International Scholarly Research Network
ISRN Applied Mathematics
Volume 2012, Article ID 197383, 14 pages
doi:10.5402/2012/197383

Research Article

A Family of Even-Point Ternary Approximating Schemes

Abdul Ghaffar and Ghulam Mustafa

Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan

Correspondence should be addressed to Ghulam Mustafa, ghulam.mustafa@iub.edu.pk

Received 27 July 2012; Accepted 8 September 2012

Academic Editors: L. Ju, F. Lamnabhi-Lagarrigue, G. Psihoyios, H. C. So, and X. Yang

We presented a general formula to generate the family of even-point ternary approximating subdivision schemes with a shape parameter for describing curves. Some sufficient conditions for C^0 to C^7 continuity and approximation order for certain ranges of parameter are discussed. The proposed even-point ternary schemes compare remarkably with existing even-point ternary schemes because they are able to generate limit functions with higher smoothness and approximation order. In addition, we measured curvature and torsion that assist the quality of subdivided curves.

1. Introduction and Preliminaries

There are numbers of binary subdivision schemes in the literature. The interest in investigating arities higher than two has been started by Hassan et al. [1, 2]. Nowadays, we have numbers of ternary schemes introduced by [3–7] and so forth, But the research communities are still gaining interest in introducing schemes higher than three arities (i.e., quaternary, quinary, senary,..., n-ary). Mustafa and Khan [8] introduced a new 4-point C^3 quaternary approximating subdivision scheme. Lian [9, 10] introduced 3-, 4-, 5-, and 6-point a-ary schemes. Lian [11] also offered $2m$-point and $(2m+1)$-point interpolating a-ary schemes for curve design.

The 2-, 3-,..., 6-point binary and ternary schemes are very common in the literature. The schemes involving convex combination of more or less than six points at coarse refinement level to insert a new point at next refinement level is introduced by Ko et al. [7]. They introduced $(2n+2)$- and $(2n+4)$-point binary schemes. Zheng et al. [12] investigated ternary interpolatory schemes with an odd number of control points, namely, $(2n-1)$-point ternary interpolatory subdivision scheme. They also investigated ternary even symmetric

$2n$-point [13] and p-ary [14] approximating subdivision scheme and presented the general ternary even symmetric $2n$-point approximating subdivision rule and design alternative smooth ternary subdivision scheme of higher order. Mustafa and Rehman [15] presented general formulae for the mask of $(2b+4)$-point n-ary approximating as well as interpolating subdivision schemes for any integers $b \geq 0$ and $n \geq 2$.

This motivates us to present the family of even-point ternary schemes with high smoothness and more degree of freedom for curve design. Proposed schemes not only provide the mask of even-point schemes but also generalize and unify several well-known schemes. Moreover, we measured curvature and torsion that can be used to describe the quality of curve. Also we compared plot of curvature and torsion, obtained by proposed schemes with the other existing schemes.

A general compact form of univariate ternary subdivision scheme S, which maps polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ is defined by

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{3j-i} f_j^k, \quad i \in \mathbb{Z}, \tag{1.1}$$

where the set $a = \{a_i : i \in \mathbb{Z}\}$ of coefficients is called the mask at k-th level of refinement. A necessary condition for the uniform convergence of subdivision scheme (1.1) is that

$$\sum_{j\in\mathbb{Z}} a_{3j} = \sum_{j\in\mathbb{Z}} a_{3j+1} = \sum_{j\in\mathbb{Z}} a_{3j+2} = 1. \tag{1.2}$$

A subdivision scheme is uniformly convergent if for any initial data $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$, there exists a continuous function f such that for any closed interval $I \subset \mathbb{R}$

$$\lim_{k\to\infty} \sup_{i\in 3^k I} \left| f_i^k - f\left(3^{-k} i\right) \right| = 0. \tag{1.3}$$

Obviously, $f = S^\infty f^0$, introducing a symbol called Laurent polynomial

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i, \tag{1.4}$$

of the mask $a = \{a_i : i \in \mathbb{Z}\}$ which plays an efficient role to analyze the convergence and smoothness of subdivision scheme. From (1.2) and (1.4) the Laurent polynomial of convergent subdivision scheme satisfies

$$a\left(e^{2i\pi/3}\right) = a\left(e^{4i\pi/3}\right) = 0, \quad a(1) = 3. \tag{1.5}$$

This condition guarantees existence of a related subdivision scheme for the divided difference of the original control points and the existence of associated Laurent polynomial $a^{(1)}(z)$

$$a^{(1)}(z) = \frac{3z^2}{1+z+z^2} a(z). \tag{1.6}$$

The subdivision scheme S_1 with Laurent polynomial $a^{(1)}(z)$ is related to scheme S with Laurent polynomial $a(z)$ by the following theorem.

Theorem 1.1 (see [1]). *Let S denote a subdivision scheme with Laurent polynomial $a(z)$ satisfying* (1.5). *Then there exists a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1}, \tag{1.7}$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = 3^k(f_{i+1}^k - f_i^k);\ i \in \mathbb{Z}\}$. Furthermore, S is a uniformly convergent if and only if $(1/3)S_1$ converges uniformly to zero function for all initial data f^0, in the sense that

$$\lim_{k\to 0}\left(\frac{1}{3}S_1\right)^k f^0 = 0. \tag{1.8}$$

A scheme S_1 satisfying (1.8) for all initial data f^0 is termed "contractive." The above theorem indicates that for any given scheme S, with mask "a" satisfying (1.2), we can prove the uniform convergence of S by deriving the mask of $(1/3)S_1$ and computing $\|((1/3)S_1)^i\|_\infty$ for $i = 1,2,3...,L$, where L is the first integer for which $\|((1/3)S_1)^L\|_\infty < 1$. If such an L exists, then S converges uniformly. Since there are three rules for computing the values at next refinement level, so we define the norm as follows:

$$\|S\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}|a_{3j}|, \sum_{j\in\mathbb{Z}}|a_{3j+1}|, \sum_{j\in\mathbb{Z}}|a_{3j+2}|\right\}, \tag{1.9}$$

$$\left\|\left(\frac{1}{3}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b_{i+3^L j}^{[n,L]}\right|;\ i = 0,1,2,\ldots,3^L-1\right\}, \tag{1.10}$$

where

$$b^{[n,L]}(z) = \frac{1}{3^L}\prod_{j=0}^{L-1} a^{(n)}\left(z^{3^j}\right), \tag{1.11}$$

$$a^{(n)}(z) = \left(\frac{3z^2}{1+z+z^2}\right)a^{(n-1)}(z) = \left(\frac{3z^2}{1+z+z^2}\right)^n a(z), \quad n \geqslant 1. \tag{1.12}$$

The paper is organized as follows: the family of even-point ternary approximating scheme and analysis of two even-point ternary schemes are presented in Section 2. Basic properties of even-point ternary schemes are discussed in Section 3. Comparison with existing even-point ternary schemes is shown in Section 4. A few remarks and future work constitute Section 5.

2. The Even-Point Ternary Approximating Schemes

Here we offer a general formula for even-point ternary approximating subdivision schemes with one parameter in the form of Laurent polynomial

$$a_{[\lambda+1]}(z) = \frac{1}{3^{\lambda+n-1}}\left(1+z+z^2\right)^{\lambda+n} \times \left\{\left(\frac{1}{12}+\omega\right)+\left(\frac{5}{12}-\omega\right)z+\left(\frac{5}{12}-\omega\right)z^2+\left(\frac{1}{12}+\omega\right)z^3\right\}, \tag{2.1}$$

where ω is shape parameter, $\lambda = 2n+1, n \geqslant 0$ and depends on n.

Despite the fact that we can generate $(\lambda+1)$-point ternary approximating schemes from (2.1) for any $\lambda = 2n+1,\ n \geqslant 0$, however for straightforwardness, we generated and discussed the smoothness of 4- and 6-point ternary schemes. The general proof regarding the support of even-point ternary schemes has been presented in this paper.

2.1. 4*-Point Ternary Scheme*

From (2.1) for $n = 1$, we get the following: Laurent polynomial for a 4-point ternary scheme

$$\begin{aligned} a_{[4]}(z) = \frac{1}{27}\Bigg\{&\left(\frac{1}{12}+\omega\right)z^0+\left(\frac{3}{4}+3\omega\right)z^1+\left(\frac{35}{12}+5\omega\right)z^2+\left(\frac{29}{4}+3\omega\right)z^3 \\ &+\left(\frac{51}{4}-3\omega\right)z^4+\left(\frac{67}{4}-9\omega\right)z^5+\left(\frac{67}{4}-9\omega\right)z^6+\left(\frac{51}{4}-3\omega\right)z^7 \\ &+\left(\frac{29}{4}+3\omega\right)z^8\left(\frac{35}{12}+5\omega\right)z^9+\left(\frac{3}{4}+3\omega\right)z^{10}+\left(\frac{1}{12}+\omega\right)z^{11}\Bigg\}. \end{aligned} \tag{2.2}$$

From the above polynomial, we suggest the following 4-point ternary approximating scheme:

$$\begin{aligned} f_{3i}^{k+1} &= \frac{1}{27}\left[\left(\frac{35}{12}+5\omega\right)f_{i-1}^k+\left(\frac{67}{4}-9\omega\right)f_i^k+\left(\frac{29}{4}+3\omega\right)f_{i+1}^k+\left(\frac{1}{12}+\omega\right)f_{i+2}^k\right], \\ f_{3i+1}^{k+1} &= \frac{1}{27}\left[\left(\frac{3}{4}+3\omega\right)f_{i-1}^k+\left(\frac{51}{4}-3\omega\right)f_i^k+\left(\frac{51}{4}-3\omega\right)f_{i+1}^k+\left(\frac{3}{4}+3\omega\right)f_{i+2}^k\right], \\ f_{3i+2}^{k+1} &= \frac{1}{27}\left[\left(\frac{1}{12}+\omega\right)f_{i-1}^k+\left(\frac{29}{4}+3\omega\right)f_i^k+\left(\frac{67}{4}-9\omega\right)f_{i+1}^k+\left(\frac{35}{12}+5\omega\right)f_{i+2}^k\right]. \end{aligned} \tag{2.3}$$

From (1.12) and (2.2), we have

$$\begin{aligned} a_{[4]}^{(1)}(z) = \frac{z^2}{9}\Bigg\{&\left(\frac{1}{12}+\omega\right)+\left(\frac{2}{3}+2\omega\right)z+\left(\frac{13}{6}+2\omega\right)z^2+\left(\frac{53}{12}-\omega\right)z^3 \\ &+\left(\frac{37}{6}-4\omega\right)z^4+\left(\frac{37}{6}-4\omega\right)z^5+\left(\frac{53}{12}-\omega\right)z^6+\left(\frac{13}{6}+2\omega\right)z^7 \\ &+\left(\frac{2}{3}+2\omega\right)z^8+\left(\frac{1}{12}+\omega\right)z^9\Bigg\}, \end{aligned}$$

$$
\begin{aligned}
a_{[4]}^{(2)}(z) &= \frac{z^4}{3}\left\{\left(\frac{1}{12}+\omega\right)z^0+\left(\frac{7}{12}+\omega\right)z^1+3z^2+\left(\frac{7}{3}-2\omega\right)z^3\right.\\
&\quad\left.+\left(\frac{7}{3}-2\omega\right)z^4+3z^5+\left(\frac{7}{12}+\omega\right)z^6+\left(\frac{1}{12}+\omega\right)z^7\right\},\\
a_{[4]}^{(3)}(z) &= \left(\frac{1}{12}+\omega\right)+\frac{1}{2}z+\left(\frac{11}{12}-\omega\right)z^2+\left(\frac{11}{12}-\omega\right)z^3+\frac{1}{2}z^4+\left(\frac{1}{12}+\omega\right)z^5,\\
a_{[4]}^{(4)}(z) &= 3z^2\left\{\left(\frac{1}{12}+\omega\right)+\left(\frac{5}{12}-\omega\right)z+\left(\frac{5}{12}-\omega\right)z^2+\left(\frac{1}{12}+\omega\right)z^3\right\}.
\end{aligned}
\tag{2.4}
$$

If $S_m^{[4]}$ is the scheme corresponding to $a_{[4]}^{(m)}$, then for C^{m-1} continuity, we require that $a_{[4]}^{(m)}$ satisfies (1.2), which it does and $\|((1/3)S_m^{[4]})^L\|_\infty < 1$, for $m = 1,2,3,4$.

Since by (1.10), for $-71/24 < \omega < 91/24$, $-11/6 < \omega < 8/3$, $-13/12 < \omega < 23/12$, and $-5/12 < \omega < 7/12$, we have

$$
\begin{aligned}
\left\|\frac{1}{3}S_1^{[4]}\right\|_\infty &= \max\left\{\frac{2}{27}\left|\frac{1}{12}+\omega\right|+\frac{2}{27}\left|\frac{53}{12}-\omega\right|,\frac{1}{27}\left|\frac{2}{3}+2\omega\right|\right.\\
&\quad\left.+\frac{1}{27}\left|\frac{37}{6}-4\omega\right|+\frac{1}{27}\left|\frac{13}{6}+2\omega\right|\right\}<1,\\
\left\|\frac{1}{3}S_2^{[4]}\right\|_\infty &= \max\left\{\frac{1}{9}\left|\frac{1}{12}+\omega\right|+\frac{1}{9}\left|\frac{7}{4}-2\omega\right|+\frac{2}{9}\left|\frac{7}{12}+\omega\right|,\frac{2}{3}\right\}<1,\\
\left\|\frac{1}{3}S_3^{[4]}\right\|_\infty &= \max\left\{\frac{1}{3}\left|\frac{1}{12}+\omega\right|+\frac{1}{3}\left|\frac{11}{12}-\omega\right|,\frac{1}{3}\right\}<1,\\
\left\|\frac{1}{3}S_4^{[4]}\right\|_\infty &= \max\left\{2\left|\frac{1}{12}+\omega\right|,\left|\frac{5}{12}-\omega\right|,\left|\frac{5}{12}-\omega\right|\right\}<1.
\end{aligned}
\tag{2.5}
$$

In order to obtain a C^4 scheme, we substitute $\omega = 1/12$ in $a_{[4]}^{(4)}(z)$, we have

$$
a_{[4]}^{(4)}(z) = \frac{z^2}{2}\left(1+2z+2z^2+z^3\right). \tag{2.6}
$$

Using (1.12), we get

$$
a_{[4]}^{(5)}(z) = \left(\frac{3z^2}{1+z+z^2}\right)a^4(z) = \frac{3}{2}z^4\{1+z\}. \tag{2.7}
$$

If $S_m^{[4]}$ is the scheme corresponding to $a_{[4]}^{(m)}$, then for C^{m-1} continuity, we require that $a_{[4]}^{(m)}$ satisfy (1.2), which it does and $\|((1/3)S_m^{[4]})^L\|_\infty < 1$, for $m = 1,2,3,4$.

For $m = 5$ and $L = 1$, we have

$$\begin{aligned}\left\|\frac{1}{3}S_5^{[4]}\right\|_\infty &= \frac{1}{3}\max\left\{\sum_{j\in\mathbb{Z}}|a_{3j}|, \sum_{j\in\mathbb{Z}}|a_{3j+1}|, \sum_{j\in\mathbb{Z}}|a_{3j+2}|\right\},\\ &= \frac{1}{3}\max\left\{\frac{3}{2}, \frac{3}{2}, 0\right\} = \frac{1}{2} < 1.\end{aligned} \tag{2.8}$$

Therefore $(1/3)S_5^{[4]}$ is contractive, that is, $\|(1/3)S_5^{[4]}\|_\infty < 1$ and the scheme is C^4.

2.2. 6-Point Ternary Scheme

From (2.1) for $n = 2$, we get the following 6-point ternary approximating scheme:

$$\begin{aligned}f_{3i}^{k+1} &= \frac{1}{729}\left[\left(\frac{17}{3}+20\omega\right)f_{i-2}^k + \left(\frac{371}{3}+56\omega\right)f_{i-1}^k + \left(\frac{4373}{12}-127\omega\right)f_i^k\right.\\ &\quad\left.+\left(\frac{2569}{12}+7\omega\right)f_{i+1}^k + \left(\frac{253}{12}+43\omega\right)f_{i+2}^k + \left(\frac{1}{12}+\omega\right)f_{i+3}^k\right],\\ f_{3i+1}^{k+1} &= \frac{1}{729}\left[(6\omega+1)f_{i-2}^k + \left(\frac{231}{4}+63\omega\right)f_{i-1}^k + \left(\frac{1223}{4}-69\omega\right)f_i^k\right.\\ &\quad\left.+\left(\frac{1223}{4}-69\omega\right)f_{i+1}^k + \left(\frac{231}{4}+63\omega\right)f_{i+2}^k + (1+6\omega)f_{i+3}^k\right],\\ f_{3i+2}^{k+1} &= \frac{1}{729}\left[\left(\frac{1}{12}+\omega\right)f_{i-2}^k + \left(\frac{253}{12}+43\omega\right)f_{i-1}^k + \left(\frac{2569}{12}+7\omega\right)f_i^k\right.\\ &\quad\left.+\left(\frac{4373}{12}-127\omega\right)f_{i+1}^k + \left(\frac{371}{3}+56\omega\right)f_{i+2}^k + \left(\frac{17}{3}+20\omega\right)f_{i+3}^k\right].\end{aligned} \tag{2.9}$$

The continuity of this scheme can be computed in a similar way as we did for the 4-point ternary scheme.

From above discussion, we reach the conclusion shown in Table 1.

Remark 2.1. (i) For $n = 1$, $\omega = -1/4+9u$, it becomes 4-point ternary subdivision scheme [14].

(ii) If we set $n = 1$, $\omega = -1/4 + 9w$, it becomes 4-point ternary subdivision scheme [13].

(iii) In case $n = 2$, $\omega = -11/12 + (81/7)u$, we get mask of 6-point ternary subdivision scheme [13].

3. Basic Properties of the Schemes

In this section, we discuss approximation order and support of basic limit function of even-point ternary approximating schemes.

Table 1: The order of continuity of proposed 4-point and 6-point ternary schemes for certain ranges of parameter.

Scheme	Parameter	Continuity	Scheme	Parameter	Continuity
4-point	$-\frac{71}{24} < \omega < \frac{91}{24}$	C^0	6-point	$-\frac{329}{60} < \omega < \frac{1489}{204}$	C^0
	$-\frac{11}{6} < \omega < \frac{8}{3}$	C^1		$-\frac{59}{12} < \omega < \frac{509}{12}$	C^1
	$-\frac{13}{12} < \omega < \frac{23}{12}$	C^2		$-\frac{149}{36} < \omega < \frac{175}{36}$	C^2
	$-\frac{5}{12} < \omega < \frac{7}{12}$	C^3		$-\frac{71}{24} < \omega < \frac{91}{24}$	C^3
	$\omega = \frac{1}{12}$	C^4		$-\frac{11}{6} < \omega < \frac{8}{3}$	C^4
				$-\frac{13}{12} < \omega < \frac{23}{12}$	C^5
				$-\frac{5}{12} < \omega < \frac{7}{12}$	C^6
				$\omega = \frac{1}{12}$	C^7

3.1. Approximation Order

Here we only find the approximation order of proposed 4-point ternary scheme. The approximation order of other even-point ternary schemes can be computed in a similar fashion.

Theorem 3.1. *A 4-point ternary approximating scheme has approximating order 4 for* $\omega \in (-5/12 < \omega < 7/12)$ *and 5 for* $\omega = 5/12$.

A 6-point ternary approximating scheme has approximating order 7 for $\omega \in (-5/12 < \omega < 7/12)$ *and 8 for* $\omega = 1/12$.

Proof. We carry out this result by taking our origin the middle of an original span with ordinate $\ldots, (-5)^n, (-3)^n, (-1)^n, 1^n, 3^n, 5^n, \ldots$.

If $y = x^n$, then we have

$$\begin{aligned}
[y] = {} & \ldots, a_1(-5)^n + a_2(-3)^n + a_3(-1)^n + a_4(1)^n \\
& + a_5(-5)^n + a_6(-3)^n + a_6(-1)^n + a_5(1)^n \\
& + a_4(-5)^n + a_3(-3)^n + a_2(-1)^n + a_1(1)^n \\
& + a_1(-3)^n + a_2(-1)^n + a_3(1)^n + a_4(3)^n \\
& + a_5(-3)^n + a_6(-1)^n + a_6(1)^n + a_5(3)^n \\
& + \cdots + \\
& a_5(-1)^n + a_6(1)^n + a_6(3)^n + a_5(5)^n \\
& + a_4(-1)^n + a_3(1)^n + a_2(3)^n + a_1(5)^n, \ldots,
\end{aligned} \tag{3.1}$$

where $a_1 = (1/27)(35/12 + 5\omega)$, $a_2 = (1/27)(67/4 - 9\omega)$, $a_3 = (1/27)(29/4 + 3\omega)$, $a_4 = (1/27)(1/12 + \omega)$, $a_5 = (1/27)(3/4 + 3\omega)$, $a_6 = (1/27)(51/3 - 3\omega)$.

If $y = x^1$, then

$$\begin{gathered}[y] = \ldots, \frac{-8}{3}, -2, \frac{-4}{3}, \frac{-2}{3}, 2, \frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \ldots \\ [\delta y] = \ldots, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \ldots, \\ \left[\delta^2 y\right] = 0,\end{gathered} \tag{3.2}$$

where δ represents the differences of the vertices.

If $y = x^2$, then

$$\begin{aligned}[y] = {} & \ldots, \frac{77}{9} + \frac{16}{9}\omega, \frac{49}{9} + \frac{16}{9}\omega, \frac{29}{9} + \frac{16}{9}\omega, \frac{17}{9} + \frac{16}{9}\omega, \frac{13}{9} + \frac{16}{9}\omega, \frac{17}{9} + \frac{16}{9}\omega, \\ & \frac{29}{9} + \frac{16}{9}\omega, \frac{49}{9} + \frac{16}{9}\omega, \frac{77}{9} + \frac{16}{9}\omega, \ldots.\end{aligned} \tag{3.3}$$

Taking further differences, we get $[\delta^3 y] = 0$.

If $y = x^3$, then

$$\begin{aligned}[y] = {} & \ldots, \frac{-824}{27} - \frac{128}{9}\omega, \frac{-50}{3} - \frac{32}{3}\omega, \frac{-220}{27} - \frac{64}{9}\omega, \frac{-86}{27} - \frac{32}{9}\omega, 0, \frac{86}{27} + \frac{32}{9}\omega, \\ & \frac{220}{27} + \frac{64}{9}\omega, \frac{50}{3} + \frac{32}{3}\omega, \frac{824}{27} + \frac{128}{27}\omega, \ldots.\end{aligned} \tag{3.4}$$

This implies that $[\delta^4 y] = 0$.

If $y = x^4$, then

$$\begin{aligned}[y] = {} & \ldots, \frac{3187}{27} + \frac{800}{9}\omega, \frac{505}{9} + \frac{544}{9}\omega, \frac{659}{27} + 32\omega, \frac{89}{9} + \frac{160}{9}\omega, \frac{49}{9} + \frac{160}{9}\omega, \frac{89}{9} + \frac{160}{9}\omega, \\ & \frac{659}{27} + 32\omega, \frac{505}{9} + \frac{544}{9}\omega, \frac{3187}{27} + \frac{800}{9}\omega, \ldots.\end{aligned} \tag{3.5}$$

By taking further differences, we have

$$\begin{gathered}\left[\delta^5 y\right] = \ldots, \frac{160}{27} - \frac{128}{9}\omega, \frac{160}{27} - \frac{128}{9}\omega, 0, \frac{160}{27} - \frac{128}{9}\omega, \frac{160}{27} - \frac{128}{9}\omega, \ldots, \\ \left[\delta^5 y\right] = 0, \text{ at } \omega = \frac{5}{12}.\end{gathered} \tag{3.6}$$

Since by [8], 4-point ternary scheme has cubic precision for $\omega \in (-5/12 < \omega < 7/12)$ and quartic at $\omega = 5/12$ then 4-point ternary scheme has approximating order 4 for $\omega \in (-5/12 < \omega < 7/12)$ and 5 for $\omega = 5/12$. The proof of other part is similar. □

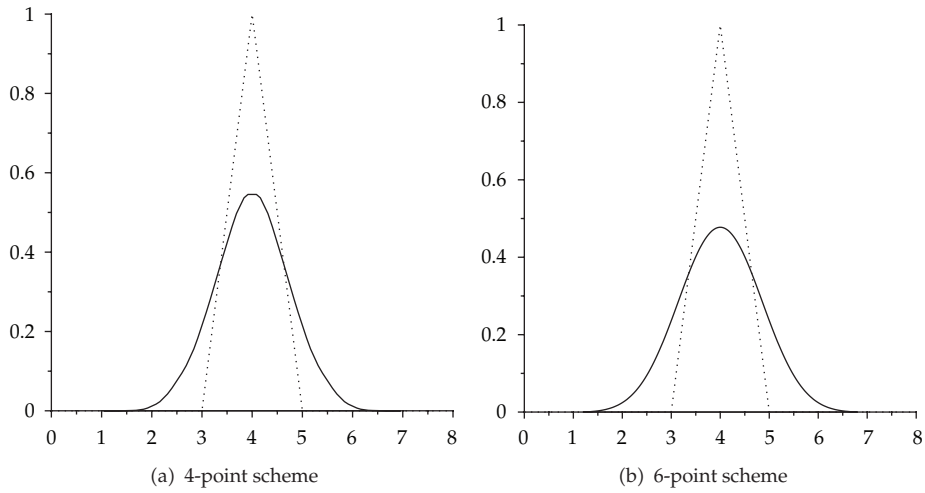


Figure 1: The basic limit functions of proposed schemes at $\omega = 1/12$.

3.2. *Support of Basic Limit Function*

The basic limit functions, $\varphi_{[\lambda+1]} = (S^{[\lambda+1]})^{\infty} f_i^0$ of proposed $(\lambda+1)$-point ternary approximating schemes, for $\lambda = 2n+1$, $n = 1, 2$ are presented in Figures 1(a) and 1(b). The following theorem is related to the support of basic limit functions of even point ternary schemes.

Theorem 3.2. *The basic limit functions $\varphi_{[\lambda+1]}$ of proposed $(\lambda + 1)$-point ternary approximating schemes have support width $s = (2(\lambda + n) + 3)/2$, where $\lambda = 2n + 1$, $n \geqslant 0$, which implies that it vanishes outside the interval $[-(2(\lambda + n) + 3)/4, (2(\lambda + n) + 3)/4]$.*

Proof. The support width "s" of the basic limit functions can be determined by computing how for the effect of the nonzero vertex f_0^0 will propagate along by. As the mask of $(\lambda + 1)$ scheme is $3(\lambda + 1)$-long sequence by centering it on that vertex, the distances to the last of its left and right nonzero coefficients are equal to $\lambda + n + 2$ and $\lambda + n + 1$, respectively. At the first subdivision step we see that the vertices on the left and right sides of f_0^1 at $(\lambda + n + 2)/3$ and $(\lambda + n + 1)/3$ are the furthest nonzero new vertices. At each refinement, the distances on both sides are reduced by the factor $1/3$. At the next step of the scheme this will propagate along by $((\lambda+n+2)/3)\times 1/3$ on the left and $((\lambda+n+1)/3)\times 1/3$ on the right. Hence after k subdivision steps the furthest nonzero vertex on the left will be at $(\lambda + n + 2)(1/3 + 1/3^2 + \cdots + 1/3^k) = ((\lambda + n + 2)/3)(\sum_{j=0}^{k-1}(1/3^j))$ and on the right will be at $(\lambda + n + 1)(1/3 + 1/3^2 + \cdots + 1/3^k) = ((\lambda + n + 1)/3)(\sum_{j=0}^{k-1}(1/3^j))$. So the total support width is $((\lambda + n + 2)/3)(\sum_{j=0}^{\infty}(1/3^j)) + ((\lambda + n + 1)/3)(\sum_{j=0}^{\infty}(1/3^j)) = (2(\lambda + n) + 3)/2$. □

4. Comparison and Application

In order to show the performance of the proposed schemes, we compare continuity, support, approximation order, and shape of limit curves. We also discuss curvature and torsion.

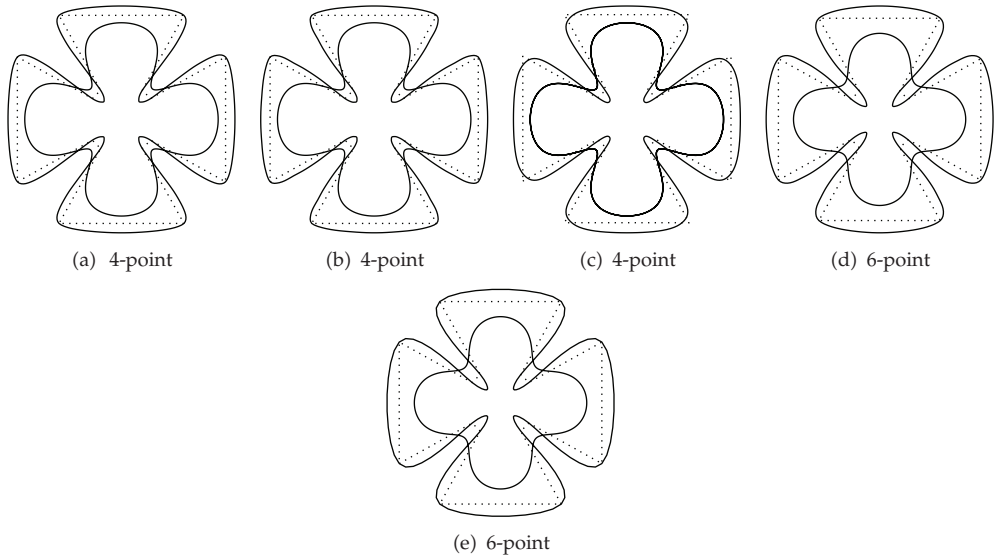


Figure 2: Comparison: bold solid continuous curves are generated by proposed 4-point and 6-point ternary approximating schemes (a) Ko et al. [7] and proposed 4-point ternary schemes (b) [5, 9] and proposed 4-point ternary scheme (c) Hassan and Dodgson [1] and proposed 4-point ternary schemes (d) [5, 9] and proposed 6-point ternary scheme (e) Khan and mustafa [6] and proposed 6-point ternary schemes.

Table 2: Comparison of 4-point ternary schemes.

Scheme	Type	Support	Order	C^n
Ternary 4-point [3]	Interpolating	5	3	2
Ternary 4-point [4]	Interpolating	5	3	2
Ternary 4-point [4]	Interpolating	5	4	1
Ternary 4-point [5, 9]	Interpolating	5	3	1
Ternary 4-point [7]	Approximating	5.5	4	2
Ternary 4-point [14]	Approximating	5	4	3
Ternary 4-point [15]	Approximating	5.5	4	2
Ternary 4-point proposed	Approximating	5.5	5	4

Comparison of various 4- and 6-point ternary schemes are given in Tables 2 and 3. Figure 2 shows visual comparison of 4- and 6-point ternary interpolating and approximating schemes with the proposed ternary approximating schemes.

4.1. Curvature and Torsion

The quality of subdivided curves can be assessed quantitatively by measuring curvature and torsion, as functions of cumulative chord length. Curvature is the amount by which a geometric object deviates from being flat, or straight in the case of a line, while in the elementary geometry of space curves, torsion measures the rate at which a twisted curve tends to depart from its osculating plane. When the torsion is zero, osculating plane never changes, and we have a plane curve. We used the method described in [16] to determine the curvature and torsion. The appearance of curvature and torsion obtained by our schemes

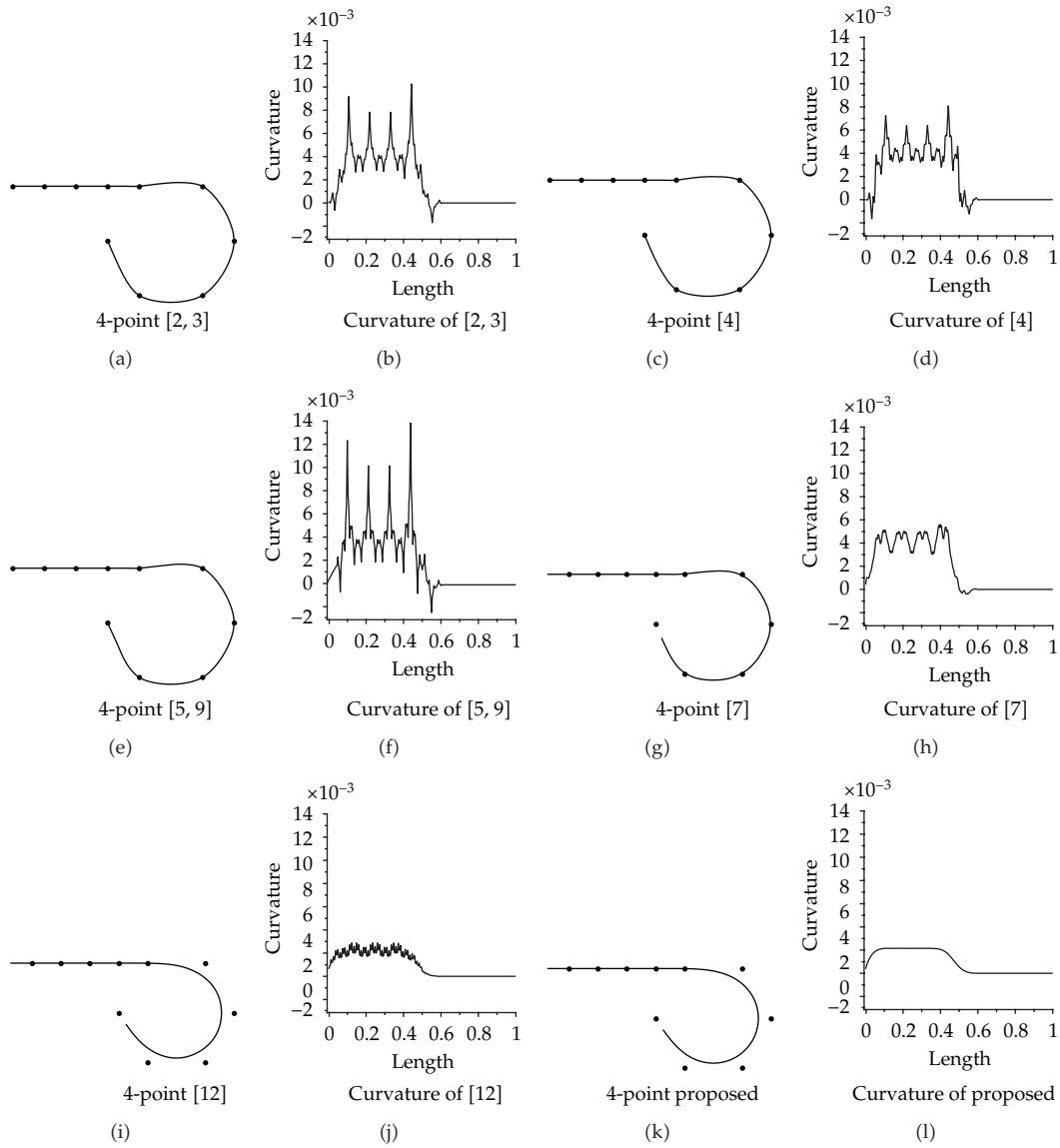


Figure 3: Comparison of existing and proposed 4-point schemes sampled from control polygon by using the 5th iteration. The results are shown on the left together with their corresponding curvature on the right.

Table 3: Comparison of 6-point ternary schemes.

Scheme	Type	Support	Order	C^n
Ternary 6-point [5, 9]	Interpolating	8	5	2
Ternary 6-point [6]	Interpolating	8	4	2
Ternary 6-point proposed	Approximating	8.5	8	7

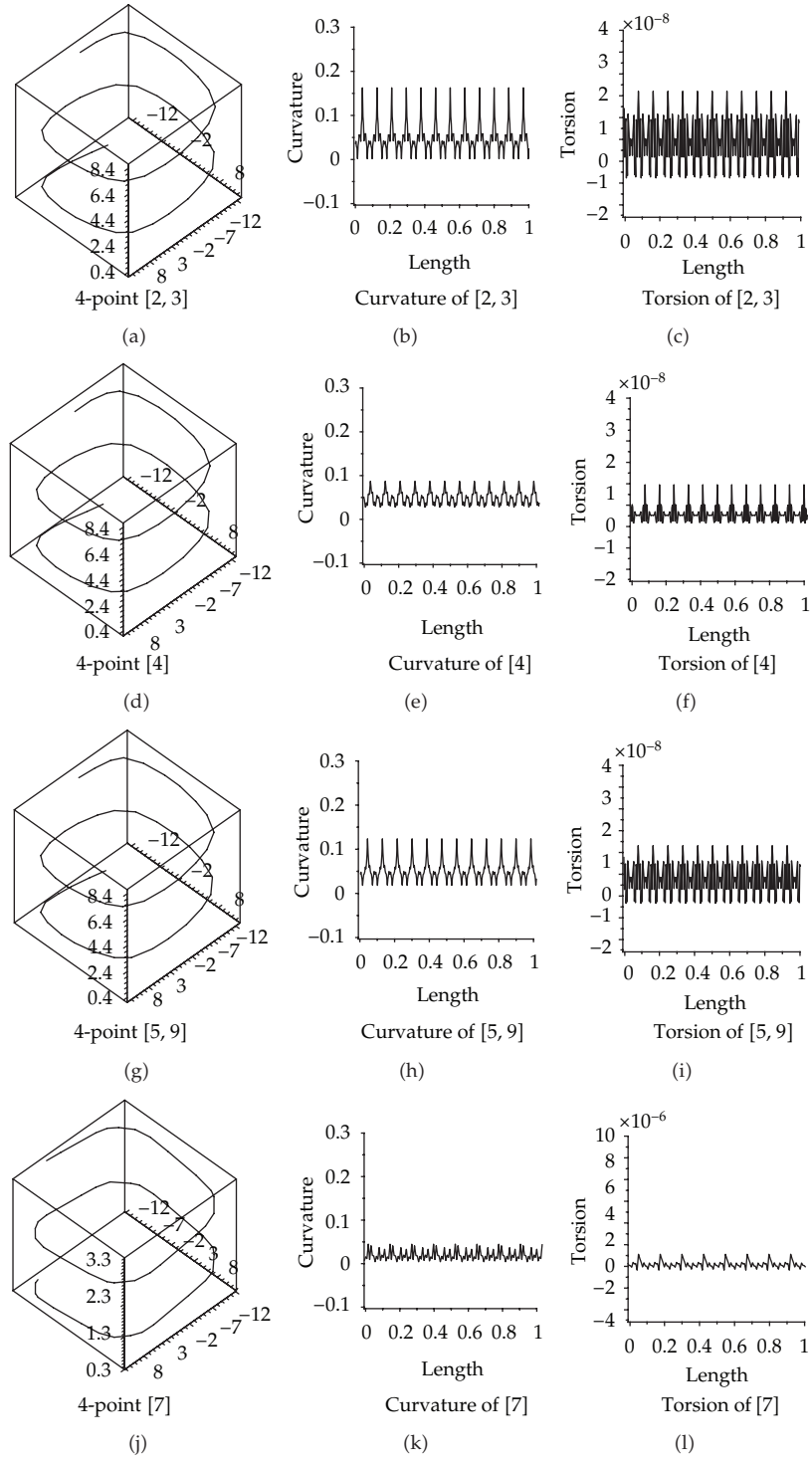
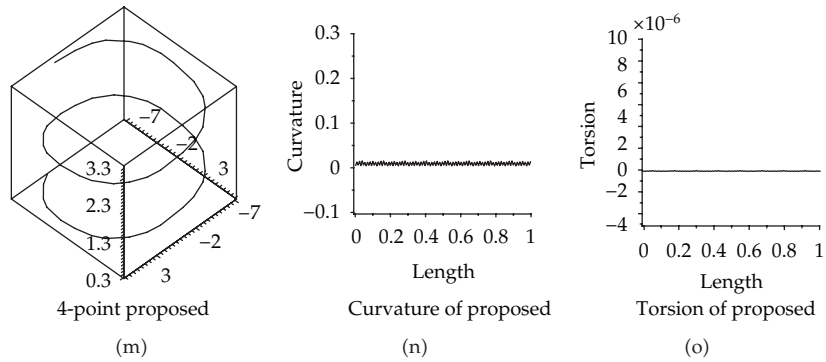


Figure 4: Continued.



4-point proposed (m) Curvature of proposed (n) Torsion of proposed (o)

Figure 4: Left: control data sampled from a space curve and four iteration of 4-point subdivision scheme. The corresponding curvature and torsion are shown in the center and right column, respectively.

is measured and compared with the plot or appearance generated by other schemes. The examples in Figure 3 show the results of curvature and torsion, together with their control polygon. The appearance of curvature of the schemes [2, 3, 5, 9] is quite similar and plot of curvature of the schemes [4, 7, 14] is different. Figure 4 shows detailed view of space curve, curvature and torsion generated by existing schemes [2–5, 7, 9] and proposed schemes. Overall one can observe that the plot of space curve, curvature, and torsion of proposed schemes are more smoothing than other existing schemes.

5. Conclusion

The family of even-point approximating schemes for curve design has been established. The 4- and 6-point ternary schemes introduced by Zheng et al. [13, 14] are special cases of our proposed even-point approximating schemes. Smoothness and approximation order of proposed 4- and 6-point ternary schemes have been discussed. Support of family of even-point ternary schemes has been computed in general. It has been shown that proposed schemes are much better than existing ternary schemes in the sense of smoothness, approximation order, curvature, and torsion. The family of odd-point ternary approximating schemes will be studied in detail in the forthcoming paper.

Acknowledgment

This work is supported by the Indigenous Ph.D Scholarship Scheme of Higher Education Commission (HEC) Pakistan.

References

[1] M. F. Hassan and N. A. Dodgson, "Ternary and three-point univariate subdivision schemes," in *Curve and Surface Fitting: Saint-Malo, 2002*, A. Cohen, J. L. Marrien, and L. L. Schumaker, Eds., pp. 199–208, Nashboro Press, Brentwood, Calif, USA, 2003.

[2] M. F. Hassan, I. P. Ivrissimitzis, N. A. Dodgson, and M. A. Sabin, "An interpolating 4-point C^2 ternary stationary subdivision scheme," *Computer Aided Geometric Design*, vol. 19, no. 1, pp. 1–18, 2002.

[3] C. Beccari, G. Casciola, and L. Romani, "An interpolating 4-point C^2 ternary non-stationary subdivision scheme with tension control," *Computer Aided Geometric Design*, vol. 24, no. 4, pp. 210–219, 2007.

[4] C. Beccari, G. Casciola, and L. Romani, "Shape controlled interpolatory ternary subdivision," *Applied Mathematics and Computation*, vol. 215, no. 3, pp. 916–927, 2009.

[5] G. Deslauriers and S. Dubuc, "Symmetric iterative interpolation processes," *Constructive Approximation*, vol. 5, no. 1, pp. 49–68, 1989.

[6] F. Khan and G. Mustafa, "Ternary six-point interpolating subdivision scheme," *Lobachevskii Journal of Mathematics*, vol. 29, no. 3, pp. 153–163, 2008.

[7] K. P. Ko, B.-G. Lee, and G. J. Yoon, "A ternary 4-point approximating subdivision scheme," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1563–1573, 2007.

[8] G. Mustafa and F. Khan, "A new 4-point C^3 quaternary approximating subdivision scheme," *Abstract and Applied Analysis*, vol. 2009, Article ID 301967, 14 pages, 2009.

[9] J.-A. Lian, "On *a*-ary subdivision for curve design. I. 4-point and 6-point interpolatory schemes," *Applications and Applied Mathematics*, vol. 3, no. 1, pp. 18–29, 2008.

[10] J.-A. Lian, "On *a* subdivision for curve design. II. 3-point and 5-point interpolatory schemes," *Applications and Applied Mathematics*, vol. 3, no. 2, pp. 176–187, 2008.

[11] J.-a. Lian, "On *a*-ary subdivision for curve design. III. $2m$-point and $2m + 1$-point interpolatory schemes," *Applications and Applied Mathematics*, vol. 4, no. 2, pp. 434–444, 2009.

[12] H. Zheng, M. Hu, and G. Peng, "Constructing 2n-1-point ternary interpolatory subdivision schemes by using variation of constants," in *Proceedings of the International Conference on Computational Intelligence and Software Engineering (CISE '09)*.

[13] H. Zheng, M. Hu, and G. Peng, "Ternary even symmetric 2n-point subdivision," in *Proceedings of the International Conference on Computational Intelligence and Software Engineering (CISE '09)*.

[14] H. Zheng, M. Hu, and G. Peng, "P-ary subdivision seneralizing B-splines," in *Proceedings of the 2nd International Conference on Computer and Electrical Engineering (ICCEE '09)*, 2009.

[15] G. Mustafa and N. A. Rehman, "The mask of 2b + 4-point *n*-ary subdivision scheme," *Computing. Archives for Scientific Computing*, vol. 90, no. 1-2, pp. 1–14, 2010.

[16] M. A. Sabin and N. A. Dodgson, "A circle-preserving variant of the four-point subdivision scheme," in *Mathematical Methods for Curve and Surfaces. Troms 2004*, M. Dhlen, K. Mrken, Schumaker, and L, Eds., Modern Methods in Mathematics, pp. 275–286, Nashboro Press, Brentwood, Calif, USA, 2005.