On Pointlike Interaction between Three Particles: Two Fermions and Another Particle

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The problem of construction of self-adjoint Hamiltonian for quantum system consisting of three pointlike interacting particles (two fermions with mass 1 plus a particle of another nature with mass \( m > 0 \)) was studied in many works. In most of these works, a family of one-parametric symmetrical operators \( \{ H_\varepsilon, \varepsilon \in \mathbb{R}^1 \} \) is considered as such Hamiltonians. In addition, the question about the self-adjointness of \( H_\varepsilon \) is equivalent to the one concerning the self-adjointness of some auxiliary operators \( \{ T_l, l = 0, 1, \ldots \} \) acting in the space \( L_2(\mathbb{R}^3, r^2 dr) \). In this work, we establish a simple general criterion of self-adjointness for operators \( T_l \) and apply it to the cases \( l = 0 \) and \( l = 1 \).

It turns out that the operator \( T_0 \) is self-adjoint for any \( m \), while the operator \( T_1 \) is self-adjoint for \( m > m_0 \), where the value of \( m_0 \) is given explicitly in the paper.

1. Introduction and Statement of the Problem

This paper is continuation of works [1–4] studying the problem of construction of Hamiltonian for a quantum system which consists of two fermions with mass 1 interacting pointwise with a particle of another nature having mass \( m \).

Originally, the construction of such Hamiltonian begins with introduction of the symmetric operator:

\[
H_0 = -\frac{1}{2} \left( \frac{1}{m} \Delta_y + \Delta_{x_1} + \Delta_{x_2} \right) \tag{1.1}
\]

acting in a Hilbert space \( \mathcal{H} = L_2(\mathbb{R}^3) \otimes L_2^{\text{sym}} (\mathbb{R}^3 \times \mathbb{R}^3) \). Here, \( x_1, x_2 \in \mathbb{R}^3 \) are the positions of fermions, \( y \) is the position of a separate particle, and \( \Delta_y, \Delta_{x_1}, \) and \( \Delta_{x_2} \) are Laplacians with respect to \( y, x_1, \) and \( x_2 \), respectively. The domain of definition of \( H_0, D(H_0) \subset \mathcal{H} \) consists
of smooth rapidly decreasing functions \( \varphi(y, x_1, x_2) \in \mathcal{A} \) on infinity, antisymmetrical with respect to \( x_1, x_2 \) and satisfying the following conditions:

\[
\varphi(y, x_1, x_2) |_{x_1 = y} = 0, \quad i = 1, 2. \tag{1.2}
\]

Usually, some family \( \{ H_\varepsilon, \varepsilon \in \mathbb{R}^1 \} \) of symmetric extensions of the operator \( H_0 \) is proposed as a possible “true” Hamiltonian of the system (the so-called Ter-Martirosian-Skornyakov extensions, see [5]). These extensions were constructed in [1–3]. For some values of mass \( m \), the extensions of Ter-Martirosian-Skornyakov are self-adjoint (for all values of the parameter \( \varepsilon \)); however, for the other values of \( m \) they are only symmetric with nonzero deficiency indexes (equal for all \( \varepsilon \)). It turns out (see [3]) that the self-adjointness of all operators \( \{ H_\varepsilon \} \) is equivalent to the one for some auxiliary symmetric operator \( \mathcal{T} \) acting in the space \( L_2(\mathbb{R}^3) \) (see below). This operator commutes with the operators \( \{ U_g, g \in O_3 \} \) of the representation of the rotation group \( O_3 \) that acts in \( L_2(\mathbb{R}^3) \) by the usual formula:

\[
(U_g f)(k) = f(g^{-1}k), \quad g \in O_3, \quad f \in L_2(\mathbb{R}^3). \tag{1.3}
\]

Let us denote by \( \mathcal{A}_l \subset L_2(\mathbb{R}^3) \) the maximal subspace, where the representation (1.3) is multiplied by the irreducible representation of \( O_3 \) with weight \( l, l = 0, 1, 2, \ldots \) (see [6]). Evidently, the space \( \mathcal{A}_l \) is invariant with respect to the operator \( \mathcal{T} \), and the restriction \( \mathcal{T}_l = \mathcal{T}|_{\mathcal{A}_l} \) of this operator to the space \( \mathcal{A}_l \) is symmetric operator. The operator \( \mathcal{T} \) is self-adjoint if all the operators \( \{ \mathcal{T}_l, l = 0, 1, \ldots \} \) are self-adjoint. In this paper, we find general simple conditions of self-adjointness of \( \mathcal{T}_l \) and the form of the defect subspaces (with small exclusions) when these conditions are broken. Then, we apply these conditions to the cases \( l = 0 \) and \( l = 1 \) and get that the operator \( \mathcal{T}_{i=0} \) is self-adjoint for all values of \( m > 0 \), while the operator \( \mathcal{T}_{i=1} \) is self-adjoint for \( m > m_0 \) and has nonzero deficiency indexes for \( m \leq m_0 \), the constant \( m_0 > 0 \) is indicated below (see (5.4)).

By the way, we note that the value of \( m_0 \) obtained in this paper differs from that one given by mistake in [2].

### 2. A Short Explanation of the Constructions from Papers [1–3]

(1) After Fourier transformation:

\[
\varphi(y, x_1, x_2) \longrightarrow \tilde{\varphi}(q, k_1, k_2)
\]

\[
= \frac{1}{2\pi^{9/2}} \int_{(\mathbb{R})^3} \varphi(y, x_1 x_2) \exp\{-i(q, y) - i(k_1, x_1) - i(k_2, x_2)\} \, dy \, dx_1 \, dx_2 \quad \tag{2.1}
\]

\[
\equiv (\mathcal{F}\varphi)(q, k_1, k_2),
\]

and change of variables:

\[
P = q + k_1 + k_2, \quad p_j = \frac{P}{m + 2} - k_j, \quad j = 1, 2. \tag{2.2}
\]
the operator
\[ \widetilde{H}_0 = \mathcal{F} H_0 \mathcal{F}^{-1}, \]  
\[ (2.3) \]
can be represented as a tensor sum:
\[ \widetilde{H}_0 = \tilde{H}_0^{(1)} + \frac{m}{m+1} \tilde{H}_0^{(2)}, \]  
\[ (2.4) \]
where \( H_0^{(1)} \) is a self-adjoint operator in \( L_2(\mathbb{R}^3) \):
\[ \left( \tilde{H}_0^{(1)} f \right) (P) = \frac{p^2}{m+2} f(P), \quad P \in \mathbb{R}^3, \ f \in L_2(\mathbb{R}^3), \]  
\[ (2.5) \]
and \( \tilde{H}_0^{(2)} \) acts in \( L_2^{\text{asym}} (\mathbb{R}^3 \times \mathbb{R}^3) \) by formula
\[ \left( \tilde{H}_0^{(2)} g \right) (p_1, p_2) = G(p_1, p_2) g(p_1, p_2), \quad g \in L_2^{\text{asym}} (\mathbb{R}^3 \times \mathbb{R}^3), \]  
\[ (2.6) \]
with
\[ G(p_1, p_2) = p_1^2 + p_2^2 + \frac{2}{m+1} (p_1, p_2) > 0. \]  
\[ (2.7) \]
The operator \( \tilde{H}_0^{(2)} \) is symmetric, and its domain is
\[ D \left( \tilde{H}_0^{(2)} \right) = \left\{ g \in L_2^{\text{asym}} (\mathbb{R}^3 \times \mathbb{R}^3) : \int_{\mathbb{R}^3} g(p_1, p_2) dp_j = 0, \ j = 1, 2 \right\}, \]  
\[ (2.8) \]
(2) the deficiency subspace \( \mathcal{R}_{-1} \subset L_2^{\text{asym}} (\mathbb{R}^3 \times \mathbb{R}^3) \) of the operator \( \tilde{H}_0^{(2)} \) consists of the functions of the form:
\[ U_{\hat{p}}(p_1, p_2) = \frac{\hat{p}(p_1) - \hat{p}(p_2)}{G(p_1, p_2) + 1}, \]  
\[ (2.9) \]
where the function \( \hat{p}(p) \) belongs to Hilbert space
\[ \mathcal{L} = \left\{ \hat{p} : \int_{\mathbb{R}^3} \frac{\left| \hat{p}(p) \right|^2}{\sqrt{p^2 + 1}} dp < \infty \right\}, \]  
\[ (2.10) \]
with inner product
\[ \langle \hat{p}_1, \hat{p}_2 \rangle = (U_{\hat{p}_1}, U_{\hat{p}_2})_{L_2(\mathbb{R}^3 \times \mathbb{R}^3)} = (W_{\hat{p}_1}, W_{\hat{p}_2})_{L_2(\mathbb{R}^3)}, \]  
\[ (2.11) \]
Here $W$ is some positive operator acting in $L_2(\mathbb{R}^3)$ (see [3]). The domain of the operator $(\tilde{H}_0^{(2)})^*$, that is, a conjugate to $\tilde{H}_0^{(2)}$, is

$$D\left((\tilde{H}_0^{(2)})^*\right) = \left\{ g \in L_2^{\text{asym}}(\mathbb{R}^3 \times \mathbb{R}^3) : g(p_1,p_2) = f(p_1,p_2) + U_p(p_1,p_2) + \frac{U_p(p_1,p_2)}{G(p_1,p_2) + 1} \right\},$$

(2.12)

where $f \in D(\tilde{H}_0^{(2)})$, $\tilde{g}, \tilde{q} \in \mathcal{L}$. In addition, the operator $(\tilde{H}_0^{(2)})^*$ acts by the formula:

$$\left((\tilde{H}_0^{(2)})^* g\right)(p_1,p_2) = G(p_1,p_2)g(p_1,p_2) - (\tilde{g}(p_1) - \tilde{g}(p_2)),$$

(2.13)

where $\tilde{g}$ is defined by (2.12).

The following asymptotics holds for vectors $g \in D((\tilde{H}_0^{(2)})^*) N \to \infty$:

$$\int_{|p_1| < N} g(p_1,p_2) dp_1 = 4\pi N \tilde{g}(p_2) + b(p_2) + o(1).$$

(2.14)

Here

$$b(p) = -(T\tilde{g})(p) + (Wq)(p),$$

(2.15)

where the operator $W$ is defined in (2.11), and $(T\tilde{g})(p)$ is given by the following expression ($\mu = 2/(m + 1)$)

$$(T\tilde{g})(p) = 2\pi^2 \sqrt{\left(1 - \frac{\mu^2}{4}\right)p^2 + 1} \tilde{g}(p) + \int_{\mathbb{R}^3} \frac{\tilde{g}(t)}{G(t,p) + 1} dt,$$

(2.16)

defined on the set:

$$D(T) = \left\{ \tilde{g} \in L_2\left(\mathbb{R}^3\right) : |p|\tilde{g}(p) \in L_2\left(\mathbb{R}^3\right) \right\}.$$

(2.17)

The above-mentioned Ter-Martirosian-Skornyakov’s extension $\tilde{H}_0^{(2)}$ of the operator $\tilde{H}_0^{(2)}$ is obtained by requiring

$$b(p) = \varepsilon \tilde{g}(p),$$

(2.18)

where $\varepsilon \in \mathbb{R}^3$ is an arbitrary parameter.

**Lemma 2.1.** The operator $T$ defined in the space $L_2(\mathbb{R}^3)$ by (2.16) is symmetric, and the self-adjointness of the operators $H_\varepsilon$ (for all $\varepsilon$) is equivalent to the self-adjointness of the operator $T$ (see [2, 3, 5]).
The operator $T$ can be represented as a sum of two operators:

$$T = \mathcal{T} + T',$$  \hfill (2.19)

where the symmetric operator $\mathcal{T}$ (with the domain $D(\mathcal{T}) = D(T)$) acts as follows:

$$(\mathcal{T}\hat{\varphi})(p) = 2\pi^2 \sqrt{1 - \frac{\mu^2}{4}} |p|\hat{\varphi}(p) + \int_{\mathbb{R}^3} \frac{\hat{\varphi}(t)dt}{G(t,p)}$$  \hfill (2.20)

and $T'$ is a bounded self-adjoint operator. Since the deficiency indexes of $T$ coincide with the ones of $\mathcal{T}$ (see [7]), we shall study the conditions of self-adjointness for the operator $\mathcal{T}$;

(3) as we said, the space $\mathcal{E}_l \subset L_2(\mathbb{R}^3)$ is invariant with respect to $\mathcal{T}$; it has the form:

$$\mathcal{E}_l = L_2(\mathbb{R}_+, r^2dr) \otimes L_2^1(S),$$  \hfill (2.21)

where $L_2^1(S) \subset L_2(S)$ is the space of spherical functions of weight $l$ (see [6]) on the unit sphere $S \subset \mathbb{R}^3$. In addition, the operator $\mathcal{T}_l = \mathcal{T}|_{\mathcal{E}_l}$ has the form

$$\mathcal{T}_l = M_l \otimes E_l,$$  \hfill (2.22)

where $E_l$ is the unit operator in $L_2^1(s)$, and $M_l$ acts in $L_2(\mathbb{R}_+, r^2dr)$ by the formula:

$$(M_l f)(r) = 2\pi^2 \sqrt{1 - \frac{\mu^2}{4}} rf(r) + 2\pi \int_{-1}^1 dx P_l(x) \int_0^\infty \frac{(r')^2 f(r')dr'}{r^2 + (r')^2 + \mu rr'x},$$  \hfill (2.23)

on the domain

$$D(M_l) \equiv V = \left\{ u \in L_2(\mathbb{R}_+, r^2dr) : ru(r) \in L_2(\mathbb{R}_+, r^2dr) \right\}. $$  \hfill (2.24)

Here $P_l(x), l = 0, 1, 2, \ldots, x \in [-1,1]$, are orthogonal polynomials (Legendre polynomials) satisfying $P_l(1) = 1$:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \quad x \in (-1,1).$$  \hfill (2.25)

The operators $\{M_l, l = 0, 1, \ldots\}$ are symmetric in $L_2(\mathbb{R}_+, r^2dr)$, and the self-adjointness of $M_l$ is equivalent to the self-adjointness of $\mathcal{T}_l$. Later on, we shall study the operators $M_l$ and derive a condition of self-adjointness.
3. Preparatory Constructions

For every function $u \in V \subset L_2(\mathbb{R}^1, r^2 dr)$, we consider the family of functions

$$\mathcal{h}(u) = \{u_\alpha = r^{i\alpha}u, \alpha \in [0, 1], \ u_0 = u\},$$ \hspace{1cm} (3.1)

which we call a chain (with initial element $u = u_0$ and the final one $u_1$). All functions $u_\alpha \in \mathcal{h}(u)$ belong to $L_2(\mathbb{R}^1, r^2 dr)$ and have a uniformly bounded norm:

$$\|u_\alpha\|^2 \leq \|u_0\|^2 + \|u_1\|^2, \quad \alpha \in [0, 1].$$ \hspace{1cm} (3.2)

Consider the unitary map (Mellin’s transformation [8]):

$$\omega : L_2\left(\mathbb{R}^1_+, r^2 dr\right) \rightarrow L_2\left(\mathbb{R}^1_+, ds\right) : f(r) \rightarrow \tilde{f}(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} r^{-i\pi s+1/2} f(r) dr, \quad s \in \mathbb{R}^1$$ \hspace{1cm} (3.3)

and its inverse:

$$\left(\omega^{-1}\tilde{f}\right)(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{is-3/2} \tilde{f}(s) ds.$$ \hspace{1cm} (3.4)

For every set of functions $B \subset L_2(\mathbb{R}^1_+, r^2 dr)$, we denote by $\tilde{B} \subset L_2(\mathbb{R}^1_+, ds)$ the set of their Mellin’s transformations:

$$\tilde{B} = \omega B.$$ \hspace{1cm} (3.5)

For every chain $\mathcal{h}(u)$, we denote by $\Gamma_u$ the family of functions:

$$\Gamma_u = \mathcal{h}(u) = \{\gamma_\alpha(s), \alpha \in [0, 1]\},$$ \hspace{1cm} (3.6)

where $\gamma_\alpha(s) = (\omega u_\alpha)(s), u_\alpha \in \mathcal{h}(u)$. The family $\Gamma_u$ can be represented as a function $\Gamma_u(z)$ of a complex variable $z = s + ia$ in the strip:

$$I = \left\{ z \in \mathbb{C}^1 : \Re z \in [0, 1] \right\},$$

$$\Gamma_u(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} r^{-i\pi s-1/2+ia} u(r) dr = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} r^{-is} u(r) dr.$$ \hspace{1cm} (3.7)

The function $\Gamma_u$ is said to be associated with the chain $\mathcal{h}(u)$, and its values $\{\gamma_\alpha(s)\}$ on the lines $\xi_\alpha = \{z = s + ia, s \in \mathbb{R}^1, 0 \leq \alpha \leq 1\} \subset I$ are called the sections of $\Gamma_u$. 
Proposition 3.1. For every chain $h(u)$, $u \in V$, the associated function $\{\Gamma_u(z), z \in I\}$ is continuous in a closed strip $I$ and analytic inside this strip. Moreover, its sections $\{v_\alpha\}$ satisfy the following inequality:

$$\sup_{0 \leq \alpha \leq 1} \|v_\alpha(\cdot)\|_{L_2(\mathbb{R}^1)} < \infty. \quad (3.8)$$

Inversely, any function $\{\Gamma(z), z \in I\}$ which possesses these properties is associated with some (unique) chain $h(\nu) : \Gamma = \Gamma_\nu$, $\nu \in V$. Let call this chain generated by $\Gamma$. In addition, the functions $\{v_\alpha, \alpha \in [0,1]\}$ of the chain $h(\nu)$ are obtained by the inverse Mellin's transformation from the sections of $\Gamma = \{v_\alpha\}$:

$$v_\alpha = \omega^{-1}v_\alpha. \quad (3.9)$$

The proof of this proposition can be obtained by using the arguments given in the book by Paley and Wiener (see [9], Chapter I), which are related to the Fourier transformation of functions analytical in a strip in a complex plane. It is not difficult to reformulate these arguments in terms of Mellin’s transformation.

Note that the estimate (3.8) for $\{v_\alpha\}$ follows from the estimate (3.2) and the unitary Mellin’s transformation. Denote by $\mathcal{G}$ a linear space of functions $\Gamma$ satisfying conditions of Proposition 3.1. Let us introduce two maps:

$$\Omega : h(u) \longrightarrow \Gamma_u \in \mathcal{G}, \quad \Omega^{-1} : \Gamma_u \longrightarrow h(u). \quad (3.10)$$

Let $N(z)$, $z \in I$, be a bounded, continuous function in the strip $I$, which is analytic inside $I$. This function generates the family $\tilde{k}_N^\alpha$, $\alpha \in [0,1]$ of bounded operators in $L_2(\mathbb{R}^1)$ which act as multiplication on the functions $n_\alpha^N(s) = N(z)|_{z = \omega + i\alpha}, s \in \mathbb{R}^1, 0 \leq \alpha \leq 1$:

$$(\tilde{k}_N^\alpha \varphi)(s) = n_\alpha^N(s)\varphi(s), \quad \varphi \in L_2(\mathbb{R}^1). \quad (3.11)$$

Evidently, for any $\Gamma \in \mathcal{G}$, the function $N(z)\Gamma(z)$ belongs to $\mathcal{G}$. If the chain $h(u)$ is generated by $\Gamma = \Gamma_u$ and the chain $h(\nu)$ is generated by $N(z)\Gamma(z) = \Gamma_\nu(z)$, then

$$v_\alpha = \kappa_\alpha^N u_\alpha, \quad \alpha \in [0,1], \quad u_\alpha \in h(u), \quad (3.12)$$

where

$$\kappa_\alpha^N = \omega^{-1}\tilde{k}_N^\alpha \varphi. \quad (3.13)$$

Denote by $\Pi$ the following self-adjoint operator in $L_2(\mathbb{R}^1, r^2dr)$:

$$(\Pi f)(r) = rf(r), \quad (3.14)$$

with the domain $D(\Pi) = V$. 
It is clear that for any $u \in V$, the power $\Pi^\beta$, $0 \leq \beta \leq 1$ of the operator $\Pi$ is applicable to an element $u_\alpha \in h(u)$ if $\beta + \alpha \leq 1$ and

$$\Pi^\beta u_\alpha = u_{\alpha+\beta}. \quad (3.15)$$

For the function $\Gamma_u$ that is associated with $h(u)$, the action of the operator $\tilde{\Pi}^\beta = \omega \Pi^\beta \omega^{-1}$ on the sections $\{\gamma_\alpha\}$ of $\Gamma_u$ has the form:

$$\tilde{\Pi}^\beta \gamma_\alpha = \gamma_{\alpha+\beta}. \quad (3.16)$$

(again if $\alpha + \beta \leq 1$).

4. The Operator $M_l$

The operator $M_l$ (see (2.23)) can be represented as

$$M_l = \Pi^{1/2} \kappa_l^l \Pi^{1/2}, \quad (4.1)$$

where $\kappa_l^l = \kappa_{1/2}^l$ is an operator in $L_2(\mathbb{R}^1, r^2 dr)$ acting by the formula:

$$\left(\kappa_{1/2}^l f\right)(r) = 2\pi^2 \sqrt{1 - \mu^2/4} f(r) + 2\pi \int_{-1}^1 dx \left(\int_0^\infty \frac{(r')^2 f(r') dr'}{(rr')^{1/2} (r^2 + (r')^2 + \mu xx')}. \quad (4.2)

Lemma 4.1. Operator $\kappa_{1/2}^l$ is bounded and self-adjoint in $L_2(\mathbb{R}^1, r^2 dr)$.

Proof. Pass to the operator:

$$\tilde{\kappa}_{1/2}^l = \omega \kappa_{1/2}^l \omega^{-1}, \quad (4.3)$$

acting in $L_2(\mathbb{R}^1)$. It follows from calculations in [2, 3] that $\tilde{\kappa}_{1/2}^l$ is the operator of multiplication on the function:

$$n_{1/2}^l(s) = 2\pi^2 \left(\sqrt{1 - \mu^2/4} + \lambda_{1/2}^l(s)\right), \quad (4.4)$$

where

$$\lambda_{1/2}^l(s) = \begin{cases} \int_0^1 P_l(x) \frac{\text{ch}(sv(x)) dx}{\text{ch}(s\pi/2) \cos(v(x))} & \text{for even } l, \\ - \int_0^1 P_l(x) \frac{\text{sh}(sv(x)) dx}{\text{sh}(s\pi/2) \cos(v(x))} & \text{for odd } l, \end{cases} \quad (4.5)$$
and $v(x) = \arcsin x / 2$, $0 \leq x \leq 1$. As we see the function $n_{1/2}^l(s), s \in \mathbb{R}$, is bounded and real. The lemma is proved.

We see from (4.4) and (4.5) that the functions $n_{1/2}^l(s)$ and $\lambda_{1/2}^l$ are continued up to bounded, analytical functions $N^i(z)$ and $\Lambda'(z)$ correspondingly, defined in the strip $\tilde{I} = \{z \in \mathbb{C}^1 : -1/2 \leq \Im z \leq 1/2\}$. Let us define the functions $\tilde{N}^i(z) = \overline{N^i}(z - i/2)$ which we shall consider in the strip $I = \{z \in \mathbb{C} : 0 \leq \Im z \leq 1\}$. The operator $\tilde{\kappa}_{1/2}^l$ coincides with the operator $\tilde{\kappa}_{1/2}^{4.9}$ from the family $\{\tilde{\kappa}_{1/2}^{4.9}\}$ generated by the function $\tilde{N}^i$ (see (3.11)). Any other operator of this family acts as multiplication on the function:

$$\tilde{n}_a^l(s) = \tilde{N}^i(z) \bigg|_{z = s + i\sigma}.$$  (4.6)

Denote by $\kappa_a^l$ the operators

$$\kappa_a^l = \omega^{-1} \kappa_{a}^l \omega,$$  (4.7)

acting in $L_2(\mathbb{R}, r^2 dr)$.

Note that

$$\left(\kappa_a^l\right)^* = \kappa_{-a}^l,$$  (4.8)

It is convenient to represent the operator $M_l$ in form of three sequential maps

$$M_l : u_0 \in \mathfrak{h}(u_0) \longrightarrow \Pi^{1/2} u_0 = u_{1/2} \longrightarrow \kappa_{1/2}^l u_{1/2} = v_{1/2} \longrightarrow \Pi^{1/2} v_{1/2} = v_1 \in \mathfrak{h}(v),$$  (4.9)

where $v = v_0, v_{1/2}, v_1$ are elements of the chain $\mathfrak{h}(v)$ generated by the function $\Gamma_v = \tilde{N}^i \Gamma_u \in \mathcal{G}$. Note that the chain (4.9) can be rewritten in the following way:

$$u_0 \in \mathfrak{h}(u_0) \xrightarrow{\Omega} \Gamma_{u_0} \longrightarrow \Gamma_v = \tilde{N}^i \Gamma_{u_0} \xrightarrow{\Omega^{-1}} \mathfrak{h}(v) \longrightarrow v_1 \in \mathfrak{h}(v).$$  (4.10)

From (4.1) and self-adjointness of $\kappa_{1/2}^l$ it follows that the operator $M_l$ with the domain $D(M_l) = V$ is symmetric. For any $a \in [0, 1]$, a representation of $M_l$ similar to (4.1) is valid:

$$M_l = \Pi^{1-a} \kappa_a^l \Pi^a$$  (4.11)

as well as decomposition like (4.9).

Let us now describe the domain $D(M_l^*) \supseteq V$ of the operator $M_l^*$ conjugated to $M_l$. Let $g \in D(M_l^*)$ be a function from $D(M_l^*)$ and $h = M_l^* g \in L_2(\mathbb{R}, r^2 dr)$. Then for every $u \in V = D(M_l)$, we can write

$$(M_l u, g) = \left(\kappa_1^l \Pi u, g\right) = \left(\Pi u, \left(\kappa_1^l\right)^* g\right) = \left(u, \Pi \kappa_0^l g\right) = (u, h).$$  (4.12)
Here we use the representation (4.11) for $\alpha = 1$ and the equality (4.8). Denote $f(r) = h(r) - (\Pi k_0)(r)$ and apply the following evident assertion.

**Lemma 4.2.** Let a measurable function $f(r)$ satisfies condition

$$\int_0^\infty f(r)u(r)r^2dr = 0,$$

(4.13)

for any $u \in V$. Then $f = 0$.

From this and (4.12), it follows that

$$\Pi k_0^i g = h.$$  

(4.14)

Hence

$$w_0 \equiv k_0^i g \in V,$$

(4.15)

and $h = w_1 \in h(w_0)$ is the final element of the chain $h(w_0)$. Thus the domain $D(M_l^*)$ of the operator $M_l^*$ is

$$D(M_l^*) = \left\{ g \in L_2(\mathbb{R}^1, r^2dr) : k_0^i g \in V \right\}.$$  

(4.16)

In the case when the operator $k_0^i$ has the inverse one, $(k_0^i)^{-1}$, which is equivalent to the condition:

$$\tilde{n}_0^i(s) \neq 0, \quad \text{for any } s \in \mathbb{R}^1,$$

(4.17)

the following equality is true:

$$D(M_l^*) = (k_0^i)^{-1} V.$$  

(4.18)

Let $\tilde{M}_l^* = \omega M_l^* \omega^{-1}$ be an operator in $L_2(\mathbb{R}^1)$ with domain $D(\tilde{M}_l^*) = \omega D(M_l^*)$. Then for $\tilde{g} \in D(\tilde{M}_l^*)$, the following representation holds true:

$$\tilde{g}(s) = \left(\tilde{n}_0^i(s)\right)^{-1} \tilde{w}_0(s) = \left(\tilde{N}^i(z)\right)^{-1} \Gamma_{w_0}^i(z) \bigg|_{z=s},$$  

(4.19)

if condition (4.17) is fulfilled. Here $\tilde{w}_0(s) = (\omega w_0)(s)$ where $w_0$ is defined in (4.15).

**Remarks.** (1) Note that the function $\tilde{N}^i(z)$ is invariant with respect to reflection of the complex plane around the point $z = i/2$:

$$z \mapsto z^* = -z + i.$$  

(4.20)
Under this reflection, the strip $I$ is mapped onto itself; hence, for every zero $\bar{z} \in I(\bar{z} \neq i/2)$ of the function $\tilde{N}^l$, there exists another zero, $\bar{z}^* \in I$, of $\tilde{N}^l$ with the same multiplicity. The multiplicity of $\bar{z} = i/2 = \bar{z}^*$ is even;

(2) Since $\tilde{N}^l(z) \to 2\pi^2\sqrt{1 - \mu^2}/4 > 0$ as $z \to \infty$ inside $I$, the function $\tilde{N}^l(z)$ has finite number of zeros inside $I$.

We can now formulate the main criterion of self-adjointness of the operator $M_l$.

**Theorem 4.3.** The operator $M_l$ is self-adjoint if and only if the function $\tilde{N}^l(z)$ has no zeros in the closed strip $I$.

**Proof.** (1) Assume $\tilde{N}^l(z) \neq 0$ in the strip $I$. Then $(\tilde{N}^l)^{-1}(z)$ is bounded and continuous on $I$ and analytical inside $I$. Let $\tilde{g} \in \tilde{D}(M^*_l)$. Since $\hat{n}^l(s) \neq 0$ for $s \in \mathbb{R}^1$, the representation (4.19) holds true. Since

$$\left(\tilde{N}^l(z)\right)^{-1} \Gamma_{w_0}(z) = \Gamma_v \in \mathcal{G}, \quad v \in V,$$

the element $g = \omega^{-1}\tilde{g} \in D(M^*_l)$ coincides with $v \in V$, that is, $D(M^*_l) = V = D(M_l)$; it means the self-adjointness of $M_l$;

(2) assume now the function $N^l(z)$ has zeros $z_1, \ldots, z_k \in I$. Consider first the case when all zeros are lying inside $I$ and their multiplicities are equal to $p_1, \ldots, p_k$, respectively. Again, let $\tilde{g} \in \tilde{D}(M^*_l)$. Since $\hat{n}^l(s) \neq 0$, the representation (4.19) holds true. The function $(\tilde{N}^l(z))^{-1}\Gamma_{w_0}(z)$ is meromorphic in $I$ with poles $z_1, \ldots, z_k$ having the order $p_1, \ldots, p_k$ respectively. For this function the usual canonical representation [10] is true:

$$\left(\tilde{N}^l(z)\right)^{-1} \Gamma_{w_0}(z) = L^{w_0}(z) + \sum_{n=1}^{k} \sum_{m=1}^{p_n} b_m^{(n)}(w_0) \frac{z_m}{(z - z_n)^m},$$

where $L^{w_0}(z)$ is bounded, continuous function on $I$, and analytical inside $I$, and the coefficients $b_m^{(n)} = b_m^{(n)}(w_0)$ depend on $w_0$.

**Lemma 4.4.** The function $L^{w_0}(z)$ in (4.22) belongs to the space $\mathcal{G}$.

The proof of this lemma is given in The appendix.

From (4.19) and (4.22), for $g = \omega^{-1}\tilde{g} \in D(M^*_l)$, we have

$$g(r) = v(r) + \sum_{m,n} b_m^{(n)}(w_0) \left(\omega^{-1}\left(\frac{1}{\cdot - z_n}\right)^m\right)(r),$$

where the function $v \in V$ is defined from relation

$$L^{w_0}(z) = \Gamma_v(z) \in \mathcal{G},$$

$$d_{m,n}(r) := \omega^{-1}\left(\frac{1}{\cdot - z_n}\right)^m(r) = A_m^{(n)} r^{-3/2+is} (\ln r)^{-1} \chi(r),$$

for $s \in \mathbb{R}$.
where \( A_m^{(n)} \) is an absolute constant, \( z_n = s_n + it_m, 0 < t_n < 1 \) and

\[
\chi(r) = \begin{cases} 
1, & r > 1, \\
0, & r \leq 1.
\end{cases}
\] (4.25)

Since linearly independent functions \( d_{m,n} \in D(M_l^*) \) do not belong to \( V \), due to (4.23), they form the basis in the defect subspace \( \mathcal{V} \) of the operator \( M_l \) (see [7]). Since the dimension of the subspace \( \mathcal{V} \) is equal to \( \sum_{i}^{k} p_n \) and the operator \( M_l \) is real, its deficiency indexes \( n_{\pm} \) are equal and have the form:

\[
n_+ = n_- = \frac{1}{2} \sum_{i}^{k} p_n.
\] (4.26)

(It follows from Remarks that the sum \( \sum_{i}^{k} p_n \) is even). Consider now the case when one of the zeros of \( N_l(z) \), say, \( z_0 = s_0 + i \), lies on the boundary of \( I \) and has multiplicity \( p \) (in addition, there is a zero \( z_0^* = s_0 + i \)). In this case, in a neighborhood of \( z_0 \), the function \( \tilde{N}_l(z) \) has the form:

\[
\tilde{N}_l(z) = (z - z_0)^p Q(z),
\] (4.27)

where \( Q(z) \) is analytic in this neighborhood. Consider the function,

\[
G(z) = \frac{1}{(-i(z - z_0))^{1/3}} \frac{1}{(z + 2i)^{1/3}},
\] (4.28)

whereby \( (-i\omega)^{1/3} \) for \( \Im \omega > 0 \), we mean the branch of a many-valued function \( (-i\omega)^{1/3} \) that takes positive values on the positive part of the imaginary axis. Evidently, the function \( G(z) \) is analytic in the strip \( I \) and satisfies condition (3.8). However, this function is discontinuous at \( z_0 \) and does not belong to \( G \). In addition, the function \( \tilde{N}_l(z) G(z) \) now belongs to \( G \) as follows from (4.27) and (4.28). Thus

\[
\tilde{g}(s) = G(z) \big|_{z = s} \tilde{V} \tilde{V} = \omega \tilde{V},
\] (4.29)

but

\[
\tilde{n}_l(s) \tilde{g}(s) = \tilde{N}_l(z) G(z) \big|_{z = s} \in \tilde{V}.
\] (4.30)

Consequently, \( g = \omega \tilde{g} \tilde{V} \) but \( \kappa_{0} g \notin V \), that is, \( g \in D(M_l^*) \). Thus \( D(M_l) \neq V \), and the operator \( M_l \) has nonzero deficiency indexes. Theorem 4.3 is proved.

\[ \square \]

5. The Operators \( M_l \) in the Cases \( l = 0 \) and \( l = 1 \)

Here, we apply Theorem 4.3 to the cases \( l = 0 \) and \( l = 1 \).
Theorem 5.1. (1) For \( l = 0 \), the operator \( M_{l=0} \) is self-adjoint for any \( m > 0 \); (2) the operator \( M_{l=1} \) is self-adjoint for \( m > m_0 \) and has nonzero deficiency indexes for \( m \leq m_0 \). In addition, for \( m < m_0 \) these indexes are equal to \((1,1)\). The constant \( m_0 \) is a unique zero of (5.4).

Proof. We need the following properties of the functions \( \tilde{\lambda}^{l=0}(z) \) and \( \tilde{\lambda}^{l=1}(z), z \in I \).

Lemma 5.2. (1) For any \( l = 0, 1, 2, \ldots \) the function \( \tilde{\lambda}^l(z) \) is invariant with respect to reflection (4.20); (2) The point \( z = i/2 \in I \) is a nondegenerate critical point for both functions \( \tilde{\lambda}^{l=0} \) and \( \tilde{\lambda}^{l=1} \); (3) These functions take real values on the line:

\[
\tilde{\xi}_{1/2} = \left\{ z = s + \frac{i}{2}, s \in \mathbb{R} \right\},
\]

and on the segment:

\[
\tilde{\tau} = \{ z = it, 0 \leq t \leq 1 \}.
\]

Outside the set \( B = \tilde{\xi}_{1/2} \cup \tilde{\tau} \), both functions take nonreal values; (4) the real values of \( \tilde{\lambda}^l, l = 0, 1 \), are between 0 and \( \tilde{\lambda}^l(0) = \tilde{\lambda}^l(i) \). Every value of \( \tilde{\lambda}^l|_{B} \)—except \( \tilde{\lambda}^l(i/2) \)—is taken exactly at two points; (5) the extreme values of \( \tilde{\lambda}^l, l = 0, 1 \), \( \tilde{\lambda}^l(0) = \tilde{\lambda}^l(i) \) are given by

\[
\tilde{\lambda}^{l=0}(0) = 8\sqrt{2}\pi^2 \mu^{-1} \sin \left( \frac{1}{2} \arcsin \frac{\mu}{2} \right) > 0,
\]

\[
\tilde{\lambda}^{l=1}(0) = -\frac{32}{3}\sqrt{2}\pi^2 \mu^{-2} \sin^3 \left( \frac{1}{2} \arcsin \frac{\mu}{2} \right) \equiv -q(\mu) < 0,
\]

(6) the function \( q(\mu) \) increases monotonically on the interval \( 0 < \mu < 2 \). The proof of this lemma is given in The appendix.

Corollary 5.3. (1) The zeros of \( \tilde{\eta}^l(z), l = 0, 1 \) can only lie in the set \( B \); (2) \( \tilde{\eta}^{l=0}(z)|_{B} > 0 \) for any value of \( \mu \), and therefore the operator \( M_{l=0} \) is self-adjoint for all \( m \in (0,2) \); (3) The function \( \tilde{\eta}^{l=1}(z)|_{B} \) is positive if \( 2\pi^2\sqrt{1 - \mu^2/4} > q(\mu) \) and vanishes at some point \( z \in B \) (and also at \( z^* \in B \)) if \( 2\pi^2\sqrt{1 - \mu^2/4} \leq q(\mu) \).

In Figure 1, the curves corresponding to the functions \( 2\pi^2\sqrt{1 - \mu^2/4} \) and \( q(\mu) \) are depicted. We see that they intersect at a unique point with abscissa \( \mu_0 \in (0,2) \) which satisfies the following equation:

\[
2\pi^2\sqrt{1 - \frac{\mu_0^2}{4}} = q(\mu_0).
\]

Thus, for \( m > m_0 = 2/\mu_0 - 1 \) the operator \( M_{l=1} \) is self-adjoint, and for \( m < m_0 \) it has deficiency indexes \((1,1)\). For \( m = m_0 \), the operator \( M_{l=1} \) is not self-adjoint as well. Theorem 5.1 is proved.
Appendix

Proof of Lemma 4.4. The function $(\tilde{N}^i(z))^{-1}$, $z \in I$ admits the canonical representation (see [10])

$$
(\tilde{N}^i(z))^{-1} = Q_i(z) + \sum_{n=1}^{k} \sum_{m=1}^{p_n} \frac{a_m^{(n)}}{(z - \overline{z}_n)^m},
$$

(A.1)

where $\overline{z}_1, \ldots, \overline{z}_k \in I$ are zeros of $\tilde{N}^i(z)$ (with multiplicities $p_1, \ldots, p_k$), $a_m^{(n)}$ are constants, $a_m^{(n)} \neq 0$, and $Q_i(z)$ is a bounded, continuous analytic function in $I$. From this, it follows that for any $v \in V$, $Q_i(z)\Gamma_v(z) \in \mathcal{Q}$. Consider some term of the sum (A.1) and write

$$
\frac{a_m^{(n)}}{(z - \overline{z}_n)^m} \Gamma_v(z) = \left( P_{m,v}^{(n)}(z) + \sum_{d=1}^{m} \frac{c_{m-d}^{(n)}}{(z - \overline{z}_n)^d} \right) a_m^{(n)},
$$

(A.2)

where

$$
P_{m,v}^{(n)}(z) = \frac{1}{(z - \overline{z}_n)^m} \left( \Gamma_v(z) - \sum_{d=1}^{m} c_{m-d}^{(n)}(z - \overline{z}_n)^{-m-d} \right),
$$

$$
c_t^{(n)} = c_t^{(n)}(\overline{z}_n) = \frac{1}{t!} \Gamma_v^{(t)}(\overline{z}_n), \quad t = 0, 1, \ldots
$$

(A.3)

It is clear that $P_{m,v}^{(n)}(z)$ is bounded, continuous analytic function in $I$. We are going to show that this function belongs to $\mathcal{Q}$. Let $O \in I$ be a small neighborhood of $\overline{z}_n$ and $\chi_O(z)$ the characteristic function of $O$. Obviously, the bounded function $\chi_O P_{m,v}^{(n)}$ satisfies condition (3.8). Every term of the sum

$$
(1 - \chi_O) P_{m,v}^{(n)}(z) = \frac{\Gamma_v(z)}{(z - \overline{z}_n)^m} (1 - \chi_O) - \sum_{d=1}^{m} \frac{c_{m-d}^{(n)}(\overline{z}_n)}{(z - \overline{z}_n)^d} (1 - \chi_O)
$$

(A.4)

satisfies this condition as well.
Thus for fixed Σ_n and ν ∈ V,

\[ \sum_{m=1}^{p_{\nu}} a_{m}^{(n)} \Gamma_{\nu}(z) = K_{\nu}^{(n)}(z) + \sum_{d=1}^{p_{\nu}} b_{d}^{(n)}(\nu) \]

(A.5)

where

\[ K_{\nu}^{(n)}(z) = \sum_{m=1}^{p_{\nu}} a_{m}^{(n)} \Gamma_{m,\nu}(z), \]

(A.6)

\[ b_{d}^{(n)}(\nu) = \sum_{m=1}^{p_{\nu}} a_{m}^{(n)} c_{m-d}^{(n)}(\nu), \quad d = 1, \ldots, p_{\nu}. \]

(A.7)

Thus, we get the representation (4.22) where

\[ L^{(w_0)}(z) = Q_{l}(z) \Gamma_{w_0}(z) + \sum_{n=1}^{k} K_{w_0}^{(n)}(z) \in G, \]  

(A.8)

and the coefficients \( b_{d}^{(n)}(w_0) \) are given by formula (A.7). Lemma 4.4 is proved.

Proof of Lemma 5.2. (1) It is more convenient to consider the functions \( N^{l}(z) \) and \( \Lambda^{l}(z) \) in the strip \( I = \{ z : |s| < 1/2 \} \) instead of the functions \( \tilde{N}^{l}(z) \) and \( \tilde{\Lambda}^{l}(z) \) in the strip \( \tilde{I} \). Similarly, instead of the reflection \( z \rightarrow z^* \) we consider the reflection \( z \rightarrow -z \) around the point \( z_0 = 0 \). It is clear that the functions \( \Lambda^{l}(z), l = 0, 1, 2, \ldots \) are invariant with respect to the change \( z \rightarrow -z \), and it means the invariance of \( \tilde{\Lambda}^{l} \) with respect to reflection (4.20);

(2) it follows from (4.5) that \( z = 0 \) is a nondegenerated critical point of \( \Lambda^{l=0} \) and \( \Lambda^{l=1} \), if we note that \( 0 < v(x) \leq \pi/2 \). Correspondingly, \( z = i/2 \) is a nondegenerated critical point for \( \Lambda^{l}(z), l = 0, 1 \). The real line \( \xi_0 = \{ z = s; s \in \mathbb{R} \} \) coincides with the saddle-point line at \( z = 0 \) (see [10]) for \( \Lambda^{l=0} \) and \( -\Lambda^{l=1} \). More precisely, these functions take real values on \( \xi_0 \) and decrease monotonically to zero as \( |s| \) increases from zero to infinity. On the contrary, \( \Lambda^{l=0} \) and \( -\Lambda^{l=1} \) increase monotonically along imaginary axis as \( |t| \) increases from zero to \( 1/2 \). The monotonicity of \( \Lambda^{l=0} \) along real axis follows from (4.5), equality \( P_{l}(x) \equiv 1 \), and inequality

\[ \left( \frac{\text{ch}(\nu(x)s)}{\text{ch}((\pi/2)s)} \right)' < \frac{\pi \text{sh}(\pi/2 - v(x)s)}{2 (\text{ch}((\pi/2)s)^2} < 0, \]  

(A.9)

for \( s > 0 \) and a similar inequality for \( s < 0 \). The proof of monotonicity of \( \Lambda^{l=1} \) along real axis, and also monotonicity of both functions along imaginary axis is analogous if we note that \( P_{l=1}(x) \equiv x \) on \( (0, 1) \). Thus the functions \( \Lambda^{l}, l = 0, 1 \), take all values between 0 and \( \Lambda^{l}(i/2) = \Lambda^{l}(-i/2) \) and every value except \( \Lambda^{l}(0) \) which is taken exactly twice;

(3) we will show now that the values of functions \( \Lambda^{l}(z), l = 0, 1 \), on the set \( \tilde{I} \setminus B \) are nonreal. Let us represent this set as a union of four sets, \( \tilde{I}_i, i = 1, 2, 3, 4 \) as shown in Figure 2.

We consider the case \( l = 0 \); the case \( l = 1 \) is similar. Figure 3 shows the disposition of lines of levels for function \( K_{0}(z) = \Re \Lambda^{l=0}(z) \) which pass through the points i and \( -i \) between lines \( \beta \) and \( \beta^* \), \( \beta = \{ z : K_{0}(z) = 0, \Re z > 0 \}, \beta^* = \{ z : K_{0}(z) = 0, \Re z < 0 \}. \)
All these lines have common tangents at points \( i \) and \(-i\), and the line \( \beta \) (resp. \( \beta^* \)) lies above (resp., below) the strip \( \tilde{I} \). The picture represented in Figure 3 is obtained by detailed study of the explicit formula for \( \Lambda^{l=0} \):

\[
\Lambda^{l=0}(z) = \frac{4\pi^2 \operatorname{sh}(z \arcsin(\mu/2))}{\mu \operatorname{ch}(z \cdot \pi/2)},
\]

(A.10)

together with the proof that the lines \( \beta \) and \( \beta^* \) do not intersect the strip \( \tilde{I} \). This proof is given below.

From Figure 3, we see that the set \( \tilde{I}_1 \) lies inside the shaded domain \( U \) that is bounded by the real semiaxis \( \xi_0^+ = \{ z : z = s, s > 0 \} \), the segment \((0,i/2)\) on the imaginary axis and the part of line \( \beta \) which lies in the right half-plane. From (A.10), it is easy to see that the function \( w = \Lambda^{l=0}(z) \) maps the boundary \( \partial U \) of the domain \( U \) into the boundary of the right lower quadrant \( M = \{ w : \Re w > 0, \Im w < 0 \} \) of the plain \( w \). Hence, the domain \( U \) is mapped inside this quadrant, that is, all values of the function \( \Lambda^{l=0} \) in \( U \) are nonreal. It means the absence of real values of \( \Lambda^{l=0} \) in \( \tilde{I}_1 \). For the domains \( \tilde{I}_2, \tilde{I}_3, \) and \( \tilde{I}_4 \), the proof is similar. Let us now prove that \( \beta \) and \( \beta^* \) do not intersect the line \( \xi_{1/2} \). It is sufficient to prove that \( \Re \Lambda^{l=0} > 0 \) on the line \( \xi_{1/2} = \{ z : z = s + i/2, s \in \mathbb{R}^1 \} \) or, which is the same, that

\[
\Re \left. \frac{\operatorname{ch}(zv(x))}{\operatorname{ch}(z\pi/2)} \right|_{z=s+i/2} > 0,
\]

(A.11)
for any $s \in \mathbb{R}^1$ and $x \in (0, 1)$. Write

$$
\frac{\text{ch}[(s + i/2)v(x)]}{\text{ch}[(s + i/2)\pi/2]} = \frac{\text{ch}(sv(x)) \cos(v(x)/2) + i\text{sh}(sv(x)) \sin(v(x)/2)}{\text{ch}(s\pi/2) \cos(\pi/4) + i\text{sh}(s\pi/2) \sin(\pi/4)} = D(s, x). \quad (A.12)
$$

Let $s > 0$. Then the values of numerator and denominator of $D(s, x)$ lie in the right upper quadrant of a complex plain, and hence $-\pi/2 < \arg D(s, x) < \pi/2$, that is, $\Re D(s, x) > 0$. Similarly (A.11) can be proved in the case $s < 0$ and for $\Lambda^{l=0}_{z=z-i/2}$;

(4) let us find the values $\Lambda^l(i/2), l = 0, 1$:

(I) the case $l = 0$:

$$
\Lambda^{l=0}(i/2) = 2\pi^2 \int_0^1 \frac{\cos(v(x)/2)}{\cos v(x) \cos(\pi/4)} dx. \quad (A.13)
$$

After the change $v(x) = \xi$, the integral (A.13) becomes

$$
\frac{4\sqrt{2}\pi^2}{\mu} \int_0^{\arcsin \mu/2} \cos \left(\frac{\xi}{2}\right) d\xi = \frac{8\sqrt{2}}{\mu} \pi^2 \sin \left(\frac{1}{2} \arcsin \frac{\mu}{2}\right); \quad (A.14)
$$

(II) The case $l = 1$:

$$
\Lambda^{l=1}(i/2) = -2\pi^2 \int_0^1 x \frac{\sin(v(x)/2)dx}{\cos v(x) \sin(\pi/4)}. \quad (A.15)
$$

The same change $v(x) = \xi$ reduces to the integral

$$
-\frac{8\sqrt{2}\pi^2}{\mu^2} \int_0^{\arcsin \mu/2} \sin \xi \sin \left(\frac{\xi}{2}\right) d\xi = -\frac{32\sqrt{2}}{3} \pi^2 \sin^3 \left(\frac{1}{2} \arcsin \frac{\mu}{2}\right); \quad (A.16)
$$

(5) let us show that the function:

$$
q(\mu) = 2\pi^2 \int_0^1 x \frac{\sin(v(x)/2)}{\cos v(x) \sin(\pi/4)} dx \quad (A.17)
$$
decreases monotonically as $\mu$ changes from 0 to 2. We have

$$
\left(\frac{\sin(v(x)/2)}{\cos v(x)}\right)' \geq 0 \quad (A.18)
$$

because the numerator of (A.18) increases, while the denominator decreases with the growth of $\mu$. This implies that

$$
q'(\mu) \geq 0, \quad (A.19)
$$

that is, $q(\mu)$ increases monotonically. Lemma 5.2 is proved.
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References
