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Review Article

Growth for Algebras Satisfying Polynomial Identities

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The nth codimension $c_n(A)$ of a PI algebra A measures how many identities of degree n the algebra A satisfies. Growth for PI algebras is the rate of growth of $c_n(A)$ as n goes to infinity. Since in most cases there is no hope in finding nice closed formula for $c_n(A)$, we study its asymptotics. We review here such results about $c_n(A)$, when A is an associative PI algebra. We start with the exponential bound on $c_n(A)$ then give few applications. We review some remarkable properties (integer and half integer) of the asymptotics of $c_n(A)$. The representation theory of the symmetric group S_n is an important tool in this theory.

1. Introduction

We study algebras A satisfying polynomial identities (PI algebra). A natural question arises: give a quantitative description of how many identities such algebra A satisfies? We assume that A is associative, though the general approach below can be applied to nonassociative PI algebras as well.

Denote by $\mathrm{Id}(A)$ the ideal of identities of A in the free algebra $F\{x\}$. If $\mathrm{Id}(A) \neq 0$ then its dimension $\dim \mathrm{Id}(A)$ is always infinite, hence dimension by itself is essentially of no use here. In order to overcome this difficulty we now introduce the sequence of codimensions.

1.1. Growth for PI Algebras

Given n, we let V_n denote the multilinear polynomials of degree n in $x_1, \ldots, x_n$, so in the associative case $\dim V_n = n!$. Identities can always be multilinearized, hence the subset $\mathrm{Id}(A) \cap V_n$ and its dimension give a good indication as to how many identities of degree n A satisfies. In fact, in characteristic zero the ideal $\mathrm{Id}(A)$ is completely determined by the sequence $\{\mathrm{Id}(A) \cap V_n\}_{n\geq 1}$, but we make no use of that remark in the sequel.

To study $\dim(\mathrm{Id}(A) \cap V_n)$, we introduce the quotient space $V_n/(\mathrm{Id}(A) \cap V_n)$ and its dimension

$$c_n(A) = \dim\left(\frac{V_n}{\mathrm{Id}(A) \cap V_n}\right). \tag{1.1}$$

The integer $c_n(A)$ is the nth *codimension* of A. Clearly $c_n(A)$ determines $\dim(\mathrm{Id}(A)\cap V_n)$ since $\dim V_n$ is known.

The study of growth for PI algebra A is mostly the study of the rate of growth of the sequence $c_n(A)$ of its codimensions, as n goes to infinity. We have the following basic property.

Theorem 1.1 (see [1]). *In the associative case, $c_n(A)$ is always exponentially bounded.*

This theorem implies several key properties for PI algebras. And it fails in various nonassociative cases.

Various recent results indicate that in general there is no hope to find a closed formula for $c_n(A)$. Instead, one therefore tries to determine the asymptotic behavior of that sequence, as n goes to infinity. We mention here three such results.

Recall that for two sequences of numbers, $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

(1) The asymptotics for the $k \times k$ matrices $M_k(F)$, see Section 6.

Theorem 1.2 (see [2]). *When n goes to infinity,*

$$c_n(M_k(F)) \sim \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{k-1}\left(\frac{1}{2}\right)^{(k^2-1)/2} \cdot 1!2!\cdots(k-1)! \cdot k^{(k^2/2)}\right] \cdot \left(\frac{1}{n}\right)^{(k^2-1)/2} \cdot k^{2(n+1)}. \tag{1.2}$$

(2) The integrality theorem of Giambruno-Zaicev, see Section 10.

Theorem 1.3 (see [3]). *Let A be an associative PI F-algebra with* $\mathrm{char}(F) = 0$, *then the limit*

$$\lim_{n\to\infty} (c_n(A))^{1/n} \tag{1.3}$$

exists and is an integer. We denote $\exp(A) = \lim_{n\to\infty}(c_n(A))^{1/n}$, *so* $\exp(A) \in \mathbb{N}$.

(3) The "1/2" theorem of Berele, see Section 11.

Theorem 1.4 (see [4, 5]). *Let A be a PI algebra with $1 \in A$. Then as n goes to infinity, $c_n(A) \sim a \cdot n^t \cdot h^n$, moreover, $h \in \mathbb{N}$ (h is given by the previous theorem) and $t \in (1/2)\mathbb{Z}$, namely, t is an integer or a half integer.*

1.2. Structure of the Paper

The paper reviews some of the main results about the asymptotics of codimensions. It does not contain full proofs but rather, it indicates some of the key ideas in the proofs of the main results.

We start by introducing Kemer's classification of the verbally prime T-ideals. After introducing the codimensions, two proofs of their exponential bound are reviewed. The S_n character of that space, $\chi_{S_n}(V_n/(\mathrm{Id}(A)\cap V_n))$ is denoted as follows:

$$\chi_n(A) = \chi_{S_n}\left(\frac{V_n}{\mathrm{Id}(A)\cap V_n}\right) \tag{1.4}$$

and is called the nth cocharacter of A. Since $c_n(A) = \deg \chi_n(A)$, cocharacters are refinement of codimensions, and are important tool in their study. By a theorem of Amitsur-Regev and of Kemer, $\chi_n(A)$ is supported on some (k,ℓ) hook. Shirshov's Height Theorem then implies that the multiplicities in the cocharacters are polynomially bounded.

We then review the proof of the Giambruno-Zaicev Theorem in the finite dimensional case, and the proof of Berele's "1/2" Theorem in the case of a Capelli identity.

The question of the algebraicity of the generating function $\sum_n c_n(A)\cdot x^n$ is examined in Section 12.

In the last section, we review a construction of nonassociative algebras where the integrality property of the exponent fails.

2. PI Algebras and T Ideals

2.1. T-ideals

Let F be a field. In most cases we assume that $\mathrm{char}(F) = 0$. We begin by studying associative F-algebras. Analogue theories exist for nonassociative algebras. Let $F\{x\} = F\{x_1, x_2, \ldots\}$ denote the algebra of associative and noncommutative polynomials in the countable sequence of variables $x_1, x_2, \ldots$. The polynomial $f(x_1,\ldots,x_n) \in F\{x\}$ is an identity of the F-algebra A if $f(a_1,\ldots,a_n) = 0$ for every $a_1,\ldots,a_n \in A$. The algebra A satisfies polynomial identities, or in short is PI, if there exist a nonzero polynomial $0 \neq f(x) = f(x_1,\ldots,x_n) \in F\{x\}$ which is an identity of A. For example, any commutative algebra is PI since it satisfies $x_1x_2 - x_2x_1$. Applying alternating polynomials imply that every finite dimensional algebra, associative or nonassociative is PI; see Section 3.1. The class of the PI algebras is both large and interesting! We remark that the algebra $M_k(F)$ of the $k\times k$ matrices over F plays a basic role in PI theory.

Definition 2.1. (1) Let $\mathrm{Id}(A) \subseteq F\{x\}$ be the subset of the identities of the algebra A.

$$\mathrm{Id}(A) = \left\{ f \in F\{x\} \mid f(x) = 0 \text{ is an identity of } A \right\}. \tag{2.1}$$

(2) *T ideals.* The set $\mathrm{Id}(A)$ is a two sided ideal in $F\{x\}$, with the additional property that it is closed under substitutions. Such ideals in $F\{x\}$ are called *T ideals*. Thus, $\mathrm{Id}(A)$ is a T ideal. It is easy to show that a T ideal is always the ideal of identities of some PI algebra A.

(3) *Varieties* of PI algebras. Given a T-ideal $I \subseteq F\{x\}$, the class of the PI algebras A satisfying $I \subseteq \mathrm{Id}(A)$ is the *variety* corresponding to I. We use here the language of T ideals, rather than that of varieties.

2.2. Kemer's Theory for T-Ideals (See [6])

The Specht Problem

One of the main problems in PI theory is the Specht problem: Are T ideals always finitely generated as T ideals?

Kemer [6] developed a powerful structure theory for T ideals which allowed him to prove that if $\mathrm{char}(F) = 0$ then T ideals are indeed finitely generated. We review some of the ingredients of that theory.

Amitsur [7] proved that the T ideal $I = \mathrm{Id}(M_k(F))$ is prime in the following sense. If $fg \in I$ then either $f \in I$ or $g \in I$; moreover, the only prime T ideals in $F\{x\}$ are $I = \mathrm{Id}(M_k(F))$.

Kemer introduced the notion of *verbally prime* ideals as follows.

Definition 2.2. The T ideal I is verbally prime if it satisfies the following condition: Let $f(x_1, \ldots, x_m)$ and $g(x_{m+1}, \ldots, x_{m+n})$ be polynomials in disjoint sets of variables. If

$$f(x_1, \ldots, x_m) \cdot g(x_{m+1}, \ldots, x_{m+n}) \in I \quad \text{then } f \in I \text{ or } g \in I. \tag{2.2}$$

Kemer then classified the verbally prime ideals in $F\{x\}$, see Theorem 2.3 below.

2.2.1. The Algebras $M_k(F)$, $M_k(G)$, and $M_{k,\ell}$

Let $U = \mathrm{span}_F\{u_1, u_2, \ldots\}$ be an infinite dimensional vector space, and let $G = G(U)$ be the corresponding infinite dimensional Grassmann (Exterior) algebra. Then

$$G = \mathrm{span}\{u_{i_1} \cdots u_{i_r} \mid r = 1, 2, \ldots \text{ and } 1 \leq i_1 < \cdots < i_r\}, \tag{2.3}$$

and $G = G_0 \oplus G_1$, where

$$\begin{gathered} G_0 = \mathrm{span}\{u_{i_1} \cdots u_{i_r} \mid r \text{ even}\}, \\ G_1 = \mathrm{span}\{u_{i_1} \cdots u_{i_r} \mid r \text{ odd}\}. \end{gathered} \tag{2.4}$$

Now $M_k(F)$ are the $k \times k$ matrices over F, while $M_k(G)$ are the $k \times k$ matrices over G.

The Algebra $M_{k,\ell}$

We have $M_{k,\ell} \subseteq M_{k+\ell}(G)$. The elements of $M_{k,\ell}$ are block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.5}$$

where $A \in M_k(G_0)$, $D \in M_\ell(G_0)$, B is $k \times \ell$ and C is $\ell \times k$, both with entries from G_1. For example:

$$M_{1,1} = \begin{pmatrix} G_0 & G_1 \\ G_1 & G_0 \end{pmatrix}. \tag{2.6}$$

Theorem 2.3 (see [6]). *The following are the three families of the verbally prime T ideals:* $\mathrm{Id}(M_k(F))$, $\mathrm{Id}(M_k(G))$, *and* $\mathrm{Id}(M_{k,\ell})$.

The importance of the verbally prime ideals is demonstrated in the following theorem.

Theorem 2.4 (see [6]). *Let $I \subset F\{x\}$ be a T ideal. Then there exist verbally prime T ideals $J_1, \ldots, J_r$ such that*

$$J_1 \cdots J_r \subseteq I \subseteq J_1 \cap \cdots \cap J_r. \tag{2.7}$$

3. Multilinear Polynomials

As usual, S_n denotes the nth symmetric group.

Definition 3.1. The polynomial $f(x_1, \ldots, x_n) \in F\{x\}$ is multilinear (in $x_1, \ldots, x_n$) if

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \tag{3.1}$$

for some coefficients $\alpha_\sigma \in F$. Let $V_n = V_n(x_1, \ldots, x_n)$ denote the vector space of multilinear polynomials in $x_1, \ldots, x_n$, so $\dim V_n = n!$. Extending the map $\sigma \to x_{\sigma(1)} \cdots x_{\sigma(n)}$ by linearity yields the vector space isomorphism $V_n \cong FS_n$, where FS_n is the group algebra of S_n. We will identify

$$FS_n \equiv V_n. \tag{3.2}$$

By the process of multilinearization [8, 9] one proves the following theorem.

Theorem 3.2. *Let the PI algebra A satisfy an identity of degree d, then A satisfies a multilinear identity of degree d. Moreover, if* $\mathrm{char}(F) = 0$ *then the ideal of identities* $\mathrm{Id}(A)$ *is determined by its multilinear elements.*

It follows that if A satisfies an identity of degree d then A satisfies an identity of the form

$$y_1 \cdots y_d - \sum_{1 \neq \pi \in S_d} \alpha_\pi y_{\pi(1)} \cdots y_{\pi(d)}, \quad \alpha_\pi \in F. \tag{3.3}$$

This fact is applied in Section 4.1 in proving the exponential bound for the codimensions.

3.1. Example: Standard and Capelli Identities

The polynomial $f(x_1, \ldots, x_n)$ is alternating in $x_1, \ldots, x_n$ if for every permutation $\pi \in S_n$, $f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \mathrm{sgn}(\pi) f(x_1, \ldots, x_n)$. For example, the polynomial

$$\mathrm{St}_n[x] = \mathrm{St}_n[x_1, \ldots, x_n] = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(n)} \tag{3.4}$$

is alternating. It is called the nth *Standard* polynomial.

Similarly the polynomial

$$\mathrm{Cap}_n[x;y] = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot x_{\sigma(1)} \cdot y_1 \cdot x_{\sigma(2)} \cdot y_2 \cdots y_{n-1} \cdot x_{\sigma(n)}, \tag{3.5}$$

which is called the (nth) *Capelli* polynomial, is multilinear of degree $2n-1$ and is alternating in $x_1, \ldots, x_n$.

It is rather easy to show that if $\dim A = d < \infty$ then A satisfies both $\mathrm{Cap}_{d+1}[x]$ and $\mathrm{St}_{d+1}[x]$, and hence every finite dimensional algebra is PI. And the same argument applies in the nonassociative case. In particular the algebra $M_k(F)$ of the $k \times k$ matrices satisfies $\mathrm{St}_{k^2+1}[x_1, x_2, \ldots, x_{k^2+1}]$. The celebrated Amitsur-Levitzki Theorem [10–12] states that $M_k(F)$ satisfies the standard identity $\mathrm{St}_{2k}[x_1, \ldots, x_{2k}] = 0$. Of course, for large k the degree $2k$ is much smaller than $k^2 + 1$.

Many infinite dimensional algebras are PI. For example, any infinite dimensional commutative algebra is PI. We remark that obviously, the free algebra $F\{x\}$ itself is *not* PI.

4. The Codimensions

Question. How many identities are satisfied by a given PI algebra, namely, how large are T ideals?

Computing dimensions might seem useless at first sight, since if the T-ideal I is nonzero then $\dim I = \infty$. To answer the above question we introduce below the notion of codimensions. Given a PI algebra A, we would like to study its multilinear identities of degree n, namely, the space

$$\mathrm{Id}(A) \cap V_n(x_1, \ldots, x_n) = \mathrm{Id}(A) \cap V_n, \tag{4.1}$$

V_n as in (3.2). Note that if $y_1, \ldots, y_n$ are any n variables then $\mathrm{Id}(A) \cap V_n(x) \cong \mathrm{Id}(A) \cap V_n(y)$. A first step is the study of the dimensions $\dim(\mathrm{Id}(A) \cap V_n)$. Now $(\mathrm{Id}(A) \cap V_n) \subseteq V_n$, and since $\dim V_n = n!$, clearly

$$\dim(\mathrm{Id}(A) \cap V_n) = n! - \dim\left(\frac{V_n}{\mathrm{Id}(A) \cap V_n}\right). \tag{4.2}$$

Thus, knowing $\dim(\mathrm{Id}(A) \cap V_n)$ is equivalent to knowing $\dim(V_n/(\mathrm{Id}(A) \cap V_n))$. This leads us to introduce the codimensions $c_n(A)$ as follows.

Definition 4.1. Let A be an F-algebra, then

$$c_n(A) = \dim\left(\frac{V_n}{\mathrm{Id}(A) \cap V_n}\right) \tag{4.3}$$

is called the nth codimension of A, and $\{c_n(A)\}_{n=1}^{\infty}$ is the sequence of codimensions of A.

For example, A satisfies $A^n = 0$ (i.e., is nilpotent) if and only if $c_n(A) = 0$. And A is commutative if and only if $c_n(A) \leq 1$ for all n.

Remark 4.2. Note that if A is not PI then $\mathrm{Id}(A) = 0$, hence $c_n(A) = n!$ for all n. In fact, the algebra A is PI (i.e., $\mathrm{Id}(A) \neq 0$) if and only if there exist n such that $c_n(A) < n!$. This follows directly from the definition.

4.1. Exponential Bound for the Codimensions

A basic property of the codimensions sequence $c_n(A)$ in the associative case is, that it is bounded by exponential growth. Applications of this fact are given in the sequel.

Theorem 4.3 (see [1]). *Assume the (associative) algebra A satisfies an identity of degree d, then*

$$c_n(A) \leq \left((d-1)^2\right)^n \tag{4.4}$$

for all n.

Proof. We sketch two proofs (both different from the original proof). Both proofs apply the notion of a d-good permutation, which we now review.

Call $\sigma \in S_n$ d-bad if there are indices $1 \leq i_1 < \cdots < i_d \leq n$ with $\sigma(i_1) > \cdots > \sigma(i_d)$. Otherwise σ is d-good, and we denote

$$g_d(n) = \left|\{\sigma \in S_n \mid \sigma \text{ is } d\text{-good}\}\right|. \tag{4.5}$$

By a Shirshov-Latyshev argument [13, 14], if A satisfies an identity of degree d, then A satisfies an identity (3.3) which, by a certain inductive argument implies that $c_n(A) \leq g_d(n)$.

By an argument based on Dilworth Theorem in Combinatorics [15], Latyshev then showed that $g_d(n) \leq ((d-1)^2)^n$, thus completing the (first) proof.

A second proof of the bound $g_d(n) \leq ((d-1)^2)^n$ goes as follows [16]. First, by the RSK correspondence [17],

$$g_d(n) = \sum_{\lambda \vdash n, \ell(\lambda) \leq d-1} \left(f^\lambda\right)^2, \tag{4.6}$$

where f^λ is the number of standard Young taableaux of shape λ, and $\ell(\lambda)$ is the number of parts of λ. Then by the Schur-Weyl theory,

$$\sum_{\lambda \vdash n, \ell(\lambda) \leq d-1} \left(f^\lambda\right)^2 = \dim(B(d-1,n)), \tag{4.7}$$

where $\dim U = d-1$ and $B(d-1,n) \subseteq \mathrm{End}_F(U^{\otimes n})$. The (second) proof now follows since $\dim(\mathrm{End}_F(U^{\otimes n})) = (d-1)^{2n}$. □

4.1.1. Application: The $A \otimes B$ Theorem

Codimensions where introduced, in [1], in order to prove the following theorem.

Theorem 4.4 (see [1]). *If A and B are (associative) PI algebras then $A \otimes B$ is PI.*

Proof. It is not too difficult to show that $c_n(A\otimes B) \leq c_n(A)\cdot c_n(B)$. Assume now that A satisfies an identity of degree d, and B satisfies an identity of degree h. Together with Theorem 4.3, this implies that

$$c_n(A \otimes B) \leq (d-1)^{2n}(h-1)^{2n}, \tag{4.8}$$

hence for a large enough n, $c_n(A \otimes B) < n!$. Finally by Remark 4.2, R is PI if and only if there exist n such that $c_n(R) < n!$, and the proof follows. □

Remark 4.5. For explicit identities for $A \otimes B$ see Remark 8.2 below, which also implies the following. Again let A satisfy an identity of degree d, and B an identity of degree h. Then $A \otimes B$ satisfies an identity of degree about $e(d-1)^2(h-1)^2$, where $e = 2.718\ldots$.

We remark that both Theorems 4.3 and 4.4 fail in some non-associative cases.

The above results motivate the study of the following problem.

Question 1. Given a PI algebra A, find a formula for the sequence $c_n(A)$. In most cases this seems to be too difficult, and one tries to, at least, get a "nice" asymptotic approximation of $c_n(A)$.

In Section 6 we determine such asymptotics for the algebras $M_k(F)$. We also give such partial results for the other verbally prime algebras $M_{k,\ell}$ and $M_k(G)$.

5. Cocharacters

Recall the identification $FS_n \equiv V_n$ (see (3.2)) and let A be a PI algebra. The regular left action of S_n on V_n is as follows. Let $\sigma \in S_n$ and $g(x_1,\ldots,x_n) \in V_n$, then

$$\sigma g(x_1,\ldots,x_n) = g\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right). \tag{5.1}$$

This makes V_n into a left S_n module, with $\mathrm{Id}(A) \cap V_n$ a submodule of V_n.

Definition 5.1. The S_n-character

$$\chi_n(A) := \chi_{S_n}\left(\frac{V_n}{\mathrm{Id}(A) \cap V_n}\right) \tag{5.2}$$

of the quotient module $V_n/(\mathrm{Id}(A) \cap V_n)$ is called the nth cocharacter of A.

Cocharacters where introduced in [18] in the study of standard identities.

Since $\mathrm{char}(F) = 0$, from the ordinary representation theory of S_n [19, 20] it is well known that the irreducible S_n characters are parametrized by the partitions λ on n:

$$\mathrm{Irred}(S_n) = \left\{\chi^{\lambda} \mid \lambda \vdash n\right\}. \tag{5.3}$$

Also, FS_n decomposes as follows:

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda, \tag{5.4}$$

where the I_λ are the minimal two sided ideals of FS_n, and there are natural bijections

$$\chi^\lambda \longleftrightarrow \lambda \longleftrightarrow I_\lambda. \tag{5.5}$$

Here $\deg \chi^\lambda = f^\lambda$ and $\dim I_\lambda = (f^\lambda)^2$ where, again, f^λ is the number of standard Young taableaux of shape λ.

Given a tableau T_λ of shape $\lambda \vdash n$, it determines the semi-idempotent $e_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda}$, and $FS_n e_{T_\lambda} \subseteq I_\lambda$ is a minimal left ideal.

The S_n-character of the regular representation $FS_n \equiv V_n$ is $\chi_{S_n}(V_n) = \sum_{\lambda \vdash n} f^\lambda \chi^\lambda$. It follows that the nth cocharacter of A can be written as

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A) \chi^\lambda \tag{5.6}$$

for some multiplicities $m_\lambda(A)$, and $m_\lambda(A) \leq f^\lambda$.

Clearly, the degree of $\chi_n(A)$ is the corresponding codimension $c_n(A)$, and since $\deg(\chi^\lambda) = f^\lambda$, hence

$$c_n(A) = \deg(\chi_n(A)) = \sum_{\lambda \vdash n} m_\lambda(A) f^\lambda. \tag{5.7}$$

For example [21] let G be the infinite dimensional Grassmann (Exterior) algebra, then

$$m_\lambda(G) = \begin{cases} 1 & \text{if } \lambda = (m - r, 1^r) \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Remark 5.2. Properties of the identification $FS_n \equiv V_n$ imply the following characterization of Capelli identities (Section 3.1) by cocharacters [22]. Let A be a PI algebra with $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A) \chi^\lambda$ its cocharacter. Then A satisfies $\mathrm{Cap}_{d+1}[x; y]$ if and only if $m_\lambda(A) = 0$ whenever $\ell(\lambda) \geq d + 1$.

As an application of cocharacters one can prove the following.

Proposition 5.3 (see [23]). *If A satisfies Cap_{d+1} and B satisfies Cap_{h+1} then $A \otimes B$ satisfies Cap_{dh+1}.*

We remark that a proof of this result without applying cocharacters is yet unknown.

Question 2. Given a PI algebra A, find a formula for the multiplicities $m_\lambda(A)$. In most cases this is too difficult, and one tries to at least get some approximate description of $m_\lambda(A)$.

Remark 5.4. The approach of codimensions and of cocharacters in the study of PI algebras applies also in the nonassociative case (though with different phenomena).

5.1. The Cocharacters of Matrix Algebras

In the case of the 2×2 matrices $M_2(F)$ there is the following nice formula for the multiplicities $m_\lambda(M_2(F))$ of $\chi_n(M_2(F))$.

Example 5.5 (see [24], [25, Theorem 12.6.5], [26]). Denote $m_\lambda(M_2(F)) = m_\lambda$. First, if $\ell(\lambda) > 4$ then $m_\lambda = 0$. So let $\lambda = (\lambda_1, \ldots, \lambda_4)$.

If $\lambda = (n)$ then $m_{(n)} = 1$.

If $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 > 0$ then $m_{(\lambda_1,\lambda_2)} = (\lambda_1 - \lambda_2 + 1) \cdot \lambda_2$.

If $\lambda = (\lambda_1, 1, 1, \lambda_4)$ (so $\lambda_4 \le 1$) then $m_{(\lambda_1,1,1,\lambda_4)} = \lambda_1 \cdot (2 - \lambda_4) - 1$.

And $m_\lambda = (\lambda_1 - \lambda_2 + 1) \cdot (\lambda_2 - \lambda_3 + 1) \cdot (\lambda_3 - \lambda_4 + 1)$ in all other cases.

Recent results of Berele [27] and of Drensky and Genov [28] indicate that when $k \ge 3$, there is no hope in getting nice formulas for the cocharacter-multiplicities $m_\lambda(M_k(F))$. A somewhat similar phenomena is discussed in Section 12.

5.2. Trace Identities, Codimensions, and Cocharacters

In the case of $k \times k$ matrices, the following is an extension of the previous theory of codimensions and cocharacters.

Instead of ordinary polynomials we can consider trace polynomials, namely, polynomials involving variables and traces, for example $x_1 \cdot x_2 \cdot \mathrm{tr}(x_3 \cdot x_4)$ is a (mixed) trace polynomial. These trace polynomials can be evaluated on the algebra $M_k(F)$ (or on any algebra with a trace) hence yielding trace identities. For example, it can be proved that the trace polynomial

$$g(x_1, x_2) = \mathrm{tr}(x_1)\,\mathrm{tr}(x_2) - x_1\,\mathrm{tr}(x_2) - x_2\,\mathrm{tr}(x_1) - \mathrm{tr}(x_1x_2) + x_1x_2 + x_2x_1 \tag{5.9}$$

is a trace identity of $M_2(F)$. This is an example of a *mixed* trace polynomial, while the polynomial

$$\begin{aligned} p(x_1, x_2, x_3) = {} & \mathrm{tr}(x_1)\,\mathrm{tr}(x_2)\,\mathrm{tr}(x_3) - \mathrm{tr}(x_1x_3)\,\mathrm{tr}(x_2) \\ & - \mathrm{tr}(x_2x_3)\,\mathrm{tr}(x_1) - \mathrm{tr}(x_1x_2)\,\mathrm{tr}(x_3) + \mathrm{tr}(x_1x_2x_3) + \mathrm{tr}(x_2x_1x_3) \end{aligned} \tag{5.10}$$

is a *pure* trace polynomial, which is also an identity of $M_2(F)$. We then have trace identities of $M_k(F)$, "pure" and "mixed," hence trace codimensions $c_n^{\mathrm{ptr}}(M_k(F))$ and $c_n^{\mathrm{mtr}}(M_k(F))$, and trace cocharacters $\chi_n^{\mathrm{ptr}}(M_k(F))$ and $\chi_n^{\mathrm{mtr}}(M_k(F))$.

The Procesi-Razmyslov theory of trace identities [29–31], together with the Schur-Weyl theory [32, 33], imply the following formula for the pure trace cocharacters $\chi_n^{\mathrm{ptr}}(M_k(F))$ of $M_k(F)$.

Theorem 5.6. *We have*

$$\chi_n^{\text{ptr}}(M_k(F)) = \sum_{\lambda\vdash n,\ell(\lambda)\leq k} \chi^\lambda \otimes \chi^\lambda. \tag{5.11}$$

This implies that the trace codimensions are given by the following formula:

$$c_n^{\text{ptr}}(M_k(F)) = \sum_{\lambda\vdash n,\ell(\lambda)\leq k} \left(f^\lambda\right)^2. \tag{5.12}$$

This formula is the starting point for computing the asymptotic formula of $c_n(M_k(F))$ given in Section 6.

6. Asymptotics of the Codimensions $c_n(M_k(F))$

For the 2×2 matrices, Procesi [34] proved the following formula for $c_n(M_2(F))$.

Theorem 6.1 (see [34]). *We have*

$$c_n(M_2(F)) = \frac{1}{n+1}\binom{2n+2}{n+1} - \binom{n}{3} + 1 - 2^n. \tag{6.1}$$

It was already mentioned that when $k \geq 3$, it most likely is impossible to find an exact formula for the multiplicities $m_\lambda(M_k(F))$, and the same is probably true about $c_n(M_k(F))$. Instead of giving up, one looks for the asymptotic of $c_n(M_k(F))$.

Here we have the folowing theorem.

Theorem 6.2 (see [2]). *When n goes to infinity,*

$$c_n(M_k(F)) \sim \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{k-1}\left(\frac{1}{2}\right)^{(k^2-1)/2}\cdot 1!2!\cdots(k-1)!\cdot k^{(k^2/2)}\right]\cdot\left(\frac{1}{n}\right)^{(k^2-1)/2}\cdot k^{2(n+1)}. \tag{6.2}$$

For example, when $k = 2$ both Theorems 6.1 and 6.2 give the same asymptotic value

$$c_n(M_2(F)) \sim \frac{4}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}\cdot 4^n, \tag{6.3}$$

compare with (12.13).

We review the major steps toward the proof of Theorem 6.2. First, it follows from deep results of Formanek [2, 26, 35] that the nth codimensions and the $n+1$-st pure trace codimensions are asymptotically equal:

$$c_n(M_k(F)) \sim c_{n+1}^{\text{ptr}}(M_k(F)). \tag{6.4}$$

We saw that by Theorem 5.6,

$$c_n^{\mathrm{ptr}}(M_k(F)) = \sum_{\lambda\vdash n,\, \ell(\lambda)\le k^2} \left(f^\lambda\right)^2. \tag{6.5}$$

Thus,

$$c_n(M_k(F)) \sim \sum_{\lambda\vdash n+1,\, \ell(\lambda)\le k^2} \left(f^\lambda\right)^2. \tag{6.6}$$

The last major step here is the computation of the asymptotic behavior of the following sum:

$$S_{k^2}^{(2)}(n) = \sum_{\lambda\vdash n,\, \ell(\lambda)\le k^2} \left(f^\lambda\right)^2. \tag{6.7}$$

The asymptotics, as $n \to \infty$, of the more general sums

$$S_h^{(\beta)}(n) = \sum_{\lambda\vdash n,\, \ell(\lambda)\le h} \left(f^\lambda\right)^\beta \tag{6.8}$$

is given in [16]. That asymptotic is of the form $S_h^{(\beta)}(n) \sim a \cdot n^b \cdot r^n$, where $a = a(\beta, h)$, $b = b(\beta, h)$ and $r = r(\beta, h) = h^\beta$, all given explicitly in [16]. We remark that the constant term a is evaluated by applying the Selberg integral (6.9).

Theorem 6.3 (see [36]). *The Selberg integral*

$$\begin{aligned}\int_0^1 \cdots \int_0^1 \prod_{i=1}^k u_i^{x-1}(1-u_i)^{y-1} \prod_{1\le i<j\le k} \left|u_i - u_j\right|^{2z} \; du_1\cdots du_k \\ = \prod_{j=0}^{k-1} \frac{\Gamma(x+jz)\Gamma(y+jz)\Gamma(1+(j+1)z)}{\Gamma(x+y+(k+j-1)z)\Gamma(1+z)}.\end{aligned} \tag{6.9}$$

Together, the above steps yield the asymptotic value of Theorem 6.2.

6.1. The Other Verbally Prime Algebras

For the other verbally prime algebras (see Section 2.2.1), we quote the following partial asymptotic results.

Theorem 6.4. *(1) [37, Theorem 7]*

$$c_n(M_{k,\ell}) \sim a \cdot \left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2} \cdot (k+\ell)^{2n}. \tag{6.10}$$

The constant a is yet unknown.

(2) [37, Theorem 8] Let G be the infinite dimensional Grassmann algebra, then

$$c_n(M_k(G)) \sim b \cdot \left(\frac{1}{n}\right)^g \cdot \left(2k^2\right)^n, \tag{6.11}$$

where the constants b and g are yet unknown, and $(k^2-1)/2 \le g \le (2k^2-1)/2$.

7. Shirshov's Height Theorem (See [13])

7.1. The Theorem

A powerful tool in the study of PI algebras is Shirshov's Height Theorem, which we now quote. We consider the alphabet $x_1, \ldots, x_\ell$ of ℓ letters; $W(x_1, \ldots, x_\ell)$ is the set of all words (i.e., monomials) in $x_1, \ldots, x_\ell$; $U(d, \ell)$ the subset of the words of length $\le d$. We consider finitely generated PI algebra $A = F\{a_1, \ldots, a_\ell\}$.

Theorem 7.1 (Shirshov's Height Theorem). *Consider a PI algebra satisfying the identity* (3.3)*:*

$$y_1 \cdots y_d - \sum_{1 \ne \pi \in S_d} \alpha_\pi y_{\pi(1)} \cdots y_{\pi(d)}. \tag{7.1}$$

There exists $h = h(d, \ell)$ large enough such that any finitely generated algebra $A = F\{a_1, \ldots, a_\ell\}$ that satisfies the identity (3.3)*, satisfies the following condition.*

Modulo $\mathrm{Id}(A)$*, $F\{x_1, \ldots, x_\ell\}$ is spanned by the elements*

$$\left\{u_1^{k_1} \cdots u_h^{k_h} \mid u_i \in U(d, \ell),\ \mathit{any}\ k_j\right\}. \tag{7.2}$$

7.2. Application: Bounds on the Cocharacters

7.2.1. The (k, ℓ) Hook Theorem

Denote by $H(k, \ell; n)$ the partitions of n in the (k, ℓ) hook:

$$H(k, \ell; n) = \{(\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{k+1} \le \ell\}, \qquad H(k, \ell) = \cup_n (H(k, \ell; n)). \tag{7.3}$$

Let χ_n be S_n characters, $n = 1, 2, \ldots$. We say that χ_n is supported on $H(k, \ell)$, and denote $\chi_n \subseteq H(k, \ell)$, if for all n,

$$\chi_n = \sum_{\lambda \in H(k, \ell; n)} m_\lambda \chi^\lambda. \tag{7.4}$$

Similar terminology applies when $H(k, \ell)$ is replaced by another family of subsets of partitions. We have the following theorem.

Theorem 7.2 (see [6, 38]). *Let A be any (associative) PI algebra, then there exist k, ℓ such that its cocharacters $\chi_n(A)$ are supported on the (k, ℓ) hook $H(k, \ell)$.*

Explicitly, let A satisfy an identity of degree d, and let $k, \ell \geq e \cdot (d-1)^4 - 1$, where $e = 2.718\ldots$ is the base of the natural logarithms. Then $\chi_n(A) \subseteq H(k, \ell)$, namely,

$$\chi_n(A) = \sum_{\lambda \in H(k,\ell;n)} m_\lambda(A) \cdot \chi^\lambda. \tag{7.5}$$

7.2.2. An Application of Shirshov's Theorem

We remark that the proof of Theorem 7.2 applies the exponential-bound of Theorem 4.3. Theorem 7.2 is the first step towards proving the following polynomial bound.

Theorem 7.3 (see [39, 40]). *For any PI algebra A, all its cocharacter-multiplicities $m_\lambda(A)$ are polynomially bounded. There exist a constant C and a power p such that for all n and $\lambda \vdash n$, $m_\lambda(A) \leq C \cdot n^p$.*

The proof of Theorem 7.3 also applies Shirshov's Height Theorem 7.1, as well as a $\mathbb{Z}_2$ version of the Schur-Weyl theory [32, 33, 41].

8. Explicit Identities

Amitsur [42] proved that any PI algebra satisfies a power of a standard identity, namely, an identity of the form $\mathrm{St}_u^v[x] = (\mathrm{St}_u[x])^v$, where $\mathrm{St}_u[x]$ is the uth standard polynomial. Amitsur's proof, which applies Structure Theory of Rings [8], yields a bound on the index u but not on v. A recent proof of Amitsur's theorem [43] applies the identification $FS_n \equiv V_n$, together with the exponential bound on $c_n(A)$ and yields a combinatorial proof of that theorem, a proof which gives bounds on both u and v. Moreover, the same arguments yield explicit identities in various other cases, for example in the $A \otimes B$ case.

Theorem 8.1 (see [38, 43]). *Let A be a PI algebra satisfying $c_n(A) \leq \alpha^n$, and let u and v be integers satisfying $u, v \geq e \cdot \alpha^2$, then $\mathrm{St}_u^v[x_1, \ldots, x_u] \in \mathrm{Id}(A)$. In particular if A satisfies an identity of degree d, and $u, v \geq e \cdot (d-1)^4$, then $\mathrm{St}_u^v[x_1, \ldots, x_u] \in \mathrm{Id}(A)$ (of degree uv which is about $e^2(d-1)^8$.)*

In fact, with these u and v A satisfies the power of the Capelli identity: $(\mathrm{Cap}_u[x; y])^v \in \mathrm{Id}(A)$.

Remark 8.2. Let A satisfy an identity of degree d and B an identity of degree h. Denote $\alpha = (d-1)^2(h-1)^2$ then $c_n(A \otimes B) \leq \alpha^n$. Thus, if $u, v \geq e \cdot ((d-1)(h-1))^4$ then $A \otimes B$ satisfies the identities $\mathrm{St}_u^v[x_1, \ldots, x_u]$ and $(\mathrm{Cap}_u[x; y])^v$. Note that here the degree of $\mathrm{St}_u^v[x_1, \ldots, x_u]$, for example, is about $e^2 \cdot ((d-1)(h-1))^8$.

Also, $A \otimes B$ satisfies some identity of degree n, where n is about $e\alpha = e((d-1)(h-1))^2$. Indeed, let $e\alpha < n$, then the classical inequality $(n/e)^n < n!$ implies that $\alpha^n < n!$. Thus, if $e(d-1)^2(h-1)^2 < n$ then, for that n,

$$c_n(A \otimes B) \leq \left((d-1)^2(h-1)^2\right)^n < n! \tag{8.1}$$

and by Remark 4.2 $A \otimes B$ satisfies an identity of degree n where n is about $e((d-1)(h-1))^2$.

9. Nonidentities for Matrices: The Polynomial $L_k[x;y]$

Usually, lower bounds for codimensions are harder to obtain than upper bounds. Given a PI algebra A, a lower bound for $c_n(A)$ can be obtained by the following technique. Find a polynomial $p = p(x_1,\ldots,x_n) \in V_n$ which is a nonidentity of A, namely, $p \notin \mathrm{Id}(A)$. In addition, with the identification $V_n \equiv FS_n = \oplus_{\lambda\vdash n} I_\lambda$ (see (5.4)), verify that $p \in I_\mu$ for some $\mu \vdash n$. Then $c_n(A) \geq f^\mu$. This follows since $0 \neq FS_n p \subseteq I_\mu$, so $J_\mu \subseteq FS_n p$ for some minimal left ideal $J_\mu \subseteq I_\mu$ with $J_\mu \cap \mathrm{Id}(A) = 0$, hence $f^\mu = \dim J_\mu \leq c_n(A)$. We now construct such a polynomial p via the polynomial $L_k(x;y)$, when $A = M_k(F)$ [44, Definition 2.3].

Corresponding to the sum $1 + 3 + \cdots + (2k-1) = k^2$, construct the monomial $N_k(x;y)$ of degree $2k^2$:

$$N_k(x;y) = (x_1)(y_1)(x_2x_3x_4)(y_2y_3y_4)(x_5\cdots x_9)(y_5\cdots y_9)\cdots. \tag{9.1}$$

For example, $N_3(x;y) = (x_1)(y_1)(x_2x_3x_4)(y_2y_3y_4)(x_5\cdots x_9)(y_5\cdots y_9)$. Now alternate the x's and alternate the y's to obtain $L_k(x;y)$:

$$L_k(x;y) = \sum_{\sigma,\pi\in S_{k^2}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi) N_k\left(x_{\sigma(1)},\ldots,x_{\sigma(k^2)};y_{\pi(1)},\ldots,y_{\pi(k^2)}\right). \tag{9.2}$$

Let T'_μ be the conjugate tableau of T_μ, and

$$\begin{aligned} T'_\mu = 1 \;\; 3 \;\; 4 \;\; 5 \;\; 9 \;\; \cdots \;\; 13 \;\; \cdots \\ 2 \;\; 6 \;\; 7 \;\; 8 \;\; 14 \;\; \cdots \;\; 18 \;\; \cdots. \end{aligned} \tag{9.3}$$

Then $L_k(x;y) = C^-_{T_\mu}$ and in that sense $L_k(x;y)$ corresponds to the tableau T_μ where $\mu = (2^{k^2})$. It is not difficult to show that $L_k(x;y)$ takes central values on $M_k(F)$.

For $k = 2$ and $k = 3$ it was verified that $L_k(x;y) \notin \mathrm{Id}(M_k(F))$, and it was conjectured that for all $kL_k(x;y) \notin \mathrm{Id}(M_k(F))$ namely, that $L_k(x;y)$ is a nonidentity of $M_k(F)$ [44]. This conjecture was verified by Formanek [35].

Theorem 9.1 (see [35]). *The plolynomial $L_k(x;y)$, which corresponds to the rectangle $\mu = (2^{k^2})$, is a nonidentity of $M_k(F)$. Hence $L_k(x;y)$ is a central polynomial.*

By Young's rule it follows that

$$L_k(x;y) = p_\mu + \sum_{\lambda\vdash 2k^2,\,\ell(\lambda)\geq k^2+1} p_\lambda, \tag{9.4}$$

where $p_\mu \in I_\mu$ and $p_\lambda \in I_\lambda$, see (5.4). Since $M_k(F)$ satisfies the Capelli identity Cap_{k^2+1}, it follows that for all λ with $\ell(\lambda) \geq k^2+1$, $p_\lambda \in \mathrm{Id}(M_k(F))$. And since $L_k(x;y) \notin \mathrm{Id}(M_k(F))$, hence $p_\mu \notin \mathrm{Id}(M_k(F))$, where μ is of the $2\times k^2$ rectangular shape $\mu = (2^{k^2})$. Thus, also p_μ is a central polynomial for $M_k(F)$). The fact that μ is a rectangle plays an important role in proving lower bounds for codimensions, since the following is applied: two or more rectangles of same height can be glued together horizontally, while two or more rectangles of same width can be glued together vertically.

For $c_n(M_k(F))$, Theorem 9.1 and further results of Formanek [2, 26, 35] imply that the ordinary cocharacters and the trace cocharacters are nearly equal, hence they have same asymptotic. Since the trace codimensions are much easier to handle than the ordinary codimensions (see Theorem 5.6), this fact allows the computation—in the next section—of the exact asymptotic of $c_n(M_k(F))$. Also, the fact that $L_k(x;y)$ is a nonidentity of $M_k(F)$ has applications in proving lower bounds for various other types of codimensions.

10. The Giambruno-Zaicev Theorem: $\exp(A) \in \mathbb{Z}$

For most PI algebras A it seems hopeless to find a precise, or even asymptotic, formula for the codimensions $c_n(A)$. We therefore ask a much more restricted question.

Question 3. Given the associative PI algebra A, what can be said about the asymptotic behavior of the codimensions $c_n(A)$?

As a first step we have the remarkable integrality property given by Theorem 10.1 below. See also Theorem 11.1 and its relation to Theorem 10.1.

Theorem 10.1 (see [3]). *Let A be an associative PI algebra (with $\mathrm{char}(F) = 0$), then the limit*

$$\lim_{n\to\infty} (c_n(A))^{1/n} \tag{10.1}$$

exists and is an integer.

We denote $\exp(A) = \lim_{n\to\infty}(c_n(A))^{1/n}$, *so* $\exp(A) \in \mathbb{N}$.

10.1. Review of the Proof When $\dim(A) < \infty$

When A is finite dimensional, the number $\exp(A)$ can be calculated as follows. We may assume that F is algebraically closed. First, by a classical theorem of Wedderburn and Malcev [45, Theorem 3.4.3], $A = B \oplus J$, where $J = J(A)$ is the Jacobson radical of A, and B is semisimple. Thus $B = B_1 \oplus \cdots \oplus B_r$ where B_j are simple, namely, $B_j \cong M_{k_j}(F)$. Consider now all possible nonzero products of the type

$$B_{i_1} J B_{i_2} J \cdots J B_{i_q} \neq 0 \quad (q \geq 1), \tag{10.2}$$

where the B_{i_j} are distinct, and for such nonzero products let

$$h := \max \dim\left(B_{i_1} \oplus \cdots \oplus B_{i_q}\right). \tag{10.3}$$

Then $h = k_{i_1}^2 + \cdots + k_{i_q}^2$, and h is the limit in Theorem 10.1:

$$\lim_{n\to\infty} (c_n(A))^{1/n} = h, \tag{10.4}$$

Namely, $\exp(A) = k_{i_1}^2 + \cdots + k_{i_q}^2$.

For example, consider the algebra of upper block triangular matrices

$$A = UT(k_1, k_2, k_3) = \begin{pmatrix} B_1 & J_1 & J_2 \\ 0 & B_2 & J_3 \\ 0 & 0 & B_3 \end{pmatrix}, \tag{10.5}$$

where for $1 \le i \le 3$, $B_i \cong M_{k_i}(F)$ and J_i are rectangular matrices of the corresponding sizes. Then the Wedderburn-Malcev decomposition of A is $A = B_1 \oplus B_2 \oplus B_3 + J$ where $B_1 \oplus B_2 \oplus B_3$ is the semisimple part and $J = J_1 + J_2 + J_3$ is the Jacobson radical. Notice that the matrix units $e_{i,j}$ in A satisfy

$$\begin{aligned} 0 \neq e_{1,1} \cdot e_{1,k_1+1} \cdot e_{k_1+1,k_1+1} \cdot e_{k_1+1,k_1+k_2+1} \\ \cdot e_{k_1+k_2+1,k_1+k_2+1} \in B_1 J_1 B_2 J_3 B_3 \subseteq B_1 J B_2 J B_3, \end{aligned} \tag{10.6}$$

so $B_1 J B_2 J B_3 \neq 0$ and also, this is the maximal such nonzero product. It follows that here

$$h = h(A) = \dim(B_1 \oplus B_2 \oplus B_3) = k_1^2 + k_2^2 + k_3^2. \tag{10.7}$$

10.1.1. The Upper Bound

In the general case the exponent $h = \exp(A) = \lim_{n\to\infty}(c_n(A))^{1/n}$ is given by (10.3). To prove this, one first proves the following upper bound.

Lemma 10.2. *There exist constants a_1, g_1 such that for all n, $c_n(A) \le a_1 \cdot n^{g_1} \cdot h^n$.*

Let $\mathrm{Par}(n)$ denote the partitions of n. Given $h \in \mathbb{N}$, define the subsets $NS_K(h,n) \subseteq \mathrm{Par}(n)$ by

$$NS_K(h,n) = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \mid \sum_{j>h} \lambda_j \le K \right\}, \tag{10.8}$$

so $NS_K(h,n)$ is nearly a strip of height h (with at most K cells below the h row). The proof of Lemma 10.2 follows by showing that the cocharacters $\chi_n(A)$ are supported on such nearly a strip $NS_K(h,n)$, and by the polynomial bound $m_\lambda(A) \le a \cdot n^b$ on the multiplicities in the cocharacters, see Theorem 7.3. Thus $c_n(A) \le an^b \sum f^\lambda$, where the sum $\sum f^\lambda$ is supported on such nearly a strip $NS_K(h,n)$. Similar to the estimates in [16], such sum

$$\sum_{\lambda \in NS_K(h,n)} f^\lambda \tag{10.9}$$

is bounded by $\overline{a} \cdot n^{\overline{b}} \cdot h^n$, and the proof of Lemma 10.2 follows.

10.1.2. The Lower Bound

Here we prove the folowing lemma.

Lemma 10.3. *There exist constants $a_2 > 0$ and g_2 such that for all n, $a_2 \cdot n^{g_2} \cdot h^n \le c_n(A)$.*

A key ingredient in proving the lower bound is the polynomial $L_k(x;y)$ which is a nonidentity for $M_k(F)$, see Section 9. We already noted that $L_k(x;y)$ has a component $p_\mu \in I_\mu$ which is central (nonidentity) for $M_k(F)$, where μ is the $2\times k^2$ rectangle (2^{k^2}), see Theorem 9.1.

We start with a single matrix algebra $B = M_k(F)$, with the corresponding central polynomials $L_k(x;y)$ and $p_{\mu^{(1)}}$. Rectangles of the same height (width) can be glued horizontally (vertically). Gluing horizontally (2^{k^2}) to itself w times yields the $2w \times k^2$ rectangle $\mu^{(w)} = ((2w)^{k^2})$, with a corresponding w-power $(p_{\mu^{(1)}})^w = p_{\mu^{(w)}}$, which is central and nonidentity for $M_k(F)$, and $p_{\mu^{(w)}} \in I_{\mu^{(w)}}$. Thus $f^{\mu^{(w)}} \le c_n(M_k(F))$. For that $\mu^{(w)} = ((2w)^{k^2}) \vdash n$, $n = 2wk^2$, the asymptotic of $f^{\mu^{(w)}}$ then yields the lower bound

$$\overline{a}_2 \cdot n^{\overline{g}_2} \cdot \left(k^2\right)^n \le c_n(M_k(F)) \tag{10.10}$$

for some constants $\overline{a}_2 > 0$ and $\overline{g}_2$.

In the general case we are given $B_{i_1} J B_{i_2} J \cdots J B_{i_q} \neq 0$ as in (10.2). To each B_{i_j} corresponds the nonidentity polynomial $L_{k_{i_j}}(x;y)$, with the corresponding rectangular tableaux $(2^{k_{i_j}{}^2})$, all with the same width = 2. These tableaux can be glued vertically, thus yielding the rectangular tableau $\rho = (2^h)$, where $h = \sum_j (k_{i_j})^2$. To that tableau there corresponds a polynomial which is essentially the product of the polynomials $L_{k_{i_j}}(x;y)$, with the corresponding component which is the product of the corresponding polynomials p_μ, hence that polynomial is central nonidentity of A, and similarly for powers of these polynomials. Similar to the above case of a single matrix algebra $B = M_k(F)$, the asymptotic of $f^{\rho^{(w)}} = f^{((2w)^h)}$ then yields the lower bound

$$a_2 \cdot n^{g_2} \cdot h^n \le c_n(A) \tag{10.11}$$

for some constants $0 < a_2$ and g_2. This proves the lower bound.

Corollary 10.4. *Putting together the lower and the upper bounds, Theorem 10.1 then follows.*

This completes our review of the proof of Theorem 10.1. For extensions of this theorem, see Remark 11.2.

11. Berele's "1/2" Theorem (See [4])

Applying Theorem 10.1, Berele proved the following remarkable theorem.

Theorem 11.1 (see [4], see also [5]). *Let A be a PI algebra with $1 \in A$. Then as n goes to infinity, $c_n(A) \sim a \cdot n^t \cdot h^n$, where, $h \in \mathbb{N}$ (given by Theorem 10.1) and $t \in (1/2)\mathbb{Z}$, namely, t is an integer or a half integer.*

Based on various examples, that theorem was conjectured for some time. It was first proved in [5] under the hypothesis that A satisfies a Capelli identity, and later in general in [4]. Here is an outline of the proof in the Capelli case.

Proof. Recall that $c_n(A) = \sum m_\lambda(A) f^\lambda$, where $m_\lambda(A)$ is the multiplicity of χ^λ. If A satisfies a Capelli identity then there exists an integer k such that $m_\lambda(A) \neq 0$ only for partitions λ with at most k parts (see Remark 5.2). The proof is now based on investigating the asymptotics of the $m_\lambda(A)$ and f^λ, and we proceed to do so.

Let

$$U_k(A) = \frac{F\{x_1, \ldots, x_k\}}{\mathrm{Id}(A) \cap F\{x_1, \ldots, x_k\}} \tag{11.1}$$

be the universal PI algebra for A in k generators. This algebra has an $\mathbb{N}$ grading by degree and a corresponding Poincaré series $P_k(t)$. It also has a finer $\mathbb{N}^k$ grading by multidegree with corresponding Poincaré series $P(t_1, \ldots, t_k)$. Belov [46] proved that $P_k(t)$ is (the Taylor series of) a rational function with coefficients in $\mathbb{Z}$ (see also [47], Theorem 9.44). It is not difficult to adapt that proof to show that $P(t_1, \ldots, t_k)$ is also a rational function with integer coefficients. The proof also implies that the denominator of this rational function can be taken to be a product of terms of the form $(1 - t_1^{a_1} \cdots t_k^{a_k})$. We call such a rational function "*nice*."

The Poincaré series $P(t_1, \ldots, t_k)$ is related to the cocharacters $\chi_n(A) = \sum m_\lambda(A)\chi^\lambda$ via

$$P(t_1, \ldots, t_k) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} m_\lambda(A) S_\lambda(t_1, \ldots, t_k), \tag{11.2}$$

where S_λ is the Schur function of λ.

Quite a lot is known about Taylor series of nice rational functions, see for example [48]. If $F(t_1, \ldots, t_k)$ is any nice rational function with Taylor series $\sum b(n_1, \ldots, n_k) t_1^{n_1} \cdots t_k^{n_k}$, then the coefficients $b(n_1, \ldots, n_k)$ can be described using a finite set of polynomials. Namely, $\mathbb{N}^k$ can be partitioned into regions $R_1, \ldots, R_m$ with corresponding polynomials $q_1, \ldots, q_m$ such that for each i,

$$c(n_1, \ldots, n_k) = q_i(n_1, \ldots, n_k) \quad \text{for } (n_1, \ldots, n_k) \in R_i. \tag{11.3}$$

Using properties of Schur functions it can be shown that an analogue formula holds for the $m_\lambda(A)$. Hence we may write

$$c_n(A) = \sum_i \left\{ q_i(\lambda) f^\lambda \mid \lambda \in R_i \right\}. \tag{11.4}$$

The regions R_i, in general, are defined by linear inequalities and modular linear equations. Here is a simple example to make this more clear. Let $F(t_1, t_2) = (1 - t_1^2)^{-1}(1 - t_2)^{-1}(1 - t_1 t_2)^{-1}$.

Then $b(n_1, n_2)$ is given by

$$b(n_1, n_2) = \begin{cases} \dfrac{n_2}{2} + 1 & \text{if } n_1 \geq n_2,\ n_2 \text{ is even,} \\[2ex] \dfrac{(n_2 + 1)}{2} & \text{if } n_1 \geq n_2,\ n_2 \text{ is odd,} \\[2ex] \dfrac{n_1}{2} + 1 & \text{if } n_1 < n_2,\ n_1 \text{ is even,} \\[2ex] \dfrac{(n_1 + 1)}{2} & \text{if } n_1 < n_2,\ n_1 \text{ is odd.} \end{cases} \tag{11.5}$$

Now back to the general case. There exists a number $h \leq k$ called the *essential height* of the cocharacter defined to be the largest number such that there exist nonzero multiplicities $m_\lambda(A)$ in which $\lambda = (\lambda_1, \ldots, \lambda_k)$ and λ_h can be taken arbitrarily large. Giambruno and Zaicev proved that $h = \lim_{n\to\infty}(c_n(A))^{1/n}$. This number, denoted $\exp(A)$, is also important for the proof of Theorem 11.1. Let v_1 be the vector in $\mathbb{N}^k$ whose coordinates are h ones followed by $k - h$ zeros. Then in (11.4) certain summands will be on regions of the form

$$\{v_0 + \alpha_1 v_1 + \cdots + \alpha_d v_d \mid \alpha_i, \ldots, \alpha_d \in \mathbb{N}\} \cap L, \tag{11.6}$$

where L is an integer lattice. Let $R'_1, \ldots, R'_p$ be the regions of this form. Then in the computation of $c_n(A)$ these terms dominate and we can refine (11.4) to

$$c_n(A) = \sum_i \left\{ q_i(\lambda) f^\lambda \mid \lambda \in R'_i \right\}. \tag{11.7}$$

The rest of the proof closely immitates the computation of [16]. The main theorem is that if R' is as above and $q(x_1, \ldots, x_k)$ is a polynomial then $\sum q(\lambda) f^\lambda$ summed over partitions of n in R' will be asymptotic to a constant—times $\sqrt{n}$ to an integer power—times h^n. Hence, the cocharacter $c_n(A)$ will be a sum of such terms.

There remains the problem that the powers of n might not be equal, and the way around this difficulty is to use the fact that if $1 \in A$ then the cocharacter sequence is Young derived, see [49]. Namely, the Poincaré series $P(t_1, \ldots, t_k)$ can be written as $(1 - t_1)^{-1} \cdots (1 - t_k)^{-1} g(t_1, \ldots, t_k)$, where g has all the nice properties of P. □

We conclude this section with the following general remark.

Remark 11.2. Recent works extended the above theorems of Sections 10 and 11 to graded polynomial identities and to PI algebras with the action of Hopf algebra, see, for example, [50–59].

12. Algebraicity of Some Generating Functions

As we show below, Theorems 10.1 and 11.1 are related to the question of whether or not the generating function of the codimensions is algebraic. We begin with the following definition.

Definition 12.1 (see [60, 61]). (1) Given the sequence a_n, then $F(x) = \sum_{n\geq 0} a_n x^n$ is its corresponding *ordinary generating function*. In particular $C_A(x) = \sum_{n\geq 0} c_n(A)x^n$ is the generating function of the codimensions of the algebra A.

(2) The function $F(x)$ is *algebraic* if there exist polynomials $P_0(x), \ldots, P_r(x)$ such that

$$P_r(x)F^r(x) + \cdots + P_1(x)F(x) + P_0(x) = 0. \tag{12.1}$$

Algebraicity or nonalgebraicity of the generating function $F(x) = \sum_{n\geq 0} a_n x^n$ is an indication of the complexity of the sequence a_n.

Example 12.2 (see [34], [25, Theorem 12.6.8]). For the 2×2 matrices $M_2(F)$ we have

$$c_n(M_2(F)) = \frac{1}{n+2}\binom{2n+2}{n+1} - \binom{n}{3} + 1 - 2^n. \tag{12.2}$$

This implies that

$$C_{M_2(F)}(x) = \frac{1}{x^2}\left(1 - 2x - \sqrt{1-4x}\right) - \frac{x^3}{(1-x)^4} + \frac{1}{1-x} - \frac{1}{1-2x}, \tag{12.3}$$

which is clearly algebraic.

Note that when $k = 1$, $M_1(F) = F$ and $c_n(F) = 1$, hence

$$C_F(x) = \frac{1}{1-x} \tag{12.4}$$

which is algebraic.

12.1. Nonalgebraicity of Some Generating Functions

We quote here a classical theorem of Jungen [62], see also [63].

Theorem 12.3 (see [62]). *Let $f : \mathbb{N} \to \mathbb{C}$, $F(x) = \sum_{n\geq 0} f(n)x^n$, and assume that as n goes to infinity,*

$$f(n) \sim b \cdot n^{-g} \cdot a^n, \tag{12.5}$$

where b and a are complex constants and g is a real number. For $F(x)$ to be algebraic it is necessary that g be rational; and if $g > 0$ then g must also be non-integral.

Applying this theorem to the codimensions of matrices we deduce the following theorem.

Theorem 12.4. *Let*

$$C_{M_k(F)}(x) = \sum_{n\geq 0} c_n(M_k(F)) \cdot x^n \tag{12.6}$$

be the generating functions of the (ordinary) codimensions of $M_k(F)$. If $k \geq 3$ and k is odd then $C_{M_k(F)}(x)$ is not algebraic.

Proof. By Theorem 6.2

$$c_n(M_k(F)) \sim b \cdot n^{-g} \cdot k^{2n}, \tag{12.7}$$

where $g = (k^2 - 1)/2$. The proof now follows by Theorem 12.3, since $g = (k^2 - 1)/2$ is an integer when k is odd. □

Obviously, Example 12.2 and Theorem 12.4 motivate the following conjecture.

Conjecture 12.5. *If $k \geq 3$ then the generating function*

$$C_{M_k(F)}(x) = \sum_{n \geq 0} c_n(M_k(F)) \cdot x^n \tag{12.8}$$

is not algebraic.

For the other verbally prime algebras (see Section 2.2.1), recall from Theorem 6.4 the following partial asymptotic results:

$$c_n(M_{k,\ell}) \sim a \cdot \left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2} \cdot (k+\ell)^{2n}, \tag{12.9}$$

where the constant a is yet unknown. Also,

$$c_n(M_k(G)) \sim b \cdot \left(\frac{1}{n}\right)^{g} \cdot \left(2k^2\right)^{n}, \tag{12.10}$$

where the constants b and g are yet unknown, and $(k^2 - 1)/2 \leq g \leq (2k^2 - 1)/2$.

Corollary 12.6. *If $k \not\equiv \ell \pmod 2$ then the generating function $C_{M_{k,\ell}}(x)$ of the codimensions $c_n(M_{k,\ell})$ in not algebraic.*

Proof. Indeed, $k \not\equiv \ell \pmod 2$ implies that $k^2 + \ell^2 - 1$ is even, and the proof follows from Theorems 12.3 and 6.4. □

Example 12.7 (see [64]). We apply here a theorem due to Kemer [6] which says that the algebras $G \otimes G$ and $M_{1,1}$ have the same identities, hence the same codimensions. Now

$$c_0(M_{1,1}) = 1, \qquad c_n(M_{1,1}) = \frac{1}{2}\binom{2n}{n} + n + 1 - 2^n, \quad n = 1, 2, \ldots \tag{12.11}$$

see [65], and hence

$$C_{M_{1,1}}(x) = \frac{1}{2} + \frac{1}{2\sqrt{1-4x}} + \frac{x}{(1-x)^2} + \frac{1}{1-x} + \frac{1}{1-2x}, \tag{12.12}$$

which is clearly algebraic. Note that

$$c_n(M_{1,1}) \sim \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 4^n, \tag{12.13}$$

compare with (6.3) of Section 6. See also [64].

Conjecture 12.8. *If $k, \ell \geq 1$ and $(k, \ell) \neq (1,1)$ then $C_{M_{k,\ell}}(x)$ is not algebraic.*

13. Nonassociative A with $\exp(A)$ a Non Integer

As remarked before, the notions of codimensions and cocharacters also apply to nonassociative PI algebras. Here, in the most general case, the free associative algebra $F\{x\} = F\{x_1, x_2, \ldots\}$ is replaced by the free algebra $F\langle x_1, x_2, \ldots\rangle = F\langle x\rangle$. Now A is a nonassociative PI algebra and again, $\mathrm{Id}(A) \subseteq F\langle x\rangle$ are the identities of A. Since we are dealing now with nonassociative polynomials, hence different parenthesizes in a monomial yield different monomials. It follows that in the nonassociative case, the space V_n of the multilinear polynomials of degree n is now of dimension $\dim V_n = C_n \cdot n!$, where C_n is the nth Catalan number (which counts the number of different parenthesizes of a monomial of degree n). The definition of the codimensions $c_n(A)$ is, formally, the same as that in the associative case.

Definition 13.1.

$$c_n(A) = \dim\left(\frac{V_n}{\mathrm{Id}(A) \cap V_n}\right), \tag{13.1}$$

compare with Definition 4.1.

Similarly for the cocharacters, the action of S_n on V_n is again given by (5.1), and one introduces cocharacters precisely as in the associative case, see Definition 5.1. However, some phenomena here are rather different from those in the associative case. For example, a counter-example to Theorem 4.4, hence also to Theorem 4.3, was given in [66].

A counter-example to Theorem 10.1 in the nonassociative case was first constructed in [67]. Recently, Giambruno et al. [68] constructed a family of nonassociative PI algebras A such that for every $1 < \alpha < 2$ there is such an algebra A for which the limit $\lim_{n\to\infty}(c_n(A))^{1/n}$ exists and is equal to α. We briefly describe that construction.

13.1. *The Algebra* $A(m, w)$

Let $w = w_1 w_2 \cdots$ be an infinite word on the alphabet $\{0, 1\}$. Given an integer $m \geq 2$, define the sequence $\{k_i\}_{i\geq 1} = K_{m,w}$ by

$$k_i = w_i + m. \tag{13.2}$$

Construct the algebra $A = A(m, w) = A(K_{m,w})$ as follows. Its basis is

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \cdots, \tag{13.3}$$

where

$$Z_i = \left\{ z_j^{(i)} \mid 1 \leq j \leq k_i \right\}, \quad i = 1, 2, \ldots, \tag{13.4}$$

and multiplication is given as follows:

$$\begin{gathered} z_1^{(i)} a = z_2^{(i)}, z_2^{(i)} a = z_3^{(i)}, \ldots, z_{k_i-1}^{(i)} a = z_{k_i}^{(i)}, \quad i = 1, 2, \ldots, \\ z_{k_i}^{(i)} b = z_1^{(i+1)}, \quad i = 1, 2, \ldots \end{gathered} \tag{13.5}$$

and all other products are zero.

Let $w = w_1 w_2 \cdots$ be an infinite word. The notion of the *complexity* of w is classical. For each n, $\text{complexity}_w(n)$ is the number of distinct subwords of w of length n. The algebra $A = A(m, w)$ depends on the integer m and on the complexity of the word w, and the following theorem is proved.

Theorem 13.2 (see [68]). *Given $1 \leq \alpha \leq 2$, we can choose $m \geq 2$ and a word w such that*

$$\lim_{n \to \infty} (c_n(A(m, w)))^{1/n} \quad \text{exists and is equal to } \alpha. \tag{13.6}$$

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