Research Article

On Generalized \((\sigma, \tau)\)-Derivations in Semiprime Rings

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Let \(R\) be a semiprime ring, \(I\) a nonzero ideal of \(R\), and \(\sigma, \tau\) two epimorphisms of \(R\). An additive mapping \(F : R \to R\) is generalized \((\sigma, \tau)\)-derivation on \(R\) if there exists a \((\sigma, \tau)\)-derivation \(d : R \to R\) such that \(F(xy) = F(x)\sigma(y) + \tau(x)d(y)\) holds for all \(x, y \in R\). In this paper, it is shown that if \(\tau(I)\sigma(I) \neq 0\), then \(R\) contains a nonzero central ideal of \(R\), if one of the following holds: (i) \(F[x, y] = \pm(x \circ y)_{\sigma, \tau}\); (ii) \(F(x \circ y) = \pm[x, y]_{\sigma, \tau}\); (iii) \(F[x, y] = \pm[F(x), y]_{\sigma, \tau}\); (iv) \(F(x \circ y) = \pm(F(x) \circ y)_{\sigma, \tau}\); (v) \(F[x, y] = \pm[\sigma(y), G(x)]\) for all \(x, y \in I\).

1. Introduction

Throughout the present paper, \(R\) always denotes an associative semiprime ring with center \(Z(R)\). For any \(x, y \in R\), the commutator and anticommutator of \(x\) and \(y\) are denoted by \([x, y]\) and \(x \circ y\) and are defined by \(xy - yx\) and \(xy + yx\), respectively. Recall that a ring \(R\) is said to be prime, if for \(a, b \in R\), \(aRb = 0\) implies either \(a = 0\) or \(b = 0\) and is said to be semiprime if for \(a \in R\), \(aRa = 0\) implies \(a = 0\). An additive mapping \(d : R \to R\) is said to be derivation if \(d(xy) = d(x)y + xd(y)\) holds for all \(x, y \in R\). The notion of derivation is extended to generalized derivation. The generalized derivation means an additive mapping \(F : R \to R\) associated with a derivation \(d : R \to R\) such that \(F(xy) = F(x)y + xd(y)\) holds for all \(x, y \in R\). Then every derivation is a generalized derivation, but the converse is not true in general.

A number of authors have studied the commutativity theorems in prime and semiprime rings admitting derivation and generalized derivation (see e.g., [1–8]; where further references can be found).

Let \(\alpha\) and \(\beta\) be two endomorphisms of \(R\). For any \(x, y \in R\), set \(x, y]_{\alpha, \beta} = x\alpha(y) - \beta(y)x\) and \((x \circ y)_{\alpha, \beta} = x\alpha(y) + \beta(y)x\). An additive mapping \(d : R \to R\) is called a \((\alpha, \beta)\)-derivation if \(d(xy) = d(x)\alpha(y) + \beta(x)d(y)\) holds for all \(x, y \in R\). By this definition, every
(1, 1)-derivation is a derivation, where 1 means the identity map of \( R \). In the same manner the concept of generalized derivation is also extended to generalized \((\alpha, \beta)\)-derivation as follows. An additive map \( F : R \to R \) is called a generalized \((\alpha, \beta)\)-derivation if there exists a \((\alpha, \beta)\)-derivation \( d : R \to R \) such that \( F(xy) = F(x)\alpha(y) + \beta(x)d(y) \) holds for all \( x, y \in R \). Of course every generalized (1, 1)-derivation is a generalized derivation of \( R \), where 1 denotes the identity map of \( R \).

There is also ongoing interest to study the commutativity in prime and semiprime rings with \((\alpha, \beta)\)-derivations or generalized \((\alpha, \beta)\)-derivations (see [9–17]). The present paper is motivated by the results of [17]. In [17], Rehman et al. have discussed the commutativity of a prime ring on generalized \((\alpha, \beta)\)-derivation, where \( \alpha \) and \( \beta \) are automorphisms of \( R \). More precisely, they studied the following situations: (i) \( F[x, y] = \pm(x \circ y)_{\alpha,\beta} \); (ii) \( F(x \circ y) = \pm[x, y]_{\alpha,\beta} \); (iii) \( F[x, y] = \pm[F(x), y]_{\alpha,\beta} \); (iv) \( F(x \circ y) = \pm(F(x) \circ y)_{\alpha,\beta} \); (v) \( F[x, y] = \pm[\sigma(y), G(x)] \) for all \( x, y \in I \), where \( I \) is a nonzero ideal of \( R \).

The main objective of the present paper is to extend above results for generalized \((\alpha, \beta)\)-derivations in semiprime ring \( R \), where \( \alpha \) and \( \beta \) are considered as epimorphisms of \( R \).

To prove our theorems, we will frequently use the following basic identities:

\[
[x, y, z]_{\alpha,\beta} = x[z, y]_{\alpha,\beta} + [x, y(z)] = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y,
\]

\[
[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z),
\]

\[
(x \circ (yz))_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z),
\]

\[
((xy) \circ z)_{\alpha,\beta} = xy \circ z - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)].
\]

2. Main Results

**Theorem 2.1.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \), \( \sigma \) and \( \tau \) two epimorphisms of \( R \) and \( F \) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \( d \) of \( R \) such that \( \tau(I) d(I) \neq 0 \). If \( F([x, y]) = \pm(x \circ y)_{\sigma,\tau} \) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal.

**Proof.** First we consider the case

\[
F([x, y]) = (x \circ y)_{\sigma,\tau}
\]

for all \( x, y \in I \). Replacing \( y \) by \( xy \) in (2.1) we get

\[
F([x, y])\sigma(x) + \tau([x, y])d(x) = (x \circ y)_{\sigma,\tau}\sigma(x) - \tau(y)[x, x]_{\sigma,\tau},
\]

Using (2.1), it reduces to

\[
\tau([x, y])d(x) = -\tau(y)[x, x]_{\sigma,\tau}
\]

(2.3)
for all $x, y \in I$. Again replacing $y$ by $ry$ in (2.3), we get
\[
\{ \tau(r)([x, y]) + \tau([x, r])\tau(y) \}d(x) = -\tau(r)\tau(y)[x, x]_{\sigma, \tau}
\] (2.4)
for all $x, y \in R$ and $r \in R$. Left multiplying (2.3) by $\tau(r)$ and then subtracting from (2.4) we have
\[
\tau([x, r])\tau(y)d(x) = 0
\] (2.5)
for all $x, y \in I$ and $r \in R$. Replacing $y$ with $sy$, $s \in R$, we get $\tau([x, r])\tau(s)\tau(y)d(x) = 0$ for all $x, y \in I$ and $r, s \in R$. Since $\tau$ is an epimorphism of $R$, we can write
\[
[R, \tau(x)]R\tau(I)d(x) = 0
\] (2.6)
for all $x \in I$.

Since $R$ is semiprime, it must contain a family $\Omega = \{P_{\alpha} \mid \alpha \in \Lambda\}$ of prime ideals such that $\cap P_{\alpha} = \{0\}$. If $P$ is a typical member of $\Omega$ and $x \in I$, it follows that
\[
[R, \tau(x)] \subseteq P \quad \text{or} \quad \tau(I)d(x) \subseteq P.
\] (2.7)

Construct two additive subgroups $T_1 = \{x \in I \mid [R, \tau(x)] \subseteq P\}$ and $T_2 = \{x \in I \mid \tau(I)d(x) \subseteq P\}$. Then $T_1 \cup T_2 = I$. Since a group cannot be a union of two its proper subgroups, either $T_1 = I$ or $T_2 = I$, that is, either $[R, \tau(I)] \subseteq P$ or $\tau(I)d(I) \subseteq P$. Thus both cases together yield $[R, \tau(I)]\tau(I)d(I) \subseteq P$ for any $P \in \Omega$. Therefore, $[R, \tau(I)]\tau(I)d(I) \subseteq \bigcap_{\alpha \in \Lambda} P_{\alpha} = 0$, that is, $[R, \tau(I)]\tau(I)d(I) = 0$. Thus
\[
0 = [R, \tau(IR)]\tau(RI)d(I) = [R, R\tau(I)R]R\tau(I)d(I)
\] (2.8)
and so $0 = [R, R\tau(I)d(I)R]R\tau(I)d(I)$. This implies $0 = [R, J]RF$, where $J = R\tau(I)d(I)R$ is a nonzero ideal of $R$, since $\tau(I)d(I) \neq 0$. Then $0 = [R, J]RF[R, J]$. Since $R$ is semiprime, it follows that $0 = [R, J]$, that is, $J \subseteq Z(R)$.

Similarly, we can obtain the same conclusion when $F([x, y]) = -(x \circ y)_{\sigma, \tau}$ for all $x, y \in I$. \qed

**Theorem 2.2.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $F$ a generalized $(\sigma, \tau)$-derivation associated with a $(\sigma, \tau)$-derivation $d$ of $R$ such that $\tau(I)d(I) \neq 0$. If $F(x \circ y) = \pm [x, y]_{\sigma, \tau}$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

**Proof.** We begin with the case
\[
F(x \circ y) = [x, y]_{\sigma, \tau}
\] (2.9)
for all $x, y \in I$. Replacing $y$ by $yx$ in (2.9) we get
\[
F(x \circ y)\sigma(x) + \tau(x \circ y)d(x) = \tau(y)[x, x]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(x).
\] (2.10)
Right multiplying (2.9) by \(\sigma(x)\) and then subtracting from (2.10) we get
\[
\tau(x \circ y)d(x) = \tau(y)[x, x]_{\sigma, \tau}
\tag{2.11}
\]
for all \(x, y \in I\).

Now replacing \(y\) by \(ry\) in (2.11), we obtain
\[
\tau(r)x \circ yd(x) + \tau([x, r]) \tau(y)d(x) = \tau(r)\tau(y)[x, x]_{\sigma, \tau}
\tag{2.12}
\]
for all \(x, y \in I\) and for all \(r \in R\). Left multiplying (2.11) by \(\tau(r)\) and then subtracting from (2.12), we get
\[
\tau([x, r]) \tau(y)d(x) = 0
\tag{2.13}
\]
for all \(x, y \in I\) and for all \(r \in R\). This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar results hold in case \(F(x \circ y) = -[x, y]_{\sigma, \tau}\) for all \(x, y \in I\).

\[\square\]

**Theorem 2.3.** Let \(R\) be a semiprime ring, \(I\) a nonzero ideal of \(R\), \(\sigma\) and \(\tau\) two epimorphisms of \(R\) and \(F\) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \(d\) of \(R\) such that \(\tau(I)d(I) \neq 0\).

If \(F[x, y] = \pm[F(x), y]_{\sigma, \tau}\) for all \(x, y \in R\), then \(R\) contains a nonzero central ideal.

**Proof.** We assume first that \(F[x, y] = [F(x), y]_{\sigma, \tau}\) for all \(x, y \in I\). This implies
\[
F(x)\sigma(y) + \tau(x)d(y) - F(y)\sigma(x) - \tau(y)d(x) = [F(x), y]_{\sigma, \tau}.
\tag{2.14}
\]
Replacing \(y\) by \(yx\) in (2.14) we have
\[
F(x)\sigma(y)\sigma(x) + \tau(x)\{d(y)\sigma(x) + \tau(y)d(x)\} - F(y)\sigma(x) - \tau(y)d(x)\sigma(x) - \tau(y)\tau(x)d(x) = \tau(y)[F(x), x]_{\sigma, \tau} + [F(x), y]_{\sigma, \tau}\sigma(x).
\tag{2.15}
\]
Right multiplying (2.14) by \(\sigma(x)\) and then subtracting from (2.15), we get
\[
\tau(x)\tau(y)d(x) - \tau(y)\tau(x)d(x) = \tau(y)[F(x), x]_{\sigma, \tau}.
\tag{2.16}
\]
Now replacing \(y\) by \(ry\), where \(r \in R\), in (2.16), we obtain
\[
\tau(x)\tau(r)\tau(y)d(x) - \tau(r)\tau(y)\tau(x)d(x) = \tau(r)\tau(y)[F(x), x]_{\sigma, \tau}.
\tag{2.17}
\]
Left multiplying (2.16) by \(\tau(r)\) and then subtracting from (2.17), we get that
\[
[\tau(x), \tau(r)]\tau(y)d(x) = 0,
\tag{2.18}
\]
that is,
\[ \tau([x, r]) \tau(y) d(x) = 0 \] (2.19)

for all \( x, y \in I \) and for all \( r \in R \). This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar results hold in case \( F[x, y] = -[F(x), y]_{\sigma, \tau} \) for all \( x, y \in I \).

\[ \square \]

**Theorem 2.4.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \), \( \sigma \) and \( \tau \) two epimorphisms of \( R \) and \( F \) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \( d \) of \( R \) such that \( \tau(I) d(I) \neq 0 \). If \( F(x \circ y) = \pm(F(x \circ y)_{\sigma, \tau} \) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal.

**Proof.** By our assumption first consider \( F(x \circ y) = (F(x) \circ y)_{\sigma, \tau} \) for all \( x, y \in I \). This gives
\[ F(x)\sigma(y) + \tau(x)d(y) + F(y)\sigma(x) + \tau(y)d(x) = (F(x) \circ y)_{\sigma, \tau}. \] (2.20)

Replacing \( y \) by \( yx \) in (2.20), we have
\[ F(x)\sigma(y)\sigma(x) + \tau(x) \{ d(y)\sigma(x) + \tau(y)d(x) \} + \{ F(y)\sigma(x) + \tau(y)d(x) \}\sigma(x) + \tau(y)\tau(x)d(x) = (F(x) \circ y)_{\sigma, \tau} - \tau(y) \{ F(x), x \}_{\sigma, \tau}. \] (2.21)

Right multiplying (2.20) by \( \sigma(x) \) and then subtracting from (2.21), we obtain that
\[ \tau(x)\tau(y)d(x) + \tau(y)\tau(x)d(x) = -\tau(y) \{ F(x), x \}_{\sigma, \tau}. \] (2.22)

Now replacing \( y \) by \( ry \), where \( r \in R \), in (2.22) and by using (2.22), we obtain
\[ \tau([x, r])\tau(y)d(x) = 0 \] (2.23)

for all \( x, y \in I \) and for all \( r \in R \). This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

Similar argument can be adapted in case \( F(x \circ y) = -(F(x) \circ y)_{\sigma, \tau} \) for all \( x, y \in I \).

\[ \square \]

**Theorem 2.5.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \), \( \sigma \) and \( \tau \) two epimorphisms of \( R \) and \( F \) a generalized \((\sigma, \tau)\)-derivation associated with a nonzero \((\sigma, \tau)\)-derivation \( d \) of \( R \) such that \( \tau(I) d(I) \neq 0 \). If \( F[x, y] = \pm[\sigma(y), G(x)] \) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal.

**Proof.** We begin with the situation
\[ F[x, y] = [\sigma(y), G(x)] \] (2.24)

for all \( x, y \in I \). Replacing \( y \) by \( yx \) in (2.24), we get
\[ F([x, y])\sigma(x) + \tau([x, y])d(x) = [\sigma(y)\sigma(x), G(x)]. \] (2.25)
Right multiplying (2.24) by $\sigma(x)$ and then subtracting from (2.25), we obtain that

$$\tau([x,y]) d(x) = \sigma(y)[\sigma(x), G(x)]$$  \hspace{1cm} (2.26)$$

for all $x, y \in I$. Now replacing $y$ by $ry$ in (2.26), where $r \in R$, and by using (2.26), we obtain

$$\tau([x,r]) \tau(y) d(x) = 0$$  \hspace{1cm} (2.27)$$

for all $x, y \in I$ and for all $r \in R$. This is same as (2.5) in Theorem 2.1. Thus, by same argument of Theorem 2.1, we can conclude the result here.

In case $F[x,y] = -[\sigma(y), G(x)]$ for all $x, y \in I$, the similar argument can be adapted to draw the same conclusion.

We know the fact that if a prime ring $R$ contains a nonzero central ideal, then $R$ must be commutative (see Lemma 2 in [18]). Hence the following corollary is straightforward.

**Corollary 2.6.** Let $R$ be a prime ring, $\sigma$ and $\tau$ two epimorphisms of $R$ and $F$ a generalized $(\sigma, \tau)$-derivation associated with a nonzero $(\sigma, \tau)$-derivation $d$ of $R$ satisfying any one of the following conditions:

1. $F([x,y]) = (x \circ y)_{\sigma,\tau}$ for all $x, y \in R$ or $F([x,y]) = -(x \circ y)_{\sigma,\tau}$ for all $x, y \in R$;
2. $F(x \circ y) = [x,y]_{\sigma,\tau}$ for all $x, y \in R$ or $F(x \circ y) = -[x,y]_{\sigma,\tau}$ for all $x, y \in R$;
3. $F[y,x] = [F(x),y]_{\sigma,\tau}$ for all $x, y \in R$ or $F[x,y] = -[F(x),y]_{\sigma,\tau}$ for all $x, y \in R$;
4. $F(x \circ y) = (F(x) \circ y)_{\sigma,\tau}$ for all $x, y \in R$ or $F(x \circ y) = -(F(x) \circ y)_{\sigma,\tau}$ for all $x, y \in R$;
5. $F[x,y] = [\sigma(y),G(x)]$ for all $x, y \in R$ or $F[x,y] = -[\sigma(y),G(x)]$ for all $x, y \in R$;

then $R$ must be commutative.

**References**


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