus, for $1 \leq i < j$, we get

$$\begin{aligned} 0 \leq \text{Cov}(Y_i, Y_j) &= \frac{1}{\sigma_1^2 \sqrt{ij}} \left(\text{Var} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu) \right) \right. \\ &\quad + \text{Cov} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu); b_{i+1,j} \sum_{k=1}^i (X_k - \mu) \right) \\ &\quad \left. + \text{Cov} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu); \sum_{k=i+1}^j b_{k,j} (X_k - \mu) \right) \right). \end{aligned} \quad (3.35)$$

We shall find the bounds for each of the above terms. By Lemma 3.1, we have

$$\text{Var} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu) \right) = \sigma_i^2 \leq C_1 i, \quad (3.36)$$

for some constant $C_1 > 0$ and every $i \in \mathbb{N}$. From the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\text{Cov} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu); b_{i+1,j} \sum_{k=1}^i (X_k - \mu) \right) \\ &\leq \sqrt{\text{Var} \left(\sum_{k=1}^i b_{k,i} (X_k - \mu) \right)} \cdot b_{i+1,j} \sqrt{\text{Var} \left(\sum_{k=1}^i (X_k - \mu) \right)}. \end{aligned} \quad (3.37)$$

Stationarity, association and our assumption (2.1) imply

$$\lim_{i \rightarrow \infty} \frac{1}{i} \text{Var} \left(\sum_{k=1}^i (X_k - \mu) \right) = \sigma_1^2, \quad (3.38)$$

thus,

$$\text{Var}\left(\sum_{k=1}^i (X_k - \mu)\right) \leq C_2 i, \quad (3.39)$$

for some constant $C_2 > 0$ and every $i \in \mathbb{N}$. Furthermore, $i + 1 \leq j$, thus,

$$\begin{aligned} b_{i+1,j} &= \sum_{v=1}^j \frac{1}{v} - \sum_{v=1}^i \frac{1}{v} \leq 1 + \log j - \gamma_E - \log i \\ &\leq 1 + \log \frac{j}{i} \leq p \left(\frac{j}{i}\right)^{1/p}, \end{aligned} \quad (3.40)$$

what follows from the inequalities

$$\begin{aligned} \gamma_E &\leq \sum_{v=1}^n \frac{1}{v} - \log n \leq 1, \\ 1 + \log x &< p x^{1/p}, \quad p > 1, \quad x \geq 1, \end{aligned} \quad (3.41)$$

where $\gamma_E = \lim_{n \rightarrow \infty} (\sum_{k=1}^n (1/k) - \log n)$ denotes the Euler constant. From (3.36)–(3.40), we get

$$\text{Cov}\left(\sum_{k=1}^i b_{k,i} (X_k - \mu); b_{i+1,j} \sum_{k=1}^i (X_k - \mu)\right) \leq \sqrt{C_1 C_2} \cdot p \cdot i \cdot \left(\frac{j}{i}\right)^{1/p}. \quad (3.42)$$

Now, let us estimate the last term in (3.35):

$$A := \text{Cov}\left(\sum_{k=1}^i b_{k,i} (X_k - \mu); \sum_{k=i+1}^j b_{k,j} (X_k - \mu)\right) = A_1 + A_2, \quad \text{say}, \quad (3.43)$$

where

$$\begin{aligned} A_1 &= \sum_{v=2}^i \sum_{s=1}^{\min(i-1, j-i)} \text{Cov}(X_{v+s-1}; X_{i+s}) b_{v+s-1,i} b_{i+s,j}, \\ A_2 &= \sum_{v=1}^{j-i} \sum_{s=1}^{\min(i, j-i)} \text{Cov}(X_s; X_{v+s+i-1}) b_{s,i} b_{v+s+i-1,j}, \end{aligned} \quad (3.44)$$

for $i = 1$, we put $A_1 = 0$.

By the monotonicity properties of the coefficients $b_{n,m}$ with respect to n and fixed m , Cauchy-Schwarz inequality, (3.40), and formula (3.7), we get

$$\begin{aligned}
\sum_{s=1}^{\min(i-1,j-i)} b_{v+s-1,i} b_{i+s,j} &\leq \sum_{s=1}^{\min(i-1,j-i)} b_{s,i} b_{i+s,j} \\
&\leq \sqrt{\sum_{s=1}^{\min(i-1,j-i)} b_{s,i}^2} \cdot \sqrt{\sum_{s=1}^{\min(i-1,j-i)} b_{i+s,j}^2} \\
&\leq \sqrt{\sum_{s=1}^i b_{s,i}^2} \cdot \sqrt{\sum_{s=1}^{\min(i-1,j-i)} b_{i+1,j}^2} \leq p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p},
\end{aligned} \tag{3.45}$$

and by stationarity

$$\begin{aligned}
A_1 &\leq p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p} \sum_{v=2}^i \text{Cov}(X_{i+1}; X_v) \\
&= p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p} \sum_{v=2}^i \text{Cov}(X_1; X_v).
\end{aligned} \tag{3.46}$$

Similarly, we deal with A_2 . We have

$$\begin{aligned}
\sum_{s=1}^{\min(i,j-i)} b_{s,i} b_{v+s+i-1,j} &\leq \sqrt{\sum_{s=1}^{\min(i,j-i)} b_{s,i}^2} \cdot \sqrt{\sum_{s=1}^{\min(i,j-i)} b_{v+s+i-1,j}^2} \\
&\leq \sqrt{\sum_{s=1}^i b_{s,i}^2} \cdot \sqrt{\sum_{s=1}^i b_{i+1,j}^2} \leq p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p},
\end{aligned} \tag{3.47}$$

and by stationarity

$$A_2 \leq p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p} \sum_{v=1}^{j-i} \text{Cov}(X_1; X_{v+i}). \tag{3.48}$$

Therefore, from (3.46), (3.48), and (2.1), we get

$$A \leq p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p} \sum_{v=2}^j \text{Cov}(X_1; X_v) \leq \sigma_1^2 p\sqrt{2} \cdot i \cdot \left(\frac{j}{i}\right)^{1/p}. \tag{3.49}$$

Clearly, $\text{Cov}(Y_i, Y_j)$ is bounded by some constant for all $i \geq 1$. So that from (3.35), (3.36), (3.42) and (3.49), we obtain

$$\text{Cov}(Y_i, Y_j) \leq C_3 \left(\frac{i}{j} \right)^{1/2-1/p}, \quad (3.50)$$

for all $1 \leq i \leq j$ and $p > 1$, where the constant $C_3 > 0$ is independent of i and j but may depend on p . Now, we shall check that the condition (3.27) holds. By (3.50) with $p > 2$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{i=1}^j \frac{1}{ij \log^2 j} \text{Cov}(Y_i, Y_j) &\leq C_3 \sum_{j=1}^{\infty} \frac{1}{j \log^2 j} \frac{1}{j^{(1/2)-(1/p)}} \sum_{i=1}^j \frac{1}{i} i^{(1/2)-(1/p)} \\ &\leq C_4 \sum_{j=1}^{\infty} \frac{1}{j \log^2 j} j^{(1/2)-(1/p)} j^{(1/2)-(1/p)} < \infty, \end{aligned} \quad (3.51)$$

for some $C_4 > 0$, since $\sum_{i=1}^j i^{-(1/2)-(1/p)}$ is of order $j^{(1/2)-(1/p)}$. By (3.28), we get

$$\begin{aligned} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I}[Y_i \leq t + \varepsilon] &\longrightarrow \Phi\left(\frac{t + \varepsilon}{\sqrt{2}}\right) \text{ a.s., as } n \longrightarrow \infty, \\ \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I}[Y_i \leq t - \varepsilon] &\longrightarrow \Phi\left(\frac{t - \varepsilon}{\sqrt{2}}\right) \text{ a.s., as } n \longrightarrow \infty, \end{aligned} \quad (3.52)$$

where $\Phi(x)$ is the standard normal distribution. By the inequality $|\Phi(t) - \Phi(t + a)| \leq a/\sqrt{2\pi}$, we get

$$\Phi\left(\frac{t + \varepsilon}{\sqrt{2}}\right) < \Phi\left(\frac{t}{\sqrt{2}}\right) + \varepsilon, \quad \Phi\left(\frac{t - \varepsilon}{\sqrt{2}}\right) > \Phi\left(\frac{t}{\sqrt{2}}\right) - \varepsilon. \quad (3.53)$$

Combining the above considerations, we get for every $\varepsilon > 0$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I}\left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i}\right)^{1/\gamma \sqrt{i}} \leq x\right] &\leq \Phi\left(\frac{t}{\sqrt{2}}\right) + \varepsilon \text{ a.s.,} \\ \liminf_{n \rightarrow \infty} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I}\left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i}\right)^{1/\gamma \sqrt{i}} \leq x\right] &\geq \Phi\left(\frac{t}{\sqrt{2}}\right) - \varepsilon \text{ a.s.,} \end{aligned} \quad (3.54)$$

since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbf{I}\left[\left(\frac{\prod_{k=1}^i S_k}{i! \mu^i}\right)^{1/\gamma \sqrt{i}} \leq x\right] = \Phi\left(\frac{\log x}{\sqrt{2}}\right) \text{ a.s.} \quad (3.55)$$

But $\Phi(\log x / \sqrt{2}) = F(x)$ is the distribution of $\exp(\sqrt{2}\mathcal{N})$. By continuity of F and the same arguments as in [8], we get the uniform convergence and the conclusion follows. $\square$

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