Research Article
On Some Volterra and Fredholm Problems via the Unified Integrodifferential Quadrature Method

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Received 6 September 2011; Accepted 20 October 2011
Academic Editors: V. Rai and K. Wang
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We present a new approach based on the formulation of the integrodifferential quadrature method (hereafter called IDQ) to handle Volterra’s and Fredholm’s equations. This approach is constructed and tested with some realistic numerical examples using the basic computational aspects.

1. Introduction
In the present work, we present a formulation of a new numerical approach which is based on the generalized integrodifferential quadrature method and applied to weakly singular Volterra and Fredholm integrodifferential equations in the linear case. This method studies the situation in which the unknown function is identified as the Lagrange polynomial and the interpolating points of the Tchebychev type are used. The accuracy and efficiency of solving integrodifferential equations are still an ongoing research in numerical analysis. Several different methods have also been modeled for the solutions of linear and nonlinear problem [1–27].

The central argument for the interest of the type of this problem comes naturally from its wide applications almost in any branches of science and engineering described by systems of ODEs and PDEs [28–38] which in some situations can be transformed into a set of integral equations. Much attention has been devoted to the investigation of new mathematical models and numerical approaches to evaluate the solutions of the integral equations.

New calculations are performed for the construction of the solution by a suitable choice of the interpolating points using the unified integrodifferential quadrature method. Based on this fact, this technique is a very promising and powerful methodology in successfully locating best solutions of the problem. Our main purpose is to develop a general numerical schema for the integrodifferential equations that is universally applicable.

The contents of this paper are organized as follows. In Section 2, a formulation adapted to the problem of construction of the solution of the one-dimensional weakly singular integrodifferential equation by using a summary limited to some aspects of the unified integrodifferential quadrature method is qualitatively presented. Section 3 exposes some essential examples for the construction of the unknown solution. The comments and conclusion are given in Section 4.

2. Unified Integrodifferential Quadrature Method
The description of generalized integrodifferential quadrature method is summarized as follows: the starting point of the weakly singular integrodifferential equation is written as

$$\sum_{k=1}^{l} y_k(x) \frac{d^k f(x)}{dx^k} = a(x) f(x) + \beta(x) + \lambda W(x, z), \quad a \leq x \leq T,$$

where

$$W(x, z) \equiv \int_{a}^{z} (x - s)^{-\gamma} K(x, s) f(s) ds,$$

In Section 2, a formulation adapted to the problem of construction of the solution of the one-dimensional weakly singular integrodifferential equation by using a summary limited to some aspects of the unified integrodifferential quadrature method is qualitatively presented. Section 3 exposes some essential examples for the construction of the unknown solution. The comments and conclusion are given in Section 4.
and we set

\[ W(x, z) = \begin{cases} W(x, x), & \text{for Volterra type,} \\ W(x, T), & \text{for Fredholm type,} \end{cases} \]

(3)

with prescribed boundary condition \( f(a) = f_a \).

\( \lambda \) and \( \nu \) are parameters with \( 0 < \nu < 1 \). \( a(x), \beta(x), \) and \( y_k(x) (k = 0, \ldots, m) \) are given functions, and \( K(x, s) \) is a regular kernel of the integrodifferential equation. It is assumed that the function involved in (1) is sufficiently regular. The above equations, for \( \nu = 0 \), become regular integrodifferential equations.

In order to avoid the singularity, (1) is integrated by part with little effort; we obtain

\[
W(x, z) = \frac{1}{(\nu - 1)} \left[ (x - z)^{1-\nu} K(x, z) f(z) \right] - (x - a)^{1-\nu} K(x, a) f(a) \]

\[ - \frac{1}{(\nu - 1)} \int_a^z (x - s)^{1-\nu} \frac{d[K(x, s) f(s)]}{ds} ds. \]

(4)

Now the relation (1) becomes

\[
\sum_{k=1}^l y_k(x) \frac{d^k f(x)}{dx^k} = \alpha(x) f(x) + \beta(x, x, z) + \frac{\lambda}{(1 - \nu)} \int_a^z (x - s)^{1-\nu} \frac{d[K(x, s) f(s)]}{ds} ds,
\]

(5)

where \( \beta(x, z) = \beta(x) + (\lambda/(\nu - 1))( (x - z)^{1-\nu} K(x, z) f(z) - (x - a)^{1-\nu} K(x, a) f(a) ) \), and \( z = x \) and \( z = T \) for Volterra type and Fredholm type, respectively.

At this stage, we introduce the quadrature aspect of solutions: the derivatives and integral in expression (6) are approximated by

\[
\frac{d^k f(x_m)}{dx^k} = \sum_{j=0}^N D_{mj}^k f(x_j),
\]

(6)

\[
\int_a^{x_m} (x_m - s)^{1-\nu} \frac{d[K(x_m, s) f(s)]}{ds} ds = \sum_{j=0}^N I_{mj} f(x_j),
\]

(7)

for Volterra type,

\[
\int_a^{T} (x_m - s)^{1-\nu} \frac{d[K(x_m, s) f(s)]}{ds} ds = \sum_{j=0}^N I_{mj} f(x_j),
\]

(8)

for Fredholm type,

where \( \{x_k\}_{k=0}^N \) are interpolating points, taken as the points of Tchebychev of the form \( x_j = (1/2)T[1 - \cos((2j + 1)/(2N + 2) \pi)], 0 \leq j \leq N \). \( D_{ij} \) and \( I_{mj} \) are the weighting coefficients linked with Lagrange interpolated polynomials \( P_{N,j}(x) \) and are explicitly given by

\[
D_{mj}^k = \frac{d^k P_{N,j}(x_m)}{dx^k},
\]

(9)

\[
I_{mj} = \int_0^{x_m} (x_m - s)^{1-\nu} \frac{d[K(x_m, s) P_{N,j}(s)]}{ds} ds,
\]

(10)

\[
J_{mj} = \int_0^{T} (x_m - s)^{1-\nu} \frac{d[K(x_m, s) P_{N,j}(s)]}{ds} ds.
\]

With little effort the original problems (1) can be converted to an algebraic system and take, respectively, the following compact forms:

\[
\alpha(x_m) f(x_m) + \beta(x_m, x_m) = \sum_{j=0}^N U_{mj}^l f(x_j), \quad m = 0, \ldots, N,
\]

(11)

for Volterra type, and

\[
\alpha(x_m) f(x_m) + \beta(x_m, T) = \sum_{j=0}^N U_{mj}^l f(x_j), \quad m = 0, \ldots, N,
\]

(12)

for Fredholm type, where

\[
U_{mj}^l = \sum_{k=0}^l y_k(x_m) D_{mj}^k - \frac{\lambda}{(1 - \nu)} I_{mj},
\]

(13)

\[
U_{mj}^l = \sum_{k=0}^l y_k(x_m) D_{mj}^k - \frac{\lambda}{(1 - \nu)} I_{mj}.
\]

(14)

The relations (12) and (13) are real systems of \( M \times M \) equations in the \( M \times M \) unknown real coefficients \( U_{ij}^l \) and \( U_{ij}^l \), respectively, where \( M = N + 1 \). In order to avoid unnecessary calculation and to optimize the computational aspect, it is therefore more convenient to get the desired coefficients \( D_{ij} \) and \( I_{ij} \) in the following forms:

\[
D_{ij} = - \sum_{j=0}^N D_{ij}^k, \quad \text{for } i = 0, \ldots, N, \quad k = 1, \ldots, l,
\]

(15)

\[
I_{ii} = \int_a^{x_i} (x_i - s)^{1-\nu} \frac{dK(x_i, s)}{ds} ds - \sum_{j=0}^N I_{ij}, \quad \text{for } i = 0, \ldots, N,
\]

(16)

\[
J_{ii} = \int_a^{T} (x_i - s)^{1-\nu} \frac{dK(x_i, s)}{ds} ds - \sum_{j=0}^N I_{ij}, \quad \text{for } i = 0, \ldots, N.
\]

(17)

Now the expressions (14), (15), (16), (17), and (18) provide the charmingly practical formulae for the weighting coefficients. Once the function values at all grid points are
obtained, it is then easy to determine the function values in the overall domain in terms of polynomial approximation, such that

\[ f(x) = \sum_{j=0}^{N} f(x_j) P_{N,j}(x). \]  

(18)

The practical part of this study is examined in the following section.

3. Worked Examples

We shall consider the solution of some crack situations in order to show the application and the effectiveness of the method described in Section 2. Four examples are considered for this purpose.

We write (1) in the form

\[
\alpha_p f_p + \beta_{v,pp} = \sum_{j=0}^{N} U^T_{pj} f_j, \quad p = 0, \ldots, N, \tag{19}
\]

\[
\alpha_f f_p + \beta_{v,pt} = \sum_{j=0}^{N} U^T_{pj} f_j, \quad p = 0, \ldots, N, \tag{20}
\]

where \( f_j \) and \( a_i \) stand for the values of \( f(x_i) \) and \( a(x_i) \), respectively, \( \beta_{v,pp} \) and \( \beta_{v,pt} \) denote the values of \( \beta_v(x_p, x_p) \) and \( \beta_v(x_p, T) \), respectively.

Evaluating (20) and (21) at \( x_p \), the linear systems of algebraic equations follow, respectively,

\[
[IA - U_1]F = B_1, \tag{21}
\]

\[
[IA - U_2]F = B_2, \tag{22}
\]

where the vectors \( F \) and \( B_i(Bl) \) have components \( f_p \) and \( \beta_{v,pp}(\beta_{v,pt}) \), respectively, \( U_i(U_2) \) is a matrix with the elements \( U_{ij} = U_{ij}^T \), and the components of \( U^T \) and \( U^T \) are given by (14) and (15) together with (16), (17), and (18). The equations (20) and (21) are explicit prescriptions that give the solutions in \( O(N^2) \) operations.

The accuracy of solutions can be checked by using the error tolerance \( E(x) = |f(x) - f_{exact}(x)| = 10^{-j} \), where “\( j \)” is a positive integer. We continue evaluating the solution until the above relation was satisfied.

In order to verify the efficiency of the method developed in the previous section, the following examples have been selected to provide a comparison with previously published work.

3.1. Example 1. In this example we use Volterra’s integrodifferential equation [39] given by

\[
\frac{df(x)}{dx} = f(x) + \beta(x) - \int_{0}^{x} (x-s)^{-1/2} f(s)ds, \quad 0 \leq x \leq 1, f(0) = 0, \tag{23}
\]

where \( \beta(x) = (3/2) \sqrt{x} + x \sqrt{x} + (3\pi/8)x^2 \), from the governing equation (1) and (24) subject to \( K(x,s) = 1 \), we deduce the following: \( I = 1, \gamma_1(x) = 1, \alpha(x) = 1, \lambda = -1, \nu = 1/2, a = 0, T = 1, \text{ and } f_0 = 0 \). The exact solution is \( f(x) = x^{1/2} \).

Illustrated in Figure 1 is the basic application of the problem (24). The numerical result displayed in Figure 1 is qualitatively in good agreement with the exact solution in which the error tolerance, \( \text{TOLV} = 10^{-5} \).

3.2. Example 2. We consider the numerical solution of the following Volterra’s integrodifferential equation [40]:

\[
\frac{d^2 f(x)}{dx^2} = f(x) + \beta(x) + \int_{0}^{x} xs f(s)ds, \quad 0 \leq x \leq 1, f(0) = 1, \tag{24}
\]

where \( \beta(x) = -x(x+1) + x(1-x) \exp(x) + 2 - x^5/4, \) and from this equation we deduce the following: \( I = 2, \gamma_1(x) = 0, \gamma_2(x) = 1, \alpha(x) = 1, \lambda = 1, \nu = 0, a = 0, T = 1, f_0 = 1, \) and the kernel \( K(x,s) = xs \).

The above equation has the exact solution \( f(x) = x^2 + \exp(x) \).

The result is displayed in Figure 2, where the error tolerance, \( \text{TOLV} = 10^{-6} \). With this tolerance and for the number of the interpolation points \( N = 10 \), the results under consideration are very satisfactory.

3.3. Example 3. Consider the Fredholm integrodifferential equation [41]:

\[
\frac{d^2 f(x)}{dx^2} + x \frac{df(x)}{dx} = xf(x) + \beta(x) + \int_{-1}^{1} e^{-s} \sin(x) f(s)ds, \quad -1 \leq x \leq 1, f(-1) = \frac{1}{e^x}, \tag{25}
\]

where \( \beta(x) = \exp(-2 \sin(x)) \), and from this equation we deduce the following: \( I = 2, \gamma_1(x) = x, \gamma_2(x) = 1, \alpha(x) = x, \lambda = 1, \nu = 0, a = -1, T = 1, f_{-1} = 1/e, \) and the kernel \( K(x,s) = e^{-s} \sin(x) \). The exact solution has the form \( f(x) = \exp(x) \).

Figure 1: Solution \( f(x) \) of (23). Solid curve: exact solution; \( f(x) = x^{1/2} \), dash curve: present work.
In Figure 3, the number of interpolation points is taken equal to \( N = 10 \) to solve this problem with \( \text{TOLV} = 10^{-8} \). We see from this Figure that the method provides accurate results in the full range, where the error tolerance, \( \text{TOLV} = 10^{-6} \).

### 3.4. Example 4.

We apply the mentioned technique to solve the fifth-order Fredholm integrodifferential equation [42]:

\[
\frac{d^5 f(x)}{dx^5} - x^2 \frac{d^3 f(x)}{dx^3} - \frac{df(x)}{dx} = xf(x) + \beta(x) + \int_{-1}^{1} f(s) ds, \\
-1 \leq x \leq 1, f(-1) = -\sin(1),
\]

(26)

where \( \beta(x) = x^2 \cos(x) - x \sin(x) \), and from this equation we deduce the following: \( l = 5, \gamma_1(x) = -1, \gamma_2(x) = \gamma_4(x) = 0, \gamma_3(x) = -x^2, \gamma_5(x) = 1, \alpha(x) = x, \lambda = 1, \nu = 0, a = -1, T = 1, f_{-1} = -\sin(1) \), and the kernel \( K(x, s) = 1 \). The exact solution has the form \( f(x) = \sin(x) \).

Figure 4 presents computer simulations of this example, that is generated using the number of interpolating points \( N = 8 \). The result is compared with the exact solution. From Figure 4, we see that the agreement between the numerical result of the present method and the exact result is very good, with an error tolerance \( \text{TOLV} = 10^{-6} \).

### 4. Comments and Conclusions

This paper has introduced a new formulation that uses the unified integrodifferential quadrature method to handle some integrodifferential equations. The main advantage of the Lagrange polynomial is that the weighting coefficients that have to be computed do not depend on the \( f(x_i) \) and it is independent on the manner the discrete points are ordered. The preliminary results, obtained through the use of this method, show that the resulting solutions are quite good for all examples which have been selected here as a testbed. We can note that similar patterns of convergence are seen for all four examples with a common tolerance \( \text{TOLV} = 10^{-6} \).

This method seems to be a powerful alternative and consequently gives very accurate solutions to the problems under consideration. It would be interesting to generalize this method to the nonlinear case. This matter deserves a further work.

### Acknowledgments

The authors gratefully acknowledge helpful conversations with Professor W. Cramer. This work was sponsored in part by the M.E.R.S (Ministère de l’Enseignement et de la Recherche Scientifique): under contract no. D01420060012.

### References


