Research Article

On Behavior of Solution of Degenerated Hyperbolic Equation

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The purpose of this paper is to learn some features of hyperbolic type of nonlinear equations. It is shown that the solution of the equation approaches to the endlessness in the inside of some initial conditions and time of the special marks. The local existence of the equation's solution has been proved and the problem of unlimited increasing on the solution of nonlinear hyperbolic equations type during the finite time is investigated.

1. Introduction

In this paper the unbounded increasing solution of the nonlinear hyperbolic-type equation for the finite times is considered. These type equations describe the processes of electron and ionic heat conductivity in plasma, diffusion of neutrons, and α -particles, and so forth. Investigation of unbounded solution or regime of peaking solutions occurs in theory of nonlinear equations where one of the essential ideas is the representation called eigenfunction of nonlinear dissipative surroundings. It is well known that even a simple nonlinearity, subject to critical of exponent, the solution of nonlinear hyperbolic type equation for the finite time may increase unboundedly, that is, there is a number $T > 0$ such that

$$\|u(x, t)\|_{L_\infty(R_n)} \longrightarrow \infty, \quad t \longrightarrow T < \infty. \quad (1.1)$$

In [1] the existence of unbounded solution for finite time with a simple nonlinearity has been proved. [2] has shown that any nonnegative solution subject to critical exponent is unbounded increasing for the finite time. Similar results were obtained in [3] and

corresponding theorems are called Fujita-Hayakawa's theorems. More detailed reviews can be found in [4–6].

The paper is organizing as follows. In Section 2 we present some definitions and auxiliary results. In Section 3, we give the main results for type nonlinear equation, where blow up solutions are obtained and the local solution exists. This results generalize the corresponding result [7].

2. Some Definitions and Auxiliary Results

Let us consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\omega(x) \left| \frac{\partial u}{\partial x_j} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + f(x, t, u), \quad (2.1)$$

in bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with nonsmooth boundary. Let's denote $\Pi_{a,b} = \{(x, t), x \in \Omega, a < t < b\}$, $\Gamma_{a,b} = \{(x, t), x \in \partial\Omega, a < t < b\}$. $\Pi_a = \Pi_{a,\infty}$, $\Gamma_a = \Gamma_{a,\infty}$. The functions $f(x, t, u)$, $\partial f(x, t, u)/\partial u$ are continuous with respect to u uniformly in $\bar{\Pi}_0 \times \{u : |u| \leq M\}$ at any $M < \infty$, $f(x, t, 0) \equiv 0$, $\partial f/\partial u|_{u=0} \equiv 0$. Besides, the function f is measurable by all arguments and does not decrease with respect to u . Let's assume the fulfillment of the following Dirichlet boundary condition:

$$u = 0, \quad x \in \partial\Omega, \quad (2.2)$$

with the initial conditions

$$u|_{t=0} = \varphi(x), \quad u'|_{t=0} = \psi(x), \quad (2.3)$$

in some domain $\Pi_{0,a}$, where $\varphi(x)$ and $\psi(x)$ are smooth functions. Let's assume that $\omega(x)$ is measurable, nonnegative function satisfying the conditions $\omega \in L_{1,\text{loc}}(\Omega)$ and for any $r > 0$ and some fixed $\theta > 1$

$$\int_{B_r} \omega^{-1/(\theta-1)} dx < \infty, \quad \text{ess sup}_{x \in B_r} \omega(x) \leq C_1 r^{n(\theta-1)} \left(\int_{B_r} \omega^{-1/(\theta-1)} dx \right)^{1-\theta}, \quad (2.4)$$

where $B_r = \{x \in \Omega; |x| < r\}$. From condition (2.4) it follows that

$$\text{ess sup}_{x \in \Omega_r} \omega \leq c_1 r^{-n} \int_{B_r} \omega dx \quad (2.5)$$

and that $\omega \in A_\theta$, that is,

$$\int_{B_r} \omega dx \left[\int_{B_r} \omega^{-1/(\theta-1)} dx \right]^{\theta-1} \leq c r^{n\theta}. \quad (2.6)$$

Condition (2.6) is called Makhenkhaupt condition (see [8]). Besides, as in [8] we will assume that $\omega \in D_\mu$, $\mu < 1 + p/n$, that is,

$$\frac{\omega(B_s)}{\omega(B_h)} \leq c_1 \left(\frac{s}{h} \right)^{n\mu}. \quad (2.7)$$

Let's introduce the weight Sobolev space

$$\|u\|_{W_{p,\omega}^1(\Omega)} = \left(\int_{\Omega} \omega(x) (|u|^p + |\nabla u|^p) dx \right)^{1/p}, \quad (2.8)$$

for any $s \geq h > 0$, where $\omega(B_s) = \int_{B_s} \omega(x) dx$ with the finite norm $W_{p,\omega}^1(\Omega)$.

As the generalized solution of problem (2.1)–(2.3) in $\Pi_{a,b}$ we will define the function $u(x, t) \in W_{p,\omega}^{1,1}(\Pi_{a,b})$ such that

$$\begin{aligned} \int_{\Pi_{a,b}} u_t \eta_t dx dt + \int_{\Omega} [u_t(x, b) \eta(x, b) - u_t(x, a) \eta(x, a)] dx \\ + \int_{\Pi_{a,b}} \sum_{i,j=1}^n \omega(x) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} dx dt = \int_{\Pi_{a,b}} f(x, t, u) \eta(x, t) dx dt, \end{aligned} \quad (2.9)$$

where $\eta(x, t) \in W_{p,\omega}^{1,1}(\Pi_{a,b})$, $\eta|_{\Gamma_{a,b}} = 0$, $0 < a < b$.

We will study the conditions at some generalized solution from Π_0

$$\lim_{t \rightarrow T-0} \max_{\Omega} |u(x, t)| = +\infty, \quad (2.10)$$

at some $T = \text{const}$.

At $f(x, t, u) = u|u|^{p-1}$, $p > 1$ and at linear main part there are many works, such as [1, 2] on investigation properties of solutions having (2.10). We will show that if $\varphi(x)$ and $\psi(x)$ are adequately big, then it holds (2.10). For small $\varphi(x)$ and $\psi(x)$ if $\lim_{t \rightarrow +\infty} u(x, t) = 0$, then $|u| < ce^{-at}$, $a = \text{const} > 0$, it doesn't depend on u . Let's construct the sufficient condition on u , at which any solution of problem (2.1)–(2.3) at $f(x, t, u)$, $\varphi(x) \geq 0$, $\psi(x) \geq 0$, $\varphi(x) \neq 0$, $\psi(x) \neq 0$ has "blowup" (without limitation of smallness on $\varphi(x)$ and $\psi(x)$).

Let's formulate some auxiliary results from [9, 10], and let's determine the harmonic operator L_p

$$L_p u = \text{div}(|\nabla u|^{p-2} \nabla u), \quad p > 1. \quad (2.11)$$

Lemma 2.1 (see [9]). *There exists the positive eigenvalue of spectral problem for the operator L_p to which corresponds the positive in Ω eigenfunction.*

Lemma 2.2 (see [10]). Let $u, \vartheta \in W_p^1(\Omega)$, $u \leq \vartheta$ on $\partial\Omega$ and for any nonnegative $\eta \in \overset{o}{W}_p^1(\Omega)$

$$\int_{\Omega} L_p(u) \eta_{x_i} dx \leq \int_{\Omega} L_p(\vartheta) \eta_{x_i} dx, \quad (2.12)$$

with $\eta \geq 0$. Then on whole domain Ω , $u \leq \vartheta$.

3. Main Results

Let $u_0(x) > 0$ be an eigenfunction of spectral problem for the operator L_p corresponding to $\lambda = \lambda_1 > 0$, $\int_{\Omega} \omega(x) u_0(x) dx = 1$. Denote $g(t) = \int_{\Omega} \omega(x) u_0(x) u(x, t) dx$. Let's assume the fulfillment of the conditions:

$$I = \int_{\Omega} \omega(x) \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \right) \frac{\partial(u_0 \omega)}{\partial x_i} dx \geq 0 \quad (3.1)$$

$$(g'(0))^2 + \lambda_1 g^2(0) - \frac{2c_1}{\sigma + 1} g^{\sigma+1}(0) > 0. \quad (3.2)$$

Theorem 3.1. Let $f(x, t, u) \geq \alpha_0 |u|^{\sigma-1} u$ at $(x, t) \in \Pi_0$, $u \geq 0$, $\sigma > 1$, $\alpha_0 = \text{const} > 0$. There exists $k = \text{const}$ such that if, $u(x, 0) \geq 0$, $u'_t(x, 0) \geq 0$, $\int_{\Omega} u(x, 0) u_0 dx \geq k$, $\int_{\Omega} u'_t(x, 0) u_0 dx \geq k$ or condition (3.2) is fulfilled. Then

$$\lim_{t \rightarrow T} \max_{\Omega} (\omega(x) \cdot u_0(x) u(x, t)) = \infty, \quad (3.3)$$

where $T = \text{const} > 0$.

Proof. Let's assume the opposite. Then $u(x, t)$ is a solution of (2.1) in Π_0 and condition (2.2) is fulfilled on Γ_0 . By the virtue of Lemma 2.2 in Π_0 , $u(x, t) > 0$. Substitute in (2.9) $\eta = \varepsilon^{-1} u_0(x) \omega(x)$, $b = a + \varepsilon$, $a > 0$, $\varepsilon > 0$, where $u_0(x) > 0$ in Ω is an eigenfunction of spectral problem for the operator L_p , corresponding to eigenvalue $\lambda_1 > 0$. Such eigenvalue exists by virtue of Lemma 2.1. As a result, we will obtain

$$\begin{aligned} & \varepsilon^{-1} \left[\int_{\Omega} \omega(x) u_t(x, a + \varepsilon) u_0(x) dx - \int_{\Omega} \omega(x) u_t(x, a) u_0(x) dx \right] \\ & + \varepsilon^{-1} \int_{\Pi_{a, a+\varepsilon}} \omega(x) \sum_{i,j=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} dx dt = \varepsilon^{-1} \int_{\Pi_{a, a+\varepsilon}} u_0 \omega f(x, t, u) dx dt. \end{aligned} \quad (3.4)$$

Let's formulate some transformations. After some simple manipulation we get

$$\varepsilon^{-1} \int_{\Pi_{a, a+\varepsilon}} \omega(x) \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial \eta}{\partial x_j} dx dt. \quad (3.5)$$

Using condition (3.1), tending ε to zero at all $t > 0$, we obtain

$$\frac{\partial}{\partial t} \int_{\Omega} u_0(x) \omega(x) u_t(x, t) dx \geq -\lambda_1 \int_{\Omega} u_0^2 \omega^2(x) dx + \int_{\Omega} u_0 \omega |u|^\sigma dx. \quad (3.6)$$

Here denoting

$$g(t) = \int_{\Omega} u_0(x) \omega(x) u(x, t) dx, \quad (3.7)$$

we have

$$g''(t) \geq -\lambda_1 \int_{\Omega} u_0^2 \omega^2(x) dx + a_0 \int_{\Omega} u_0 \omega |u|^\sigma dx. \quad (3.8)$$

Let's estimate first integral on the right-hand side of (3.8). Using the Cauchy inequality with $\varepsilon > 0$ we get

$$\int_{\Omega} \omega u u_0 dx \leq \frac{\varepsilon}{2} \int_{\Omega} u^2 dx + \frac{1}{2\varepsilon} \int_{\Omega} \omega^2 u_0^2 dx. \quad (3.9)$$

Hence

$$\int_{\Omega} \omega u u_0 dx - \frac{\varepsilon}{2} \int_{\Omega} u^2 dx < \frac{1}{2\varepsilon} \int_{\Omega} \omega^2 u_0^2 dx, \quad (3.10)$$

then

$$\lambda_1 \int_{\Omega} \omega^2 u_0^2 dx > -\lambda_1 \int_{\Omega} \omega u u_0 dx + \frac{\lambda_1 \varepsilon}{2} \int_{\Omega} u^2 dx > -\lambda_1 \int_{\Omega} \omega u u_0 dx. \quad (3.11)$$

So, from (3.8) we have

$$g''(t) \geq -\lambda_1 \int_{\Omega} \omega u_0^2 u dx + a_0 \int_{\Omega} u_0 \omega u^\sigma dx. \quad (3.12)$$

By virtue of Holder inequality we obtain

$$\left(\int_{\Omega} u u_0 \omega dx \right)^\sigma \leq \left[\left(\int_{\Omega} u^\sigma u_0 \omega dx \right)^{1/\sigma} \left(\int_{\Omega} \omega u_0 dx \right)^{(\sigma-1)/\sigma} \right]^\sigma \leq c_1 \int_{\Omega} u^\sigma u_0 \omega dx. \quad (3.13)$$

Thus, we have

$$g''(t) \geq -\lambda_1 g(t) + c_1 g^\sigma(t), \quad c = \text{const} > 0. \quad (3.14)$$

Let's multiply (3.14) by g' , then

$$\frac{1}{2} \left((g')^2 \right)' \geq c_1 g^\sigma g' - \lambda_1 g g' = c_1 g^\sigma g' - \frac{1}{2} \lambda_1 (g^2)' \quad (3.15)$$

Hence

$$\left((g')^2 \right)' \geq c_1 g^\sigma g' - \lambda_1 (g^2)' \quad (3.16)$$

Let's integrate by t from 0 to t , then from (3.16) we have

$$\begin{aligned} \int_0^t \left((g')^2 \right)' dt &\geq c_1 \int_0^t g^\sigma g' dt - \lambda_1 \int_0^t (g^2)' dt, \\ (g'(t))^2 - (g'(0))^2 &\geq \frac{2c_1}{\sigma+1} [g^{\sigma+1}(t) - g^{\sigma+1}(0)] - \lambda_1 [g^2(t) - g^2(0)], \end{aligned} \quad (3.17)$$

or

$$(g'(t))^2 \geq \frac{2c_1}{\sigma+1} g^{\sigma+1}(t) - \lambda_1 g^2(t) - \frac{2c_1}{\sigma+1} g^{\sigma+1}(0) + \lambda_1 g^2(0) - g^2(0). \quad (3.18)$$

Denote

$$A = (g'(0))^2 + \lambda_1 g^2(0) - \frac{2c_1}{\sigma+1} g^{\sigma+1}(0). \quad (3.19)$$

If the condition (3.2) is fulfilled, that is,

$$\begin{aligned} \left(\int_{\Omega} \omega(x) u_0(x) \psi(x) dx \right)^2 + \lambda_1 \left(\int_{\Omega} \omega(x) u_0(x) \varphi(x) dx \right)^2 \\ - \frac{2c_1}{\sigma+1} \left(\int_{\Omega} \omega(x) u_0(x) \varphi(x) dx \right)^{\sigma\tau+1} > 0 \end{aligned} \quad (3.20)$$

and if $g(0) > c_2 = (\sqrt{\lambda_1}/\sqrt{2c_1/(\sigma+1)})^{2/(\sigma+1)}$, then $\lim_{t \rightarrow T-0} g(t) = +\infty$. Hence we have

$$\lim_{t \rightarrow T-0} \max_{\Omega} (\omega(x) u_0(x) u(x, t)) = \infty. \quad (3.21)$$

□

Remark 3.2. This theorem generalizes the corresponding result [7].

That is why, (2.1) is not a solution in Π_0 , satisfying the boundary condition (2.2), if $u(x, 0) \geq 0$, $u'_t(x, 0) \geq 0$ is not smaller.

Now we will show that at small initial functions the solution of problem (2.1)–(2.3) exists in small in Π_0 .

Theorem 3.3. *Let one assume that $|f(x, t, u)| \leq (c_3 + c_4 t^m)|u|^\sigma$, $\sigma > 1$. There exists $\delta > 0$ such that if $|\varphi(x)| < \delta$, $|\varphi(x)| < \delta$, then the solution of problem (2.1)–(2.3) exists in small in Π_0 and $|u(x, t)| \leq c_5 e^{-\alpha t}$, $\alpha = \text{const} \Rightarrow 0$ does not depend on u .*

Proof. Let $\bar{\Omega} \subset B_R$, where $B_R = \{x : |x| \leq R\}$. Let $\vartheta > 0$ in B_R be an eigenfunction, corresponding to positive eigenvalue λ_1 of the boundary-value problem

$$L_p u + \lambda_1 u = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega. \quad (3.22)$$

Let's consider the function $V(x, t) = \varepsilon e^{-\lambda_1 t/2} \vartheta(x)$. We have

$$\begin{aligned} V_{tt} - L_p V - f(x, t, V) &= \varepsilon \left(\frac{1}{4} \lambda_1^2 + c_2 \right) e^{-\lambda_1 t/2} \vartheta(x) - (c_3 + c_4 t^m) \varepsilon^\sigma e^{-\lambda_1 t/2} \vartheta^\sigma \geq 0, \\ (x, t) &\in \Pi_0, \quad V > 0, \quad (x, t) \in \Gamma_0, \end{aligned} \quad (3.23)$$

if $\varepsilon > 0$ is sufficiently small. Inequality (3.23) is understood in weak sense (see [11]). From (3.23) and Lemma 2.2, it follows that $|u| \leq V \leq c_5 e^{-\lambda_1 t/2}$. Let us determine the class of functions $|\varphi(x)| \leq \delta$, $|\varphi(x)| \leq \delta = \varepsilon \cdot \min_{\bar{\Omega}} \vartheta(x)$ consisting from $g(x, t)$ continuous in $\bar{\Pi}_{-\infty, +\infty}$ equal to zero at $t \leq T$ and such that $|g(x, t)| \leq K e^{-ht}$ is a subset at Banach space of continuous functions in $\bar{\Pi}_{-\infty, +\infty}$ with the norm $|g| = \sup_{\Pi_{-\infty, +\infty}} |g e^{ht}|$.

Let

$$\theta(t) \in C^\infty(\mathbb{R}^1), \quad \theta(t) \equiv 1, \quad \text{if } t \leq T, \quad \theta(t) = 0, \quad \text{if } t > T + 1. \quad (3.24)$$

Let's determine the operator H on K putting $Hg = \theta(t)z$, $g \in K$, where z is a solution of linearizing problem. By virtue of obtained estimation above, H maps K to K . This follows from the obtained estimation and theorem on the solution of the hyperbolic problems in $\Pi_{-a, a}$ at the small [11].

From Leray-Schauder theorem, it follows that the operator H has a fixed point z . This shows the existence of solution in the small. The theorem is proved. $\square$

Note that the sufficient condition at which any nonnegative solution of problem (2.1)–(2.3) has "blow-up" is

$$\lim_{t \rightarrow T-0} \max_{\bar{\Omega}} (\omega(x) u_0(x) u(x, t)) = +\infty, \quad (3.25)$$

where $T = \text{const} > 0$.

Theorem 3.4. *Let $f(x, t, u) \geq c_\sigma e^{\lambda_1 \sigma t} u^\sigma$ at $(x, t) \in \Pi_0$, $u \geq 0$, $\sigma = \text{const} > 1$, and λ_1 be positive eigenvalue of problem (3.16) in Ω which corresponds to the positive in Ω eigenfunction. If $u(x, 0) \geq 0$, $u'_t(x, 0) \geq 0$, $u(x, 0) \neq 0$, where $u'_t(x, 0) \neq 0$, $u(x, t)$ is a solution of problem (2.1)–(2.3) then it holds (3.25).*

Proof. Analogously, as it is constructed in inequality (3.14), we will obtain

$$g''(t) \geq -\lambda_1 g + C_\sigma e^{\lambda_1 \sigma t} g^\sigma(t). \quad (3.26)$$

Let $g(t) = \psi(t)e^{-\lambda_1 t}$. From (3.25) it follows that $\psi'' \geq c_8 \psi^\tau$. Hence $\psi(t) \rightarrow +\infty$ at $t \rightarrow T - 0$. Thus it tends $g(t)$ to $k + \infty$ and $t \rightarrow T - 0$. Consequently, $\max_{\bar{\Omega}}(\omega(x)u(x, t))$ also tends to infinity. $\square$

Corollary 3.5. *Let $f(x, t, u) \geq c_8 e^{\lambda_1 \sigma} u^\sigma$ at $(x, t) \in \Pi_0, u \geq 0, \sigma > 1$. Then there is no positive in Π_0 solution of the problem (2.1)–(2.3).*

Let one consider the following equation:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\omega(x) u^\sigma \frac{\partial u}{\partial x_j} \right) + f(x, t, u), \quad (3.27)$$

in bounded domain $\Omega \in R^n, n \geq 2$, and let $u_0(x) > 0, \int_{\Omega} u_0(x) dx = 1$, then one will have eigenfunction of the problem $\Delta u + \lambda u = 0$ in $\Omega, u = 0$ on $\partial\Omega$, and $\lambda_1 > 0$ the corresponding eigenvalue.

Theorem 3.6. *Let $f(x, t, u) \geq \alpha_0 |u|^{\sigma+1}$ at $\alpha_0 = \text{const} \geq 0$ at $(x, t) \in \Pi_0, u \geq 0, \sigma > 1$. There exists $k = \text{const} > 0$ such that if $u(x, 0) \geq 0, u'_i(x, 0) \geq 0, \int_{\Omega} u(x, 0) u_0 dx \geq k, \int_{\Omega} u'_i(x, 0) u_0 dx \geq k$ or condition (3.2) is fulfilled, then*

$$\lim_{t \rightarrow T} \max_{\Omega} (\omega(x) u_0(x) u(x, t)) = \infty, \quad (3.28)$$

where $T = \text{const} > 0$.

Proof. Let's assume a contrary. Then $u(x, t)$ is a solution of (3.27) in Π_0 and condition (2.2) is fulfilled on Γ_0 . Substituting in corresponding integral identity $\Gamma_0 \eta = \varepsilon^{-1} u_0(x) \omega(x), b = a + \varepsilon, \varepsilon > 0$ to get

$$\begin{aligned} & \varepsilon^{-1} \left[\int_{\Omega} \omega(x) u_t(x, a + \varepsilon) u_0(x) dx - \int_{\Omega} \omega(x) u_t(x, a) u_0(x) dx \right] \\ & + \varepsilon^{-1} \int_{\Pi_{a, a+\varepsilon}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\omega(x) u^\sigma \frac{\partial u}{\partial x_j} \right) u_0(x) \omega(x) dx = \varepsilon^{-1} \int_{\Pi_{a, a+\varepsilon}} u_0 \omega f(x, t, u) dx dt, \end{aligned} \quad (3.29)$$

where $u_0(x) > 0$. Tending ε to zero, we'll obtain

$$\frac{\partial^2}{\partial t^2} g(t) \geq \int_{\Omega} u_0(x) \omega(x) \frac{\partial}{\partial x_i} \left(\omega u^\sigma \frac{\partial u}{\partial x_j} \right) dx + \int_{\Omega} u_0 \omega u^{\sigma+1} dx, \quad (3.30)$$

where $g(t) = \int_{\Omega} \omega(x) u_0(x) u(x, t) dx$. Let's consider first the integral on the right-hand side in (3.30). Let $\varphi(x) > 0, \psi(x) > 0$ in Ω and $\text{supp } \varphi(x) \subset \Omega, \text{supp } \psi(x) \subset \Omega$. Denote the surface of

degeneration of (3.27), that is, the boundary of the solution $\sup u(t, x) \equiv \theta(t)$. Since $u(t, x) \equiv 0$ on $\partial\theta(t)$ and $\Omega/\theta(t)$ from the Green's formula, we get

$$\begin{aligned} \int_{\Omega} \omega u_0 \frac{\partial}{\partial x_i} \left(\omega^{\frac{\sigma}{4}} \frac{\partial u}{\partial x_j} \right) dx &\equiv \frac{1}{\sigma+1} \int_{\theta(t)} \omega u_0 \Delta u^{\sigma+1} dx \\ &\equiv \frac{1}{\sigma+1} \left[\int_{\theta(t)} \Delta u_0(0) \omega(x)^{\sigma+1} u dx + \int_{\partial\theta(t)} u_0 \omega \frac{\partial u^{\sigma+1}}{\partial n_t} ds - \int_{\partial\theta(t)} \frac{\partial(u_0 \omega)}{\partial n_t} u^{\sigma+1} dx \right], \end{aligned} \quad (3.31)$$

where $\partial/\partial n_t$ is a derivative on direction of external norms to $\partial\theta(t)$. Since $u^{\sigma+1} = 0$ is on $\partial\theta(t)$ and by virtue of continuity of flow $\partial u^{\sigma+1}/\partial u_t \equiv \nabla u^{\sigma+1} \cdot n_t = 0$ at $x \in \partial\theta(t)$. Therefore, the last two integrals in (3.31) are equal to zero. Then we will obtain

$$\int_{\Omega} \omega u_0(x) \frac{\partial}{\partial x_i} \left(\omega u^{\sigma} \frac{\partial u}{\partial x_j} \right) dx = \frac{1}{\sigma+1} \int_{\Omega} u^{\sigma+1} \Delta \omega u_0(x) dx. \quad (3.32)$$

Using $\Delta u_0(x) \omega(x) = -\lambda_1 u_0(x) \omega(x)$ we will get

$$\frac{\partial^2 g(t)}{\partial t^2} \geq \left(1 - \frac{\lambda_1}{\sigma+1} \right) \int_{\Omega} u^{\sigma+1} u_0(x) \omega(x) dx, \quad t > 0. \quad (3.33)$$

Using the results of paper [12] we will obtain the required result. $\square$

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