Research Article

Tensor Products of Noncommutative L^p -Spaces

Somlak Utudee

Centre of Excellence in Mathematics, CHE, Si Ayutthaya RD, Bangkok 10400, Thailand

Correspondence should be addressed to Somlak Utudee, somlak.u@cmu.ac.th

Received 27 January 2012; Accepted 1 March 2012

Academic Editor: F. Kittaneh

Copyright © 2012 Somlak Utudee. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the notion of tensor product of noncommutative L^p spaces associated with finite von Neumann algebras and define the notion of tensor product of Haagerup noncommutative L^p spaces associated with σ -finite von Neumann algebras.

1. Introduction and Preliminaries

The main goal of this paper is explanation of the notion of tensor products of noncommutative L^p -spaces associated with von Neumann algebras. The notion of tensor products of noncommutative probability spaces was considered by Xu in [1]. We will generalize that notations to the cases of noncommutative L^p -spaces associated with von Neumann algebras.

In this section, we also give some necessary preliminaries on noncommutative L^p -spaces associated with von Neumann algebras and tensor product of von Neumann algebras.

1.1. Noncommutative L^p -Spaces Associated with Semifinite von Neumann Algebras

We denote by $\mathcal{M}$ an infinite-dimensional von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Let us define a trace on $\mathcal{M}^+$, the set of all positive elements of $\mathcal{M}$.

Definition 1.1. Let $\mathcal{M}$ be a von Neumann algebra.

(i) A trace on $\mathcal{M}^+$ is a function $\tau : \mathcal{M}^+ \rightarrow [0, \infty]$ satisfying the following.

- (a) $\tau(x + \lambda y) = \tau(x) + \lambda \tau(y)$ for any $x, y \in \mathcal{M}^+$ and any $\lambda \in \mathbb{R}^+$.
- (b) $\tau(xx^*) = \tau(x^*x)$ for any $x \in \mathcal{M}^+$ (tracial property).

- (ii) A trace τ is faithful if $\tau(x) = 0$ implies $x = 0$.
- (iii) A trace τ is normal if $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ for any bounded increasing net (x_i) in $\mathcal{M}^+$.
- (iv) A trace τ is semifinite if for any nonzero $x \in \mathcal{M}^+$ there exists a nonzero $y \in \mathcal{M}^+$ such that $y \leq x$ and $\tau(y) < \infty$.
- (v) A trace τ is finite if $\tau(1) < \infty$. In this case, we will often assume that it is normalized.

Recall that a von Neumann algebra $\mathcal{M}$ is called semifinite if any nonzero central projection contains a nonzero finite projection. The following theorem will always be used in our construction and can be found in many references (see, e.g., [2–4]).

Theorem 1.2. *A von Neumann algebra $\mathcal{M}$ is semifinite von Neumann algebra if and only if there exists a faithful normal semifinite trace.*

Proof. Let $\mathcal{M}$ be a von Neumann algebra and τ a faithful normal semifinite trace. For any nonzero central projection $p \in \mathcal{M}$, there exist $x \in \mathcal{M}^+$, $0 \neq x \leq p$ such that $\tau(x) < \infty$. Then, there exists a nonzero projection $e \in \mathcal{M}$ and a positive number ε such that $xe = ex \geq \varepsilon e$. Thus, e is a finite projection. Hence, $\mathcal{M}$ is semifinite.

Conversely, let $\mathcal{M}$ be a semifinite von Neumann algebra. We can assume that $\mathcal{M}$ is a uniform von Neumann algebra, that is, there exists a family $\{e_i\}_{i \in I}$ of equivalent finite mutually orthogonal projections such that $\sum_{i \in I} e_i = 1$. For each e_i , the von Neumann algebra $e_i \mathcal{M} e_i$ is finite and it then possesses a finite normal trace τ_i . Define a mapping by

$$\tau(x) = \sum_{i \in I} \tau_i(v_i^* x v_i), \quad x \in \mathcal{M}^+, \quad (1.1)$$

where $v_i \in \mathcal{M}$ is a partial isometry such that $v_i^* v_i = e_i = v_i v_i^*$. Then, τ is a semifinite normal trace on $\mathcal{M}^+$. Since the set of all semifinite normal traces on $\mathcal{M}^+$, obtained in this manner, is sufficient. Then, $\mathcal{M}$ possesses a faithful normal semifinite trace. $\square$

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ . For $0 < p < \infty$, let

$$\|x\|_p = [\tau(|x|^p)]^{1/p}, \quad \text{where } |x| = (x^* x)^{1/2}. \quad (1.2)$$

The noncommutative L^p -space $L^p(\mathcal{M}, \tau)$ associated with $(\mathcal{M}, \tau)$ is defined as the Banach space completion of $(\mathcal{M}, \|\cdot\|_p)$. We set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the norm $\|x\|_\infty = \|x\|$, the operator norm. Note that the usual (commutative) L^p -space is also in the family of noncommutative L^p -space (see, e.g., [1, 5]).

Elements of the noncommutative L^p -space $L^p(\mathcal{M}, \tau)$ may be identified with unbounded operators.

Definition 1.3. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ .

- (i) A linear operator $x : \text{dom}(x) \rightarrow \mathcal{H}$ is called affiliated with $\mathcal{M}$ if $xu = ux$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$.

- (ii) A closed densely defined operator x , affiliated with $\mathcal{M}$, is called τ -measurable if for every $\varepsilon > 0$ there exists an orthogonal projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subseteq \text{dom}(x)$ and $\tau(1-p) < \varepsilon$.

For $0 < p < \infty$, we have

$$L^p(\mathcal{M}, \tau) \cong \{x \mid x \text{ is } \tau\text{-measurable, } \tau(|x|^p) < \infty\}. \quad (1.3)$$

Note that $L^2(\mathcal{M}, \tau)$ is a Hilbert space with respect to the scalar product $\langle x, y \rangle = \tau(y^*x)$.

If τ is a normal faithful finite trace, then it is normalized, that is, $\tau(1) = 1$. In this case, $(\mathcal{M}, \tau)$ is called a *noncommutative probability space*.

1.2. Noncommutative L^p -Spaces Associated with Arbitrary von Neumann Algebras

In this subsection, we will recall the definitions of cross product (see [2]) and Haagerup noncommutative L^p -spaces. For details of the following results in Haagerup noncommutative L^p -spaces, we refer to [1, 5].

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $\text{Aut}(\mathcal{M})$ the group of all $*$ -automorphism of $\mathcal{M}$, G a locally compact group equipped with its left Haar measure dg and

$$G \ni g \mapsto \pi_g \in \text{Aut}(\mathcal{M}) \quad (1.4)$$

a homomorphism of group, such that for any $x \in \mathcal{M}$, the mapping

$$G \ni g \mapsto \pi_g(x) \in \mathcal{M} \quad (1.5)$$

is continuous for the weak operator topology in $\mathcal{M}$. Let $C_c(G, \mathcal{H})$ be the space of all norm continuous functions defined on G and taking values in $\mathcal{H}$ which have compact supports. We endow it with the inner product:

$$\langle f_1, f_2 \rangle = \int_G \langle f_1(g), f_2(g) \rangle dg, \quad (1.6)$$

and we denote by $L^2(G, \mathcal{H})$ the Hilbert space obtained by completion.

For any $x \in \mathcal{M}$, the operator $\lambda_x \in \mathcal{B}(L^2(G, \mathcal{H}))$ is defined by the relations:

$$(\lambda_x(f))(g) = \pi_g^{-1}(x)(f(g)), \quad f \in C_c(G, \mathcal{H}), \quad g \in G, \quad (1.7)$$

whereas for any $g \in G$ one defines the unitary operator $u_g \in \mathcal{B}(L^2(G, \mathcal{H}))$ by the relations

$$(u_g(f))(g') = f(g^{-1}g'), \quad f \in C_c(G, \mathcal{H}), \quad g' \in G. \quad (1.8)$$

The von Neumann algebra generated in $\mathcal{B}(L^2(G, \mathcal{H}))$ by the operators λ_x , $x \in \mathcal{M}$ and u_g , $g \in G$, is called the *cross-product* of $\mathcal{M}$ by the action π of G and it is denoted by $\mathcal{M} \rtimes_{\pi} G$ or simply by $\mathcal{M} \rtimes G$.

Remark 1.4. If $\mathcal{M}$ is a von Neumann algebra on a separable Hilbert space $\mathcal{H}$ and G is a separable abelian locally compact group acting by $*$ -automorphisms of $\mathcal{M}$, then the group $\widehat{G}$ of the character of G acts by $*$ -automorphisms of $\mathcal{M} \rtimes G$. M. Takesaki has proved that

$$(M \rtimes G) \rtimes \widehat{G} \cong \mathcal{M} \overline{\otimes} \mathcal{B}(L^2(G, \mathcal{H})). \quad (1.9)$$

In particular, if $\mathcal{M}$ is properly infinite, then $(M \rtimes G) \rtimes \widehat{G} \cong \mathcal{M}$.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal semifinite weight φ . Let us recall the noncommutative L^p -space associated with $(\mathcal{M}, \varphi)$ constructed by Haagerup (see, e.g., [1, 5]).

Let $\sigma_t = \sigma_t^\varphi$, $t \in \mathbb{R}$ denote the one parameter modular automorphism group of $\mathbb{R}$ on $\mathcal{M}$ associated with φ . The group $\{\sigma_t^\varphi\}$ is the only group of $*$ -automorphisms of $\mathcal{M}$, with respect to φ which satisfies the KMS-conditions. We consider the cross-product $\mathcal{N} = \mathcal{M} \times_{\sigma} \mathbb{R}$, that is, a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$, generated by the operators π_x , $x \in \mathcal{M}$, and the operators λ_s , $s \in \mathbb{R}$, defined by

$$\pi_x(f(t)) = \sigma_{-t}(x)f(t), \quad \lambda_s(f(t)) = f(t-s) \quad \text{for any } f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}. \quad (1.10)$$

It is well known that cross product $\mathcal{N}$ is semifinite (see [5]). By Theorem 10.29 of [2], there exists a strong operator continuous group $\{u_t\}_{t \in \mathbb{R}}$ of unitary operators in $\mathcal{M}$ such that

$$\widehat{\sigma}_t^\varphi(x) = u_t x u_t^*, \quad t \in \mathbb{R}. \quad (1.11)$$

Let τ be its (unique) faithful normal semifinite trace satisfying

$$\tau \circ \widehat{\sigma}_t = e^{-t} \tau, \quad \forall t \in \mathbb{R}, \quad (1.12)$$

The $*$ -algebra of all τ -measurable operators on $L^2(\mathbb{R}, \mathcal{H})$ affiliated with $\mathcal{N}$ is denoted by $\widetilde{\mathcal{N}}$. For each $0 < p \leq \infty$, we define the *Haagerup noncommutative L^p -spaces* by

$$L^p(\mathcal{M}, \varphi) = \left\{ x \in \widetilde{\mathcal{N}} \mid \widehat{\sigma}_t(x) = e^{-t/p} x, \quad \forall t \in \mathbb{R} \right\}. \quad (1.13)$$

We have

$$L^\infty(\mathcal{M}, \varphi) = \mathcal{M}, \quad L^1(\mathcal{M}, \varphi) = \mathcal{M}_*. \quad (1.14)$$

For $0 < p < \infty$, $x \in L^p(\mathcal{M}, \varphi)$ if and only if $|x^p| \in L^1(\mathcal{M}, \varphi)$, we then define

$$\|x\|_p = \| |x|^p \|_1^{1/p}, \quad x \in L^p(\mathcal{M}, \varphi). \quad (1.15)$$

For $1 \leq p < \infty$, $L^p(\mathcal{M}, \varphi)$ is a Banach space equipped with a norm $\|\cdot\|_p$. For $0 < p < 1$, $L^p(\mathcal{M}, \varphi)$ is a quasi-Banach space equipped with a p -norm $\|\cdot\|_p$.

It is well known that $L^p(\mathcal{M}, \varphi)$ is independent of φ up to isometric isomorphism preserving the order and modular structure of $L^p(\mathcal{M}, \varphi)$ (see [6–8]). Sometimes, we denote $L^p(\mathcal{M}, \varphi)$ simply by $L^p(\mathcal{M})$.

1.3. Tensor Products of von Neumann Algebras

Let $\mathcal{H} \bar{\otimes} \mathcal{K}$ be the Hilbert space tensor product of $\mathcal{H}$ and $\mathcal{K}$. For $x \in \mathcal{M}$ and $y \in \mathcal{N}$, the tensor product $x \bar{\otimes} y$ is the bounded linear operator on $\mathcal{H} \bar{\otimes} \mathcal{K}$ uniquely determined by

$$(x \bar{\otimes} y)(\xi \otimes \eta) = x(\xi) \otimes y(\eta) \quad \forall \xi \in \mathcal{H}, \eta \in \mathcal{K}. \quad (1.16)$$

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subset \mathcal{B}(\mathcal{K})$ be two von Neumann algebras. The algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$,

$$\mathcal{M} \otimes \mathcal{N} = \left\{ \sum_{k=1}^n x_k \otimes y_k \mid x_k \in \mathcal{M}, y_k \in \mathcal{N}, n = 1, 2, \dots \right\}, \quad (1.17)$$

is a $*$ -subalgebra of operators on $\mathcal{H} \bar{\otimes} \mathcal{K}$. The von Neumann algebra generated by $\mathcal{M} \bar{\otimes} \mathcal{N}$ in $\mathcal{B}(\mathcal{H} \bar{\otimes} \mathcal{K})$ is denoted by $\mathcal{M} \bar{\otimes} \mathcal{N}$ and it is called the tensor product of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. Since the map

$$\mathcal{M} \ni x \mapsto x \bar{\otimes} 1 \in \mathcal{M} \bar{\otimes} \mathcal{N} \quad (1.18)$$

is a $*$ -isomorphism, we can view $\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{N}$. Similarly, we can also view $\mathcal{N}$ as a von Neumann subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{N}$. By the Tomita commutation theorem, $\mathcal{M}$ and $\mathcal{N}$ commute and together generate $\mathcal{M} \bar{\otimes} \mathcal{N}$.

Example 1.5. Let $\mathbb{T}$ be the unit circle equipped with the normalized Lebesgue measure dm and $(\mathcal{M}, \tau)$ a finite von Neumann algebra. Let $(L^\infty(\mathbb{T}), \mathrm{dm}) \bar{\otimes} (\mathcal{M}, \tau)$ be consisting of all functions f such that

$$\int \tau(xf(z)) \bar{z}^n \mathrm{dm}(z), \quad \forall x \in L^1(\mathcal{M}, \tau), n \in \mathbb{Z}, n > 0. \quad (1.19)$$

Then, $H^\infty(\mathbb{T}, \mathcal{M})$ is a finite subdiagonal algebra of $(L^\infty(\mathbb{T}), \mathrm{dm}) \bar{\otimes} (\mathcal{M}, \tau)$ (see [5]).

2. Tensor Products of Noncommutative L^p -Spaces Associated with von Neumann Algebras

We first consider the simple case: finite von Neumann algebras.

2.1. Tensor Products of Noncommutative L^p -Spaces Associated with Normal Faithful Finite von Neumann Algebras

Theorem 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be finite von Neumann algebras equipped with normal faithful normalized traces τ_1 and τ_2 , respectively. Then, there exists a normal faithful trace on the tensor product von Neumann algebra $\mathcal{M} \bar{\otimes} \mathcal{N}$ such that*

$$\tau(x \otimes y) = \tau_1(x)\tau_2(y), \quad x \in \mathcal{M}, y \in \mathcal{N}. \quad (2.1)$$

Proof. Since τ_1 and τ_2 are normal faithful normalized traces, we can view $\mathcal{M}$ and $\mathcal{N}$ as von Neumann algebras acting on $\mathcal{H} = L^2(\mathcal{M}, \tau_1)$ and $\mathcal{K} = L^2(\mathcal{N}, \tau_2)$, respectively, by left multiplication. Then, τ_1 and τ_2 are the vector states associated to the identities $1_{\mathcal{M}}$ of $\mathcal{M}$ and $1_{\mathcal{N}}$ of $\mathcal{N}$, respectively. That is,

$$\tau_1(x) = \langle x(1_{\mathcal{M}}), 1_{\mathcal{M}} \rangle, \quad \tau_2(y) = \langle y(1_{\mathcal{N}}), 1_{\mathcal{N}} \rangle, \quad x \in \mathcal{M}, y \in \mathcal{N}. \quad (2.2)$$

Let τ be the vector state associated to $1_{\mathcal{M}} \otimes 1_{\mathcal{N}}$ on $\mathcal{M} \bar{\otimes} \mathcal{N}$. Then, τ is uniquely determined by $\tau(x \otimes y) = \tau_1(x)\tau_2(y)$ for all $x \in \mathcal{M}, y \in \mathcal{N}$. Therefore, τ is tracial and faithful. $\square$

τ is called *the tensor product trace* of τ_1 and τ_2 , and we denote it by $\tau_1 \bar{\otimes} \tau_2$. Then, we can define the noncommutative L^p -spaces $L^p(\mathcal{M} \bar{\otimes} \mathcal{N}, \tau_1 \bar{\otimes} \tau_2)$ and called it the *noncommutative L^p -tensor product* of $(\mathcal{M}, \tau_1)$ and $(\mathcal{N}, \tau_2)$.

Example 2.2. Let us consider two cases (see [1, 5]).

- (1) Let (Ω, P) be a probability space. We can represent $L^\infty(\Omega)$ as a von Neumann algebra on $\mathcal{H} = L^2(\Omega)$ by multiplication and the integral against P is a normal faithful normalized trace on $L^\infty(\Omega)$. Let $(\mathcal{M}, \tau)$ be a noncommutative probability space. Then, $L^p(L^\infty(\Omega) \bar{\otimes} \mathcal{M}, \int \bar{\otimes} \tau)$ is isometric to $L^p(\Omega, L^p(\mathcal{M}))$, the usual L^p -space of p -integrable functions from Ω to $L^p(\mathcal{M})$.
- (2) Let $\mathcal{B}(l^2)$ be equipped with the usual trace Tr and let $(\mathcal{M}, \tau)$ be a noncommutative probability space. Then, the element of $L^p(\mathcal{B}(l^2) \bar{\otimes} \mathcal{M}, \text{Tr} \bar{\otimes} \tau)$, the noncommutative L^p -tensor product of $(\mathcal{B}(l^2), \text{Tr})$ and $(\mathcal{M}, \tau)$ can be identified with an infinite matrix with entries in $L^p(\mathcal{M}, \tau)$.

2.2. Infinite Tensor Products of Noncommutative L^p -Spaces Associated with Finite von Neumann Algebras

For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be a von Neumann algebras. The *infinite algebraic tensor product* $\otimes_{n \geq 1} \mathcal{M}_n$ of $\mathcal{M}_n$ is the set of all finite linear combinations of elementary tensors $\otimes_{n \geq 1} x_n$, where $x_n \in \mathcal{M}_n$ and all but finitely many x_n are 1, that is,

$$\bigotimes_{n \geq 1} \mathcal{M}_n = \left\{ \sum_{k=1}^m \left(\bigotimes_{n \geq 1} x_n^{(k)} \right) \mid x_n^{(k)} \in \mathcal{M}_n \text{ and all but finitely many } x_n \text{ are } 1, m \in \mathbb{N} \right\}. \quad (2.3)$$

First, let us consider infinite tensor products of noncommutative L^p -spaces associated with finite factors.

For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be a finite factor equipped with a unique normal faithful normalized trace τ_n . We have the product state τ on $\otimes_{n \geq 1} \mathcal{M}_n$, defined by

$$\tau\left(\bigotimes_{n \geq 1} x_n\right) = \prod_{n \geq 1} \tau_n(x_n), \quad x_n \in \mathcal{M}_n. \quad (2.4)$$

The *infinite von Neumann tensor product* $\overline{\otimes}_{n \geq 1} \mathcal{M}_n$ is the weak-closure of the image of the representation of $\otimes_{n \geq 1} \mathcal{M}_n$ by the left multiplication on the Hilbert space $L^2(\otimes_{n \geq 1} \mathcal{M}_n)$. It is a finite factor with the trace $\bar{\tau}$ is the extension of τ , which is the unique normalized trace. $\bar{\tau}$ is called the *infinite tensor product trace* of τ_n and denoted by $\overline{\otimes}_{n \geq 1} \tau_n$ (see [7]). Then, we can define the noncommutative L^p -spaces $L^p(\overline{\otimes}_{n \geq 1} \mathcal{M}_n, \overline{\otimes}_{n \geq 1} \tau_n)$ and called it the *infinite noncommutative L^p -tensor product* of $(\mathcal{M}_n, \tau_n)$.

Next, let us consider the infinite tensor products of noncommutative L^p -Spaces associated with normal faithful finite von Neumann algebras.

Theorem 2.3. Let $(\mathcal{M}_m)_{m \in \mathbb{N}}$ be a sequence of finite von Neumann algebras equipped with normal faithful normalized traces τ_m . Let $\mathcal{A} = \cup_{m \geq 1} (\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_m)$. Let $\mathcal{H}$ be the completion of $\mathcal{A}$ with respect to the inner product

$$\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m \rangle = \prod_{k=1}^m \tau_k(y_k^* x_k). \quad (2.5)$$

Let $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be defined by

$$\pi(x)\Lambda(a) = \Lambda(xa), \quad x \in \mathcal{A}, \quad a \in \mathcal{A}, \quad \text{where } \Lambda : \mathcal{A} \longrightarrow \mathcal{H} \text{ is the inclusion.} \quad (2.6)$$

Let $\mathcal{N}$ be the weak*-closure of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})$. Then, there exists a normal state ν on $\mathcal{N}$ such that

$$\nu(x_1 \otimes \cdots \otimes x_m) = \prod_{k=1}^m \tau_k(x_k), \quad x_k \in \mathcal{M}_k, \quad m \in \mathbb{N}. \quad (2.7)$$

Proof. Let $\mathcal{H}_m = L^2(\mathcal{M}_m)$ and consider $\mathcal{M}_m$ as a von Neumann algebra on $\mathcal{H}_m$ by left multiplication. Let

$$\begin{aligned} \mathcal{N}_m &= \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_m, \\ \nu_m &= \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_m. \end{aligned} \quad (2.8)$$

We view $\mathcal{N}_m$ as a von Neumann subalgebra of $\mathcal{N}_{m+1}$ via the inclusion:

$$x_1 \otimes \cdots \otimes x_m \longmapsto x_1 \otimes \cdots \otimes x_m \otimes 1_{\mathcal{M}_{m+1}}. \quad (2.9)$$

Since $\tau_{m+1}(1) = 1$, $\nu_{m+1}|_{\mathcal{N}_m} = \nu_m$. Note that $\mathcal{A}$ is a unital $*$ -algebra and the traces ν_m induce a faithful normal state ν_o on $\mathcal{A}$. Since ν_o is faithful, the representation π is faithful. Therefore, $\mathcal{A}$, and all $\mathcal{N}_m$, can be viewed as subalgebras of $\mathcal{B}(\mathcal{H})$. Let ν the restriction to $\mathcal{N}$ of the vector

state given by $\Lambda(1)$. Then, ν is tracial and faithful. The trace $\nu|_{\mathcal{M}_m} = \tau_m$ and ν is the unique normal state on $\mathcal{N}$ such that

$$\nu(x_1 \bar{\otimes} \cdots \bar{\otimes} x_m) = \prod_{k=1}^m \tau_k(x_k), \quad x_k \in \mathcal{M}_k, \quad m \in \mathbb{N}. \quad (2.10)$$

□

$(\mathcal{N}, \nu)$ is called *the infinite tensor products of noncommutative L^p -spaces of $(\mathcal{M}_m, \tau_m)$* (see [1]).

Example 2.4. Let $M_2(\mathbb{C})$ be the full algebra of 2×2 matrices. Murray and von Neumann proved that the infinite tensor product

$$\overline{(\otimes_{n \geq 1} M_2(\mathbb{C}))}^{\text{WOT}}, \quad (2.11)$$

produced with respect to the unique normalized trace tr_2 on $M_2(\mathbb{C})$, is the unique AFD II_1 -factor (see, e.g., [7]).

2.3. Tensor Products of Noncommutative L^p -Spaces Associated with σ -Finite von Neumann Algebras

In the case of tensor products of σ -finite von Neumann algebras, we will apply the *reduction theorem*. This theorem was proved by Haaagerup in 1979 and can be used to reduce the problems on general noncommutative L^p -spaces to the corresponding ones on those associated with finite von Neumann algebras (see, e.g., [6, 8]).

For each $k \in \{1, 2\}$, let $\mathcal{M}_k$ be a σ -finite von Neumann algebra. Let $L^p(\mathcal{M}_k)$ be the Haagerup noncommutative L^p -spaces. By the reduction theorem, there exist a Banach space $(X_p)_k$ (a quasi Banach space if $p < 1$), a sequence $(\mathcal{R}_{k,m})_{m \in \mathbb{N}}$ of finite von Neumann algebras, each equipped with a faithful normal finite trace $\tau_{k,m}$, and for each $m \in \mathbb{N}$ an isometric embedding $J_{k,m} : L^p(\mathcal{R}_{k,m}, \tau_{k,m}) \rightarrow (X_p)_k$ such that

- (1) $J_{k,m_1}(L^p(\mathcal{R}_{k,m_1}, \tau_{k,m_1})) \subset J_{k,m_2}(L^p(\mathcal{R}_{k,m_2}, \tau_{k,m_2}))$ for all $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leq m_2$;
- (2) $\bigcup_{m \in \mathbb{N}} J_{k,m}(L^p(\mathcal{R}_{k,m}, \tau_{k,m}))$ is dense in $(X_p)_k$;
- (3) $L^p(\mathcal{M}_k)$ is isometric to a subspace $(Y_p)_k$ of $(X_p)_k$;
- (4) $(Y_p)_k$ and all $J_k(L^p(\mathcal{R}_{k,m}, \tau_{k,m}))$, $m \in \mathbb{N}$ are 1-complemented in $(X_p)_k$ for $1 \leq p < \infty$.

Here, $L^p(\mathcal{R}_{k,m}, \tau_{k,m})$ is the tracial noncommutative L^p -space associated with $(\mathcal{R}_{k,m}, \tau_{k,m})$.

Thus, we have a sequence $(\mathcal{R}_{k,m}, \tau_{k,m})$ of finite von Neumann algebras. We then have the noncommutative L^p -tensor product $(\mathcal{R}_m, \tau_m) := (\mathcal{R}_{1,m} \bar{\otimes} \mathcal{R}_{2,m}, \tau_{1,m} \bar{\otimes} \tau_{2,m})$. Applying the construction in Section 2.2, we will be able to construct the infinite tensor products of noncommutative L^p -spaces of $(\mathcal{R}_m, \tau_m)$. Hence, we have the tensor products of noncommutative L^p -spaces of $L^p(\mathcal{M}_1)$ and $L^p(\mathcal{M}_2)$.

With this setting, if $\{\mathcal{M}_k\}_{k \in \mathbb{N}}$ be a sequence of σ -finite von Neumann algebra, we will also be able to construct the infinite tensor product of noncommutative L^p -spaces associated with σ -finite von Neumann algebras.

Let $\mathcal{M}$ be an (arbitrary) von Neumann algebra. Then, $\mathcal{M}$ admits the following direct sum decomposition:

$$\mathcal{M} = \bigotimes_{j \in I} \mathcal{N}_j \bar{\otimes} B(K_j), \quad (2.12)$$

where each $\mathcal{N}_j$ is an σ -finite von Neumann algebra. Using the reduction theorem in general case, the approximation theorem can be extended to the general case as follows.

Let $\mathcal{M}$ be a general von Neumann algebra and $0 < p < \infty$. Let $L^p(\mathcal{M})$ be the Haagerup noncommutative L^p -space associated with $\mathcal{M}$. Then, there exist a Banach space X_p (a quasi Banach space if $p < 1$), a family $(\mathcal{R}_i)_{i \in I}$ of finite von Neumann algebras, each equipped with a normal faithful finite trace τ_i , and, for each $i \in I$, an isometric embedding $J_i : L^p(\mathcal{R}_i, \tau_i) \rightarrow X_p$ such that

- (1) $J_i(L^p(\mathcal{R}_i, \tau_i)) \subset J_j(L^p(\mathcal{R}_j, \tau_j))$ for all $i, j \in I$ such that $i \leq j$;
- (2) $\bigcup_{i \in I} J_i(L^p(\mathcal{R}_i, \tau_i))$ is dense in X_p ;
- (3) $L^p(\mathcal{M})$ is isometric to a subspace Y_p of X_p ;
- (4) Y_p and all $J_i(L^p(\mathcal{R}_i, \tau_i))$, $i \in I$ are 1-complemented in X_p for $1 \leq p < \infty$.

Here, $L^p(\mathcal{R}_i, \tau_i)$ is the tracial noncommutative L^p -space associated with $(\mathcal{R}_i, \tau_i)$.

If we can define the notion of (uncountable) infinite tensor products of noncommutative L^p -spaces associated with finite von Neumann algebras, we should be able to define tensor products of Haagerup noncommutative L^p -spaces.

Acknowledgment

This research is supported by the Centre of Excellence in Mathematics, the commission on Higher Education, Thailand.

References

- [1] Q. Xu, *Operator Spaces and Noncommutative L_p* , Lecture in the Summer School on Banach Spaces and Operator Spaces, Nankai University, China.
- [2] S. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Well, Kent, UK, 1979.
- [3] G. K. Pedersen, "The trace in semi-finite von Neumann algebras," *Mathematica Scandinavica*, vol. 37, no. 1, pp. 142–144, 1975.
- [4] M. Takesaki, *Theory of Operator Algebras—I*, Springer, Berlin, Germany, 2003.
- [5] G. Pisier and Q. Xu, "Non-commutative L^p -spaces," in *Handbook of the Geometry of Banach Spaces*, vol. 2, pp. 1459–1517, North-Holland, Amsterdam, The Netherlands, 2003.
- [6] U. Haagerup, " L_p -spaces associated with an arbitrary von Neumann algebra," in *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977)*, vol. 274 of *CNRS International Colloquium*, pp. 175–184, CNRS, Paris, France, 1979.
- [7] M. Terp, *L_p Spaces Associated with von Neumann Algebras Notes*, Mathematical Institute, Copenhagen University, 1981.
- [8] U. Haagerup, M. Junge, and Q. Xu, "A reduction method for noncommutative L^p -spaces and applications," *Memoirs of the American Mathematical Society*, vol. 331, pp. 691–695, 2000.