Research Article

Continuation Criterion for the 2D Liquid Crystal Flows

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We consider the 2D liquid crystal systems, which consists of Navier-Stokes system coupled with wave maps or biharmonic wave maps, respectively. By logarithmic Sobolev inequalities, we obtain a blow-up criterion $\nabla d, \partial_t d \in L^1(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^2))$ for the case with wave maps, and we prove the existence of a global-in-time strong solutions for the case with biharmonic wave maps.

1. Introduction

First, we consider the following simplified liquid crystal flows in two space dimensions [1]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \sum_k \partial_t d^k \cdot \nabla d^k, \quad (1.1)$$

$$\operatorname{div} u = 0, \quad (1.2)$$

$$\partial_t^2 d + u \cdot \nabla d - \Delta d = d \left(|\nabla d|^2 - |\partial_t d|^2 \right), \quad |d| = 1, \quad (1.3)$$

$$(u, d, \partial_t d)(x, 0) = (u_0, d_0, d_1)(x), \quad x \in \mathbb{R}^2, \quad |d_0| = 1, \quad d_0 \cdot d_1 = 0, \quad (1.4)$$

where u is the velocity, π is the pressure, and d represents the macroscopic average of the liquid crystal orientation field with values in the unit circle.

The first two equations (1.1) and (1.2) are the well-known Navier-Stokes system with the Lorentz force $\sum_k \partial_t d^k \cdot \nabla d^k$. The last equation (1.3) is the well-known wave maps when $u = 0$.

It is a simple matter to show that the system (1.1)–(1.4) has a unique local-in-time smooth solution when $u_0, \nabla d_0, d_1 \in H^{1+s}(\mathbb{R}^2)$ with $s > 0$, $\operatorname{div} u_0 = 0$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ in $\mathbb{R}^2$. The aim of this paper is to study the regularity criterion of smooth solutions to the problem (1.1)–(1.4). We will prove the following.

Theorem 1.1. *Let $u_0, \nabla d_0, d_1 \in H^{1+s}(\mathbb{R}^2)$ with $s > 0$, $\operatorname{div} u_0 = 0$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ in $\mathbb{R}^2$ and let (u, d) be a smooth solution of (1.1)–(1.4) on some interval $[0, T]$ with $0 < T < \infty$. Assume that*

$$\nabla d, \partial_t d \in L^1\left(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^2)\right). \quad (1.5)$$

Then the solution (u, d) can be extended beyond $T > 0$.

$\dot{B}_{\infty, \infty}^0$ is the homogeneous Besov space. We have $L^\infty \subset BMO \subset \dot{B}_{\infty, \infty}^0$; see Triebel [2]. In the proof of Theorem 1.1, we will use the logarithmic Sobolev inequalities [3–6]:

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1} \log^{1/2}(e + \|u\|_{H^{1+s}}), \quad (1.6)$$

$$\|\nabla d\|_{L^\infty} \leq C\left(1 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0} \log(e + \|\nabla d\|_{H^{1+s}})\right), \quad (1.7)$$

$$\|\partial_t d\|_{L^\infty} \leq C\left(1 + \|\partial_t d\|_{\dot{B}_{\infty, \infty}^0} \log(e + \|\partial_t d\|_{H^{1+s}})\right), \quad (1.8)$$

for $s > 0$, and the Gagliardo-Nirenberg inequalities:

$$\begin{aligned} \|w\|_{L^4} &\leq C\|w\|_{L^2}^\alpha \left\|\Lambda^{1+s} w\right\|_{L^2}^{1-\alpha}, \\ \|\Lambda^s w\|_{L^4} &\leq C\|w\|_{L^2}^{1-\alpha} \left\|\Lambda^{1+s} w\right\|_{L^2}^\alpha, \end{aligned} \quad (1.9)$$

with $\Lambda := (-\Delta)^{1/2}$, $\alpha := 1 - (1/2) \cdot (1/(1+s))$, and $s > 0$, and the product estimate due to Kato-Ponce [7]:

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}} \|\Lambda^s f\|_{L^{q_2}}), \quad (1.10)$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Motivated by the problem (1.1)–(1.4), we consider the following liquid crystal flows:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \sum_k \partial_t d^k \cdot \nabla d^k, \quad (1.11)$$

$$\operatorname{div} u = 0, \quad (1.12)$$

$$\partial_t^2 d + u \cdot \nabla d + (-\Delta)^2 d = -\lambda d, \quad |d| = 1, \quad (1.13)$$

$$\lambda := |\partial_t d|^2 + |\Delta d|^2 + \Delta |\nabla d|^2 + 2 \sum_k \partial_k d \cdot \Delta \partial_k d, \quad (1.14)$$

$$(u, d, \partial_t d)(x, 0) = (u_0, d_0, d_1)(x, 0), \quad x \in \mathbb{R}^2, \quad |d_0| = 1. \quad (1.15)$$

The last two equations (1.13) and (1.14) are the biharmonic wave maps. It is also a simple matter to show that the problem (1.11)–(1.15) has at least one local-in-time strong solution. The aim of this paper is to prove the global-in-time regularity. We obtain the following.

Theorem 1.2. *Let $u_0 \in H^2$, $(\nabla d_0, d_1) \in H^3 \times H^2$ with $\operatorname{div} u_0 = 0$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ in $\mathbb{R}^2$. Then there exists at least a global-in-time smooth solution:*

$$(u, \nabla d, \partial_t d) \in L^\infty(0, T; H^2) \times L^\infty(0, T; H^3) \times L^\infty(0, T; H^2) \quad (1.16)$$

for any $T > 0$.

Remark 1.3. We are unable to prove the uniqueness of strong solutions in Theorem 1.2.

2. Proof of Theorem 1.1

We only need to prove a priori estimates.

Testing (1.1) by u , using (1.2), we see that

$$\frac{1}{2} \frac{d}{dt} \int u^2 dx + \int |\nabla u|^2 dx = \int (u \cdot \nabla) d \cdot \partial_t d dx. \quad (2.1)$$

Testing (1.3) by $\partial_t d$, using $|d| = 1$ and $d \cdot \partial_t d = 0$, we find that

$$\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 + |\partial_t d|^2 dx = - \int (u \cdot \nabla) d \cdot \partial_t d dx. \quad (2.2)$$

Summing up (2.1) and (2.2), we get

$$\frac{1}{2} \frac{d}{dt} \int u^2 + |\nabla d|^2 + |\partial_t d|^2 dx + \int |\nabla u|^2 dx = 0, \quad (2.3)$$

from which we get

$$\int u^2 + |\nabla d|^2 + |\partial_t d|^2 dx + \int_0^T \int |\nabla u|^2 dx dt \leq C. \quad (2.4)$$

Applying Λ^{1+s} to (1.1), testing by $\Lambda^{1+s} u$, using (1.2) and (1.10), we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left| \Lambda^{1+s} u \right|^2 dx + \int \left| \Lambda^{2+s} u \right|^2 dx \\ &= - \int \Lambda^{1+s} \operatorname{div}(u \otimes u) \cdot \Lambda^{1+s} u dx + \int \Lambda^{1+s} (\partial_t d \cdot \nabla d) \cdot \Lambda^{1+s} u dx \\ &\leq C \|u\|_{L^\infty} \left\| \Lambda^{2+s} u \right\|_{L^2} \left\| \Lambda^{1+s} u \right\|_{L^2} \end{aligned}$$

$$\begin{aligned}
& + C \left(\|\partial_t d\|_{L^\infty} \left\| \Lambda^{2+s} d \right\|_{L^2} + \|\nabla d\|_{L^\infty} \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2} \right) \left\| \Lambda^{1+s} u \right\|_{L^2} \\
& \leq \frac{1}{2} \left\| \Lambda^{2+s} u \right\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \left\| \Lambda^{1+s} u \right\|_{L^2}^2 \\
& \quad + C \|(\partial_t d, \nabla d)\|_{L^\infty} \left(y^2 + \left\| \Lambda^{1+s} u \right\|_{L^2}^2 \right),
\end{aligned} \tag{2.5}$$

where

$$y^2 := \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2}^2 + \left\| \Lambda^{2+s} d \right\|_{L^2}^2. \tag{2.6}$$

Taking Λ^{1+s} to (1.3), testing by $\Lambda^{1+s} \partial_t d$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} y^2 &= \int \Lambda^{1+s} \left[d \left(|\nabla d|^2 - |\partial_t d|^2 \right) \right] \cdot \Lambda^{1+s} \partial_t d \, dx \\
&\quad - \int \Lambda^{1+s} (u \cdot \nabla d) \cdot \Lambda^{1+s} \partial_t d \, dx =: I_1 + I_2.
\end{aligned} \tag{2.7}$$

By using (1.10), (2.4), and (1.9), I_1 can be bounded as follows:

$$\begin{aligned}
I_1 &\leq C \left[\|d\|_{L^\infty} \left\| \Lambda^{1+s} \left(|\nabla d|^2 - |\partial_t d|^2 \right) \right\|_{L^2} \right. \\
&\quad \left. + \left\| \Lambda^{1+s} d \right\|_{L^4} \left(\|\nabla d\|_{L^\infty} \|\nabla d\|_{L^4} + \|\partial_t d\|_{L^\infty} \|\partial_t d\|_{L^4} \right) \right] \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2} \\
&\leq C \left[\left(\|\nabla d\|_{L^\infty} \left\| \Lambda^{2+s} d \right\|_{L^2} + \|\partial_t d\|_{L^\infty} \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2} \right) \right. \\
&\quad \left. + C y^\alpha \left(\|\nabla d\|_{L^\infty} y^{1-\alpha} + \|\partial_t d\|_{L^\infty} y^{1-\alpha} \right) \right] \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2} \\
&\leq C \|(\partial_t d, \nabla d)\|_{L^\infty} y^2.
\end{aligned} \tag{2.8}$$

By using (1.10), I_2 can be bounded as

$$\begin{aligned}
I_2 &\leq C \left(\|u\|_{L^\infty} \left\| \Lambda^{2+s} d \right\|_{L^2} + \|\nabla d\|_{L^\infty} \left\| \Lambda^{1+s} u \right\|_{L^2} \right) \left\| \Lambda^{1+s} \partial_t d \right\|_{L^2} \\
&\leq C \|u\|_{L^\infty} y^2 + C \|\nabla d\|_{L^\infty} \left(y^2 + \left\| \Lambda^{1+s} u \right\|_{L^2}^2 \right).
\end{aligned} \tag{2.9}$$

Combining (2.5), (2.7), (2.8), and (2.9) and using (1.6), (1.7), (1.8), and the Gronwall lemma, we arrive at

$$\begin{aligned}
\|u\|_{L^\infty(0,T;H^{1+s})} + \|u\|_{L^2(0,T;H^{2+s})} &\leq C, \\
\|(\nabla d, \partial_t d)\|_{L^\infty(0,T;H^{1+s})} &\leq C.
\end{aligned} \tag{2.10}$$

This completes the proof.

3. Proof of Theorem 1.2

For simplicity, we only present a priori estimates.

First, we still have (2.1).

Testing (1.13) by $\partial_t d$, using $d \cdot \partial_t d = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 + |\partial_t d|^2 dx = - \int (u \cdot \nabla) d \cdot \partial_t d dx. \quad (3.1)$$

Summing up (2.1) and (3.1), we get

$$\begin{aligned} \int u^2 + |\Delta d|^2 + |\partial_t d|^2 dx + \int_0^T \int |\nabla u|^2 dx dt &\leq C, \\ \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx &= - \int \Delta d \cdot \partial_t d dx \leq \|\Delta d\|_{L^2} \|\partial_t d\|_{L^2} \leq C, \end{aligned} \quad (3.2)$$

which yields

$$\int |\nabla d|^2 dx \leq C. \quad (3.3)$$

Applying Δ to (1.11), testing by Δu , using (1.2) and (1.10), we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\Delta u|^2 dx + \int |\nabla \Delta u|^2 dx &= - \int \Delta \operatorname{div}(u \otimes u) \cdot \Delta u dx + \int \Delta(\partial_t d, \nabla d) \cdot \Delta u dx \\ &\leq C \|u\|_{L^\infty} \|\nabla \Delta u\|_{L^2} \|\Delta u\|_{L^2} + C (\|\partial_t d\|_{L^\infty} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^\infty} \|\Delta \partial_t d\|_{L^2}) \|\Delta u\|_{L^2} \\ &\leq \frac{1}{2} \|\nabla \Delta u\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\Delta u\|_{L^2}^2 \\ &\quad + C \left(\|\partial_t d\|_{L^2}^{1/2} \|\Delta \partial_t d\|_{L^2}^{1/2} \|\Delta d\|_{L^2}^{1/2} \left\| \Delta^2 d \right\|_{L^2}^{1/2} + \|\nabla d\|_{L^\infty} \|\Delta \partial_t d\|_{L^2} \right) \|\Delta u\|_{L^2} \\ &\leq \frac{1}{2} \|\nabla \Delta u\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\Delta u\|_{L^2}^2 + C y^2 + C \|\Delta u\|_{L^2}^2 + C \|\nabla d\|_{L^\infty} (y^2 + \|\Delta u\|_{L^2}^2), \end{aligned} \quad (3.4)$$

where

$$y^2 := \|\Delta \partial_t d\|_{L^2}^2 + \left\| \Delta^2 d \right\|_{L^2}^2. \quad (3.5)$$

Applying Δ to (1.13), we have

$$\Delta \partial_t^2 d + \Delta^3 d = -(\lambda \Delta d + 2 \nabla \lambda \cdot \nabla d + d \Delta \lambda) - \Delta(u \cdot \nabla d). \quad (3.6)$$

Since

$$0 = \Delta(d\partial_t d) = d\Delta\partial_t d + \partial_t d\Delta d + 2\sum_k \partial_k d \partial_k \partial_t d, \quad (3.7)$$

we easily see that

$$-d\Delta\partial_t d = \partial_t d\Delta d + \partial_t |\nabla d|^2. \quad (3.8)$$

Testing (3.6) by $\Delta\partial_t d$, using (3.8), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} y^2 &= - \int \lambda \Delta d \cdot \Delta\partial_t d + \nabla \lambda [2\nabla d \cdot \Delta\partial_t d + \nabla(\partial_t d \Delta d + 2\nabla d \cdot \nabla \partial_t d)] dx \\ &\quad - \int \Delta(u \cdot \nabla d) \cdot \Delta\partial_t d dx =: J_1 + J_2. \end{aligned} \quad (3.9)$$

By the same calculations as those in [8], we have

$$J_1 \leq C(1 + \|\nabla d\|_{L^\infty}) y^2 \leq C(1 + \|\nabla d\|_{H^1} \log(e + y)) y^2. \quad (3.10)$$

By using (1.10), J_2 can be bounded as

$$\begin{aligned} J_2 &\leq C(\|\Delta u\|_{L^2} \|\nabla d\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla \Delta d\|_{L^2}) \|\Delta\partial_t d\|_{L^2} \\ &\leq C\|\nabla d\|_{L^\infty} (y^2 + \|\Delta u\|_{L^2}^2) + C\|u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} \|\Delta d\|_{L^2}^{1/2} \left\| \Delta^2 d \right\|_{L^2}^{1/2} \|\Delta\partial_t d\|_{L^2} \\ &\leq C\|\nabla d\|_{L^\infty} (y^2 + \|\Delta u\|_{L^2}^2) + Cy^2 + C\|\Delta u\|_{L^2}^2. \end{aligned} \quad (3.11)$$

Combining (3.4), (3.9), (3.10), and (3.11) and using (1.6) and the Gronwall lemma, we conclude that

$$\begin{aligned} \|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;H^3)} &\leq C, \\ \|\nabla d\|_{L^\infty(0,T;H^3)} + \|\partial_t d\|_{L^\infty(0,T;H^2)} &\leq C. \end{aligned} \quad (3.12)$$

This completes the proof.

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References

- [1] F. Lin and C. Liu, "Static and dynamic theories of liquid crystals," *Journal of Partial Differential Equations*, vol. 14, no. 4, pp. 289–330, 2001.

- [2] H. Triebel, *Theory of Function Spaces. II*, vol. 84, Birkhäuser, Basel, Switzerland, 1992.
- [3] H. Brézis and T. Gallouet, "Nonlinear Schrödinger evolution equations," *Nonlinear Analysis*, vol. 4, no. 4, pp. 677–681, 1980.
- [4] H. Brézis and S. Wainger, "A note on limiting cases of Sobolev embeddings and convolution inequalities," *Communications in Partial Differential Equations*, vol. 5, no. 7, pp. 773–789, 1980.
- [5] T. Ozawa, "On critical cases of Sobolev's inequalities," *Journal of Functional Analysis*, vol. 127, no. 2, pp. 259–269, 1995.
- [6] H. Kozono, T. Ogawa, and Y. Taniuchi, "The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations," *Mathematische Zeitschrift*, vol. 242, no. 2, pp. 251–278, 2002.
- [7] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.
- [8] J. Fan and T. Ozawa, "On regularity criterion for the 2D wave maps and the 4D biharmonic wave maps," in *Current Advances in Nonlinear Analysis and Related Topics*, vol. 32 of *GAKUTO International Series Mathematical Sciences and Applications*, pp. 69–83, Gakkōtoshō, Tokyo, Japan, 2010.