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Research Article

On Generalized (σ, τ)-Derivations in Semiprime Rings

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Let R be a semiprime ring, I a nonzero ideal of R, and σ, τ two epimorphisms of R. An additive mapping $F : R \to R$ is generalized (σ, τ)-derivation on R if there exists a (σ, τ)-derivation $d : R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. In this paper, it is shown that if $\tau(I)d(I) \neq 0$, then R contains a nonzero central ideal of R, if one of the following holds: (i) $F[x, y] = \pm(x \circ y)_{\sigma,\tau}$; (ii) $F(x \circ y) = \pm[x, y]_{\sigma,\tau}$; (iii) $F[x, y] = \pm[F(x), y]_{\sigma,\tau}$; (iv) $F(x \circ y) = \pm(F(x) \circ y)_{\sigma,\tau}$; (v) $F[x, y] = \pm[\sigma(y), G(x)]$ for all $x, y \in I$.

1. Introduction

Throughout the present paper, R always denotes an associative semiprime ring with center $Z(R)$. For any $x, y \in R$, the commutator and anticommutator of x and y are denoted by $[x, y]$ and $x \circ y$ and are defined by $xy - yx$ and $xy + yx$, respectively. Recall that a ring R is said to be prime, if for $a, b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$ and is said to be semiprime if for $a \in R$, $aRa = 0$ implies $a = 0$. An additive mapping $d : R \to R$ is said to be derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The notion of derivation is extended to generalized derivation. The generalized derivation means an additive mapping $F : R \to R$ associated with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Then every derivation is a generalized derivation, but the converse is not true in general.

A number of authors have studied the commutativity theorems in prime and semiprime rings admitting derivation and generalized derivation (see e.g., [1–8]; where further references can be found).

Let α and β be two endomorphisms of R. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. An additive mapping $d : R \to R$ is called a (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. By this definition, every

$(1,1)$-derivation is a derivation, where 1 means the identity map of R. In the same manner the concept of generalized derivation is also extended to generalized (α,β)-derivation as follows. An additive map $F : R \to R$ is called a generalized (α,β)-derivation if there exists a (α,β)-derivation $d : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Of course every generalized $(1,1)$-derivation is a generalized derivation of R, where 1 denotes the identity map of R.

There is also ongoing interest to study the commutativity in prime and semiprime rings with (α,β)-derivations or generalized (α,β)-derivations (see [9–17]).

The present paper is motivated by the results of [17]. In [17], Rehman et al. have discussed the commutativity of a prime ring on generalized (α,β)-derivation, where α and β are automorphisms of R. More precisely, they studied the following situations: (i) $F[x,y] = \pm(x\circ y)_{\sigma,\tau}$; (ii) $F(x\circ y) = \pm[x,y]_{\sigma,\tau}$; (iii) $F[x,y] = \pm[F(x),y]_{\sigma,\tau}$; (iv) $F(x\circ y) = \pm(F(x)\circ y)_{\sigma,\tau}$; (v) $F[x,y] = \pm[\sigma(y),G(x)]$ for all $x, y \in I$, where I is a nonzero ideal of R.

The main objective of the present paper is to extend above results for generalized (α,β)-derivations in semiprime ring R, where α and β are considered as epimorphisms of R.

To prove our theorems, we will frequently use the following basic identities:

$$\begin{gathered}
[xy,z]_{\alpha,\beta} = x[y,z]_{\alpha,\beta} + [x,\beta(z)]y = x[y,\alpha(z)] + [x,z]_{\alpha,\beta}y,\\
[x,yz]_{\alpha,\beta} = \beta(y)[x,z]_{\alpha,\beta} + [x,y]_{\alpha,\beta}\alpha(z),\\
(x\circ(yz))_{\alpha,\beta} = (x\circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x,z]_{\alpha,\beta} = \beta(y)(x\circ z)_{\alpha,\beta} + [x,y]_{\alpha,\beta}\alpha(z),\\
((xy)\circ z)_{\alpha,\beta} = x(y\circ z)_{\alpha,\beta} - [x,\beta(z)]y = (x\circ z)_{\alpha,\beta}y + x[y,\alpha(z)].
\end{gathered} \tag{1.1}$$

2. Main Results

Theorem 2.1. *Let R be a semiprime ring, I a nonzero ideal of R, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a (σ,τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F([x,y]) = \pm(x\circ y)_{\sigma,\tau}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. First we consider the case

$$F([x,y]) = (x\circ y)_{\sigma,\tau} \tag{2.1}$$

for all $x, y \in I$. Replacing y by yx in (2.1) we get

$$F([x,y])\sigma(x) + \tau([x,y])d(x) = (x\circ y)_{\sigma,\tau}\sigma(x) - \tau(y)[x,x]_{\sigma,\tau}. \tag{2.2}$$

Using (2.1), it reduces to

$$\tau([x,y])d(x) = -\tau(y)[x,x]_{\sigma,\tau} \tag{2.3}$$

for all $x, y \in I$. Again replacing y by ry in (2.3), we get

$$\{\tau(r)\tau([x,y]) + \tau([x,r])\tau(y)\}d(x) = -\tau(r)\tau(y)[x,x]_{\sigma,\tau} \tag{2.4}$$

for all $x, y \in R$ and $r \in R$. Left multiplying (2.3) by $\tau(r)$ and then subtracting from (2.4) we have

$$\tau([x,r])\tau(y)d(x) = 0 \tag{2.5}$$

for all $x, y \in I$ and $r \in R$. Replacing y with sy, $s \in R$, we get $\tau([x,r])\tau(s)\tau(y)d(x) = 0$ for all $x, y \in I$ and $r, s \in R$. Since τ is an epimorphism of R, we can write

$$[R, \tau(x)]R\tau(I)d(x) = 0 \tag{2.6}$$

for all $x \in I$.

Since R is semiprime, it must contain a family $\Omega = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of Ω and $x \in I$, it follows that

$$[R, \tau(x)] \subseteq P \quad \text{or} \quad \tau(I)d(x) \subseteq P. \tag{2.7}$$

Construct two additive subgroups $T_1 = \{x \in I \mid [R, \tau(x)] \subseteq P\}$ and $T_2 = \{x \in I \mid \tau(I)d(x) \subseteq P\}$. Then $T_1 \bigcup T_2 = I$. Since a group cannot be a union of two its proper subgroups, either $T_1 = I$ or $T_2 = I$, that is, either $[\tau(I), R] \subseteq P$ or $\tau(I)d(I) \subseteq P$. Thus both cases together yield $[R, \tau(I)]\tau(I)d(I) \subseteq P$ for any $P \in \Omega$. Therefore, $[R, \tau(I)]\tau(I)d(I) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = 0$, that is, $[R, \tau(I)]\tau(I)d(I) = 0$. Thus

$$0 = [R, \tau(RIR)]\tau(RI)d(I) = [R, R\tau(I)R]R\tau(I)d(I) \tag{2.8}$$

and so $0 = [R, R\tau(I)d(I)R]R\tau(I)d(I)R$. This implies $0 = [R, J]RJ$, where $J = R\tau(I)d(I)R$ is a nonzero ideal of R, since $\tau(I)d(I) \neq 0$. Then $0 = [R, J]R[R, J]$. Since R is semiprime, it follows that $0 = [R, J]$, that is, $J \subseteq Z(R)$.

Similarly, we can obtain the same conclusion when $F([x,y]) = -(x \circ y)_{\sigma,\tau}$ for all $x, y \in I$. □

Theorem 2.2. *Let R be a semiprime ring, I a nonzero ideal of R, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a (σ,τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F(x \circ y) = \pm[x,y]_{\sigma,\tau}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We begin with the case

$$F(x \circ y) = [x,y]_{\sigma,\tau} \tag{2.9}$$

for all $x, y \in I$. Replacing y by yx in (2.9) we get

$$F(x \circ y)\sigma(x) + \tau(x \circ y)d(x) = \tau(y)[x,x]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(x). \tag{2.10}$$

Right multiplying (2.9) by $\sigma(x)$ and then subtracting from (2.10) we get

$$\tau(x \circ y)d(x) = \tau(y)[x,x]_{\sigma,\tau} \tag{2.11}$$

for all $x, y \in I$.

Now replacing y by ry in (2.11), we obtain

$$\tau(r)\tau(x \circ y)d(x) + \tau([x,r])\tau(y)d(x) = \tau(r)\tau(y)[x,x]_{\sigma,\tau} \tag{2.12}$$

for all $x, y \in I$ and for all $r \in R$. Left multiplying (2.11) by $\tau(r)$ and then subtracting from (2.12), we get

$$\tau([x,r])\tau(y)d(x) = 0 \tag{2.13}$$

for all $x, y \in I$ and for all $r \in R$. This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar results hold in case $F(x \circ y) = -[x,y]_{\sigma,\tau}$ for all $x, y \in I$. □

Theorem 2.3. *Let R be a semiprime ring, I a nonzero ideal of R, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a (σ,τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F[x,y] = \pm[F(x),y]_{\sigma,\tau}$ for all $x, y \in R$, then R contains a nonzero central ideal.*

Proof. We assume first that $F[x,y] = [F(x),y]_{\sigma,\tau}$ for all $x, y \in I$. This implies

$$F(x)\sigma(y) + \tau(x)d(y) - F(y)\sigma(x) - \tau(y)d(x) = [F(x),y]_{\sigma,\tau}. \tag{2.14}$$

Replacing y by yx in (2.14) we have

$$\begin{aligned} &F(x)\sigma(y)\sigma(x) + \tau(x)\{d(y)\sigma(x) + \tau(y)d(x)\} - F(y)\sigma(x)^2 - \tau(y)d(x)\sigma(x) \\ &\quad - \tau(y)\tau(x)d(x) = \tau(y)[F(x),x]_{\sigma,\tau} + [F(x),y]_{\sigma,\tau}\sigma(x). \end{aligned} \tag{2.15}$$

Right multiplying (2.14) by $\sigma(x)$ and then subtracting from (2.15), we get

$$\tau(x)\tau(y)d(x) - \tau(y)\tau(x)d(x) = \tau(y)[F(x),x]_{\sigma,\tau}. \tag{2.16}$$

Now replacing y by ry, where $r \in R$, in (2.16), we obtain

$$\tau(x)\tau(r)\tau(y)d(x) - \tau(r)\tau(y)\tau(x)d(x) = \tau(r)\tau(y)[F(x),x]_{\sigma,\tau}. \tag{2.17}$$

Left multiplying (2.16) by $\tau(r)$ and then subtracting from (2.17), we get that

$$[\tau(x),\tau(r)]\tau(y)d(x) = 0, \tag{2.18}$$

that is,

$$\tau([x,r])\tau(y)d(x) = 0 \tag{2.19}$$

for all $x, y \in I$ and for all $r \in R$. This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar results hold in case $F[x,y] = -[F(x),y]_{\sigma,\tau}$ for all $x, y \in I$. □

Theorem 2.4. *Let R be a semiprime ring, I a nonzero ideal of R, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a (σ,τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F(x \circ y) = \pm(F(x) \circ y)_{\sigma,\tau}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By our assumption first consider $F(x \circ y) = (F(x) \circ y)_{\sigma,\tau}$ for all $x, y \in I$. This gives

$$F(x)\sigma(y) + \tau(x)d(y) + F(y)\sigma(x) + \tau(y)d(x) = (F(x) \circ y)_{\sigma,\tau}. \tag{2.20}$$

Replacing y by yx in (2.20), we have

$$\begin{aligned} &F(x)\sigma(y)\sigma(x) + \tau(x)\{d(y)\sigma(x) + \tau(y)d(x)\} + \{F(y)\sigma(x) + \tau(y)d(x)\}\sigma(x) \\ &\quad + \tau(y)\tau(x)d(x) = (F(x) \circ y)_{\sigma,\tau}\sigma(x) - \tau(y)[F(x),x]_{\sigma,\tau}. \end{aligned} \tag{2.21}$$

Right multiplying (2.20) by $\sigma(x)$ and then subtracting from (2.21), we obtain that

$$\tau(x)\tau(y)d(x) + \tau(y)\tau(x)d(x) = -\tau(y)[F(x),x]_{\sigma,\tau}. \tag{2.22}$$

Now replacing y by ry, where $r \in R$, in (2.22) and by using (2.22), we obtain

$$\tau([x,r])\tau(y)d(x) = 0 \tag{2.23}$$

for all $x, y \in I$ and for all $r \in R$. This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar argument can be adapted in case $F(x \circ y) = -(F(x) \circ y)_{\sigma,\tau}$ for all $x, y \in I$. □

Theorem 2.5. *Let R be a semiprime ring, I a nonzero ideal of R, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a nonzero (σ,τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F[x,y] = \pm[\sigma(y),G(x)]$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We begin with the situation

$$F[x,y] = [\sigma(y),G(x)] \tag{2.24}$$

for all $x, y \in I$. Replacing y by yx in (2.24), we get

$$F([x,y])\sigma(x) + \tau([x,y])d(x) = [\sigma(y)\sigma(x),G(x)]. \tag{2.25}$$

Right multiplying (2.24) by $\sigma(x)$ and then subtracting from (2.25), we obtain that

$$\tau([x,y])d(x) = \sigma(y)[\sigma(x), G(x)] \tag{2.26}$$

for all $x, y \in I$. Now replacing y by ry in (2.26), where $r \in R$, and by using (2.26), we obtain

$$\tau([x,r])\tau(y)d(x) = 0 \tag{2.27}$$

for all $x, y \in I$ and for all $r \in R$. This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

In case $F[x,y] = -[\alpha(y), G(x)]$ for all $x, y \in I$, the similar argument can be adapted to draw the same conclusion. □

We know the fact that if a prime ring R contains a nonzero central ideal, then R must be commutative (see Lemma 2 in [18]). Hence the following corollary is straightforward.

Corollary 2.6. *Let R be a prime ring, σ and τ two epimorphisms of R and F a generalized (σ,τ)-derivation associated with a nonzero (σ,τ)-derivation d of R satisfying any one of the following conditions:*

(1) $F([x,y]) = (x \circ y)_{\sigma,\tau}$ *for all* $x, y \in R$ *or* $F([x,y]) = -(x \circ y)_{\sigma,\tau}$ *for all* $x, y \in R$;

(2) $F(x \circ y) = [x,y]_{\sigma,\tau}$ *for all* $x, y \in R$ *or* $F(x \circ y) = -[x,y]_{\sigma,\tau}$ *for all* $x, y \in R$;

(3) $F[x,y] = [F(x),y]_{\sigma,\tau}$ *for all* $x, y \in R$ *or* $F[x,y] = -[F(x),y]_{\sigma,\tau}$ *for all* $x, y \in R$;

(4) $F(x \circ y) = (F(x) \circ y)_{\sigma,\tau}$ *for all* $x, y \in R$ *or* $F(x \circ y) = -(F(x) \circ y)_{\sigma,\tau}$ *for all* $x, y \in R$;

(5) $F[x,y] = [\sigma(y), G(x)]$ *for all* $x, y \in R$ *or* $F[x,y] = -[\sigma(y), G(x)]$ *for all* $x, y \in R$;

then R must be commutative.

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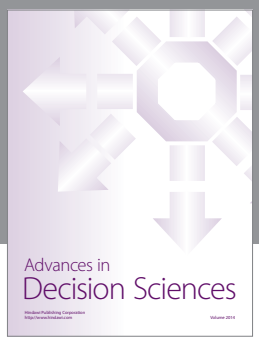

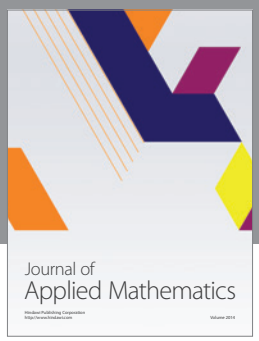

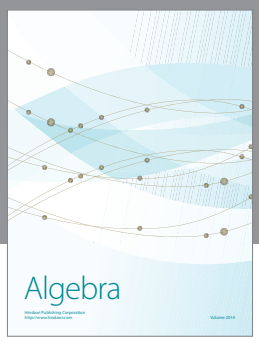

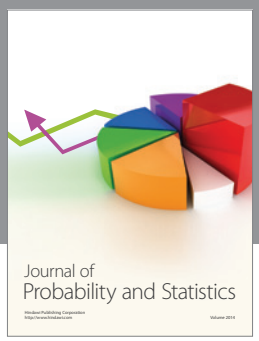

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