Research Article

Hemi-Slant Lightlike Submanifolds of Indefinite Kenmotsu Manifolds

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Received 25 April 2012; Accepted 25 June 2012

Academic Editors: T. Friedrich and M. Khalkhali

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We introduce and study hemi-slant lightlike submanifolds of an indefinite Kenmotsu manifold. We give an example of hemi-slant lightlike submanifold and establish two characterization theorems for the existence of such submanifolds. We prove some theorems which ensure the existence of minimal hemi-slant lightlike submanifolds and obtain a condition under which the induced connection ∇ on M is a metric connection. An example of proper minimal hemi-slant lightlike submanifolds is also given.

1. Introduction

Given a semi-Riemannian manifold, one can consider its lightlike submanifold whose study is important from application point of view and difficult in the sense that the intersection of normal vector bundle and tangent bundle of these submanifolds is nonempty. This unique feature makes the study of lightlike submanifolds different from the study of nondegenerate submanifolds. The general theory of lightlike submanifolds was developed by Kupeli [1]; Duggal and Bejancu [2]. On the other hand, the concepts of transversal and screen transversal lightlike submanifolds of an indefinite Kaehler manifold were given by Sahin [3, 4]. In Sasakian setting, these submanifolds were investigated by Yildirim and Sahin [5–7].

In the present paper, we introduce and study hemi-slant lightlike submanifolds of an indefinite Kenmotsu manifold. The paper is arranged as follows. In Section 2, we recall definitions for indefinite Kenmotsu manifolds and give basic information on the lightlike geometry needed for this paper. In Section 3, after defining hemi-slant lightlike submanifolds immersed in an indefinite Kenmotsu manifold, we give an example which ensures the existence of hemi-slant lightlike submanifolds. We obtain integrability conditions of distributions involved, investigate the geometry of leaves of distributions, and establish two characterization theorems for the existence of hemi-slant lightlike submanifolds in

an indefinite Kenmotsu manifold. We prove that there does not exist curvature invariant hemi-slant lightlike submanifold in an indefinite Kenmotsu space form with some condition on c and obtain a geometric condition under which the induced connection on M is a metric connection. In Section 4, we study minimal hemi-slant lightlike submanifolds and give an example of proper minimal hemi-slant lightlike submanifolds immersed in R_2^9 .

2. Preliminaries

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an indefinite contact metric manifold [8] if there exists a $(1,1)$ tensor field ϕ , a vector field V , called the characteristic vector field, and its 1-form η satisfying

$$\begin{aligned} \overline{g}(\phi X, \phi Y) &= \overline{g}(X, Y) - \epsilon \eta(X) \eta(Y), \quad \overline{g}(V, V) = \epsilon \\ \phi^2 X &= -X + \eta(X)V, \quad \overline{g}(X, V) = \epsilon \eta(X), \quad \forall X, Y \in \Gamma(TM), \end{aligned} \quad (2.1)$$

where $\epsilon = \pm 1$. It follows that $\phi V = 0$, $\eta \circ \phi = 0$, $\eta(V) = \epsilon$.

An indefinite almost contact metric manifold $\overline{M}$ is said to be an indefinite Kenmotsu manifold [9] if

$$\overline{\nabla}_X V = -X + \eta(X)V, \quad (2.2)$$

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \epsilon \eta(Y)\phi X. \quad (2.3)$$

for any $X, Y \in \Gamma(T\overline{M})$.

We follow [2] for the notation and formulae used in this paper. A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $\text{Rad } TM$ is of rank r , where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad } TM$ in TM , that is,

$$TM = \text{Rad } TM \perp S(TM). \quad (2.4)$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $\text{Rad } TM$ in $TM^\perp$. Since for any local basis $\{\xi_i\}$ of $\text{Rad } TM$, there exist a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $\text{ltr}(TM)$ locally spanned by $\{N_i\}$ [2, page 144]. Let $\text{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$\begin{aligned} \text{tr}(TM) &= \text{ltr}(TM) \perp S(TM^\perp). \\ T\overline{M}|_M &= S(TM) \perp [(\text{Rad } TM) \oplus \text{ltr}(TM)] \perp S(TM^\perp). \end{aligned} \quad (2.5)$$

The Gauss and Weingarten equations are as follows:

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \\ \bar{\nabla}_X U &= -A_U X + \nabla_X^t U, \quad \forall X \in \Gamma(TM), U \in \Gamma(\text{tr}(TM)),\end{aligned}\tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^t U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr}(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $\text{tr}(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y),\tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N),\tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W),\tag{2.9}$$

for each $X, Y \in \Gamma(TM)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^\perp))$. If we denote the projection of TM on $S(TM)$ by P , then by using (2.7)–(2.9) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y),\tag{2.10}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X).\tag{2.11}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY),\tag{2.12}$$

$$\nabla_X \xi = -A_\xi^* X + \nabla_X^{*t} \xi,\tag{2.13}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad } TM)$. By using the above equation, we get

$$\bar{g}(h^l(X, PY), \xi) = g(A_\xi^* X, PY).\tag{2.14}$$

It is important to note that the induced connection ∇ on M is not a metric connection whereas ∇^* is a metric connection on $S(TM)$.

We denote curvature tensors of $\bar{M}$ and M by $\bar{R}$ and R , respectively. The Gauss equation for M is given by

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + A_{h^l(X, Z)}Y - A_{h^l(Y, Z)}X + A_{h^s(X, Z)}Y - A_{h^s(Y, Z)}X \\ &\quad + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ &\quad + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)),\end{aligned}\tag{2.15}$$

for all $X, Y, Z \in \Gamma(TM)$.

The curvature tensor $\bar{R}$ of an indefinite Kenmotsu space form $\bar{M}(c)$ is given by [9] as follows:

$$\begin{aligned}\bar{R}(X, Y)Z = & \frac{c-3}{4} \{ \bar{g}(Y, Z)X - \bar{g}(X, Z)Y \} \\ & + \frac{c+1}{4} \{ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X, Z)\eta(Y)V - \bar{g}(Y, Z)\eta(X)V \\ & + \bar{g}(\phi Y, Z)\phi X - \bar{g}(\phi X, Z)\phi Y - 2\bar{g}(\phi X, Y)\phi Z \},\end{aligned}\quad (2.16)$$

for any $X, Y \in \Gamma(T\bar{M})$.

From now on, we denote $(M, g, S(TM), S(TM^\perp))$ by M in this paper.

3. Hemi-Slant Lightlike Submanifolds

The concept of pseudoslant (antislant) submanifolds with definite metric is given by Carriazo [10] which was renamed by Sahin as hemi-slant submanifolds and studied their warped product in Kaehler manifolds [11]. In this section, we introduce and study hemi-slant lightlike submanifolds of an indefinite Kenmotsu manifold for which we need the following lemma from [3].

Lemma 3.1. *Let M be a $2q$ -lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with constant index $2q$ such that $2q < \dim(M)$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

In view of the above lemma, we can define hemi-slant lightlike submanifolds in an indefinite Kenmotsu manifold as follows.

Definition 3.2. Let M be a $2q$ -lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$ with the structure vector field V tangent to M such that $2q < \dim M$. Then, the submanifold M is said to be hemi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

- (i) $\text{Rad } TM$ is a distribution on M such that

$$\phi(\text{Rad } TM) = \text{ltr}(TM); \quad (3.1)$$

- (ii) for all $x \in U \subset M$ and for each non zero vector field X tangent to $S(TM) = D^\theta \perp \{V\}$ if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $S(TM)$ is constant, where D^θ is the complementary distribution to V in the screen distribution $S(TM)$.

The angle $\theta(X)$ is called the slant angle of the distribution D^θ . A hemi-slant lightlike submanifold is said to be proper if $D^\theta \neq 0$ and $\theta \neq 0, \pi/2$.

From Definition 3.2, we have the following decomposition:

$$\begin{aligned} TM &= \text{Rad } TM \perp D^\theta \perp \{V\}, \\ \text{tr}(TM) &= \text{ltr}(TM) \perp FD^\theta \perp \mu, \\ \overline{TM} &= (\text{Rad } TM \oplus \text{ltr}(TM)) \perp D^\theta \perp FD^\theta \perp \mu \perp \{V\}. \end{aligned} \quad (3.2)$$

Denoting $(R_q^{2m+1}, \phi_o, V, \eta, g)$ as a manifold R_q^{2m+1} with its usual Kenmotsu structure given by

$$\begin{aligned} \eta &= dz, \quad V = \partial Z, \\ \bar{g} &= \eta \otimes \eta + e^{2z} \left\{ -\left(\sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i \right) + \sum_{i=q+1}^m dx^i \otimes dx^i + dy^i \otimes dy^i \right\}, \\ \phi_o \left(\sum_{i=1}^m (X_i \partial x^i + Y_i \partial y^i) + Z \partial z \right) &= \sum_{i=1}^m (Y_i \partial x^i - X_i \partial y^i), \end{aligned} \quad (3.3)$$

where (x^i, y^i, z) are the cartesian coordinates.

The existence of proper hemi-slant lightlike submanifolds in indefinite Kenmotsu manifolds is ensured by the following example.

Example 3.3. Let M be a submanifold of $\overline{M} = (R_2^9, \bar{g})$ defined by

$$\begin{aligned} x^1 &= s, & x^2 &= t, & x^3 &= u \sin w, & x^4 &= \sin u, \\ y^1 &= t, & y^2 &= s, & y^3 &= u \cos w, & y^4 &= \cos u, \end{aligned} \quad (3.4)$$

where $u, w \in (0, \pi/2)$ and R_2^9 is a semi-Euclidean space of signature $(-, -, +, +, +, +, +, +, +)$.

Then, a local frame of TM is given by

$$\begin{aligned} \xi_1 &= e^{-z} (\partial x_1 + \partial y_2), \\ \xi_2 &= e^{-z} (\partial x_2 + \partial y_1), \\ X_1 &= e^{-z} (\sin w \partial x_3 + \cos u \partial x_4 + \cos w \partial y_3 - \sin u \partial y_4), \\ X_2 &= e^{-z} (u \cos w \partial x_3 - u \sin w \partial y_3), \\ V &= \partial Z. \end{aligned} \quad (3.5)$$

We see that M is a 2-lightlike submanifold with $\text{Rad } TM = \text{span}\{\xi_1, \xi_2\}$ and $S(TM) = \text{span}\{X_1, X_2\} \perp \{V\}$ which is Riemannian and can be easily proved that $S(TM)$ is a slant distribution with slant angle $\theta = \pi/4$. Moreover, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$\begin{aligned} W_1 &= e^{-z} (\sin u \partial x_4 + \cos u \partial y_4), \\ W_2 &= e^{-z} (\sin w \partial x_3 - \cos u \partial x_4 + \cos w \partial y_3 + \sin u \partial y_4), \end{aligned} \quad (3.6)$$

and the transversal lightlike bundle $\text{ltr}(TM)$ is spanned by

$$\begin{aligned} N_1 &= -\frac{e^{-z}}{2}(\partial x_1 - \partial y_2), \\ N_2 &= \frac{e^{-z}}{2}(\partial x_2 - \partial y_1). \end{aligned} \quad (3.7)$$

It is not difficult to see that

$$\phi \xi_1 = N_2, \quad \phi \xi_2 = N_1. \quad (3.8)$$

Hence, M is a hemi-slant lightlike submanifold of R_2^9 .

For any $X \in \Gamma(TM)$, we write

$$\phi X = TX + FX, \quad (3.9)$$

where TX and FX are the tangential and transversal components of ϕX , respectively. Similarly, for $W \in \Gamma(\text{tr}(TM))$, we write

$$\phi W = BW + CW, \quad (3.10)$$

where BW and CW are the tangential and transversal components of ϕW , respectively. If the projections on the distributions $S(TM) = D^\theta \perp \{V\}$ and $\text{Rad } TM$ are denoted by $\overline{Q}_1$ and Q_2 , respectively, then any X tangent to M can be written as

$$X = \overline{Q}_1 X + Q_2 X. \quad (3.11)$$

We observe that $\overline{Q}_1 X = Q_1 X + \eta(X)V$. By applying ϕ to (3.11) and using (3.9), we conclude that

$$\phi X = TQ_1 X + FQ_1 X + FQ_2 X, \quad (3.12)$$

$$\phi Q_2 X = FQ_2 X, \quad TQ_2 X = 0, \quad TQ_1 X \in \Gamma(S(TM)). \quad (3.13)$$

The following two theorems are the characterizations of hemi-slant lightlike submanifolds which are similar to the characterization of slant submanifolds in indefinite Hermitian manifolds given by Sahin [12].

Theorem 3.4. *The necessary and sufficient condition for a $2q$ -lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field V tangent to M to be hemi-slant lightlike submanifold is that*

- (i) $\phi(\text{ltr}(TM))$ is a distribution on M ;

(ii) for any vector field X tangent to M linearly independent of structure vector field V , there exists a constant $\lambda \in [-1, 0]$ such that

$$(Q_1 T)^2 Q_1 X = \lambda (\overline{Q_1} X - \eta(\overline{Q_1} X) V), \quad (3.14)$$

where $\lambda = -\cos^2 \theta$ and θ is the slant angle of M .

Proof. Assume that M is a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $\phi(\text{Rad } TM) = \text{ltr}(TM)$, from which we have $\phi(\text{ltr}(TM)) = \text{Rad}(TM)$. Hence $\phi(\text{ltr}(TM))$ is a distribution on M . In order to prove (ii), we observe that, for any $X \in \Gamma(TM)$ linearly independent of structure vector field V and $Q_1 X \in \Gamma(D^\theta)$, one has

$$\cos \theta(Q_1 X) = \frac{-g(Q_1 X, (Q_1 T)^2 Q_1 X)}{|Q_1 X| |T Q_1 X|}. \quad (3.15)$$

But $\cos \theta(Q_1 X)$ can also be computed as

$$\cos \theta(Q_1 X) = \frac{|T Q_1 X|}{|\phi Q_1 X|}. \quad (3.16)$$

From (3.15) and (3.16), we arrive at

$$\cos^2 \theta(Q_1 X) = \frac{-g(Q_1 X, (Q_1 T)^2 Q_1 X)}{|Q_1 X|^2}. \quad (3.17)$$

Taking into account that $\theta(Q_1 X)$ is constant on D^θ , from (3.17) we infer that

$$(Q_1 T)^2 Q_1 X = \lambda Q_1 X = \lambda (\overline{Q_1} X - \eta(\overline{Q_1} X) V), \quad (3.18)$$

where we have used $\lambda = -\cos^2 \theta(Q_1 X)$. Clearly $\lambda \in [-1, 0]$. Converse part directly follows from (i) and (ii). $\square$

Theorem 3.5. *The necessary and sufficient condition for a $2q$ -lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field V tangent to M to be hemi-slant lightlike submanifold is that*

- (i) $\phi(\text{ltr}(TM))$ is a distribution on M ,
- (ii) for any vector field X tangent to M linearly independent of structure vector field V , there exists a constant $\mu \in [-1, 0]$ such that

$$BF \overline{Q_1} X = \mu (\overline{Q_1} X - \eta(\overline{Q_1} X) V), \quad (3.19)$$

where $\lambda = -\cos^2 \theta$ and θ is the slant angle of M .

Proof. If M is a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$, then $\phi(\text{ltr}(TM))$ is a distribution on M . For the proof of (ii), applying ϕ to (3.12) and using (3.9) and (3.10), we arrive at

$$-X + \eta(X)V = (Q_1T)^2Q_1X + FQ_1TQ_1X + BFQ_1X + CFQ_1X + BFQ_2X. \quad (3.20)$$

Comparing the components of screen distribution on both sides of the above equation, we get

$$-\overline{Q}_1X + \eta(\overline{Q}_1X)V = (Q_1T)^2Q_1X + BFQ_1X. \quad (3.21)$$

In view of Theorem 3.4 and the fact that M is a hemi-slant lightlike submanifold, we conclude that

$$(Q_1T)^2Q_1X = -\cos^2\theta(\overline{Q}_1X - \eta(\overline{Q}_1X)V). \quad (3.22)$$

Hence (ii) follows from (3.21) and (3.22) together with $FV = 0$. Conversely, assume that (i) and (ii) hold. From (ii) and (3.21), we observe that

$$(Q_1T)^2Q_1X = -(1 + \mu)Q_1X. \quad (3.23)$$

If we take $-(1 + \mu) = \lambda$, then $\lambda \in [-1, 0]$. Hence the proof of Theorem follows from Theorem 3.4. $\square$

Corollary 3.6. *Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then*

$$g(T\overline{Q}_1X, T\overline{Q}_1Y) = \cos^2\theta \Big|_{S(TM)} \left[g(\overline{Q}_1X, \overline{Q}_1Y) - \eta(\overline{Q}_1X)\eta(\overline{Q}_1Y) \right], \quad (3.24)$$

$$g(F\overline{Q}_1X, F\overline{Q}_1Y) = \sin^2\theta \Big|_{S(TM)} \left[g(\overline{Q}_1X, \overline{Q}_1Y) - \eta(\overline{Q}_1X)\eta(\overline{Q}_1Y) \right] \quad (3.25)$$

for any $X, Y \in \Gamma(TM)$.

Differentiating (3.12), taking into account that $\overline{M}$ is an indefinite Kenmotsu manifold and then comparing tangential, lightlike transversal and screen transversal parts, we obtain

$$(\nabla_X TQ_1)Y = A_{FQ_1Y}X + A_{FQ_2Y}X + Bh^l(X, Y) + Bh^s(X, Y) - g(\phi X, Y)V + \eta(Y)TQ_1X \quad (3.26)$$

$$(\nabla_X F)Q_2Y = -h^l(X, TQ_1Y) - D^l(X, FQ_1Y) + \eta(Y)FQ_2X \quad (3.27)$$

$$(\nabla_X F)Q_1Y = Ch^s(X, Y) - h^s(X, TQ_1Y) - D^s(X, FQ_2Y) + \eta(Y)FQ_1X. \quad (3.28)$$

The integrability conditions of the distributions involved in the definition of hemi-slant lightlike submanifolds are given by the following theorems.

Theorem 3.7. Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$. Then the distribution $\text{Rad } TM$ is integrable if and only if

- (i) $A_{FQ_2Y}X - A_{FQ_2X}Y = 0$;
- (ii) $D^s(X, FQ_2Y) = D^s(Y, FQ_2X)$

for any $X, Y \in \Gamma(\text{Rad } TM)$.

Proof. For any $X, Y \in \Gamma(\text{Rad } TM)$, from (3.26) we have

$$-TQ_1 \nabla_X Y = A_{FQ_2Y}X + Bh^l(X, Y) + Bh^s(X, Y) - g(\phi X, Y)V. \quad (3.29)$$

Interchanging X and Y in (3.29) and subtracting (3.29) from the resulting equation, we get

$$TQ_1[X, Y] = A_{FQ_2X}Y - A_{FQ_2Y}X + 2g(\phi X, Y)V. \quad (3.30)$$

Taking inner product of the above equation with ϕZ for $Z \in \Gamma(D^\theta)$, after simplification, we obtain

$$g([X, Y], Z) = \sec^2 \theta \{g(A_{FQ_2X}Y - A_{FQ_2Y}X), \phi Z\}. \quad (3.31)$$

Also, from (3.28) we have

$$-FQ_1 \nabla_X Y = Ch^s(X, Y) - D^s(X, FQ_2Y). \quad (3.32)$$

Interchanging the role of X and Y in (3.32) and subtracting (3.32) from the resulting equation, we get

$$FQ_1[X, Y] = D^s(X, FQ_2Y) - D^s(Y, FQ_2X). \quad (3.33)$$

Taking inner product of above equation with ϕZ , we arrive at

$$g([X, Y], Z) = \text{cosec}^2 \theta g(D^s(X, FQ_2Y) - D^s(Y, FQ_2X), \phi Z). \quad (3.34)$$

Hence our assertion follows from (3.31) and (3.34). $\square$

Theorem 3.8. Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$. Then the distribution $D^\theta \perp \{V\}$ is integrable if and only if

$$h^l(X, TQ_1Y) - h^l(Y, TQ_1X) + D^l(X, FQ_1Y) - D^l(Y, FQ_1X) = 0, \quad (3.35)$$

for any $X, Y \in \Gamma(S(TM))$.

Proof. For any $X, Y \in \Gamma(D^\theta \perp \{V\})$, from (3.27) we have

$$-FQ_2 \nabla_X Y = -h^l(X, TQ_1 Y) - D^l(X, FQ_1 Y). \quad (3.36)$$

Interchanging X and Y in (3.36) and subtracting (3.36) from the resulting equation, we obtain

$$FQ_2[X, Y] = h^l(X, TQ_1 Y) - h^l(Y, TQ_1 X) + D^l(X, FQ_1 Y) - D^l(Y, FQ_1 X), \quad (3.37)$$

which proves our assertion. $\square$

In view of the above argument, if we restrict the vector fields X and Y to D^θ , then we see that the distribution D^θ is not integrable.

The following two theorems deal with the totally geodesic foliations of the leaves of the distributions $S(TM) = D^\theta \perp \{V\}$ and $\text{Rad } TM$.

Theorem 3.9. *Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then the distribution $D^\theta \perp \{V\}$ defines totally geodesic foliation in M if and only if $A_{FQ_1 TQ_1 Y} X - BD^l(X, FQ_1 Y)$ has no component in $\text{Rad } TM$ for any $X, Y \in \Gamma(S(TM))$.*

Proof. Using (2.7), (2.9), (3.12), and (3.14), for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\text{ltr}(TM))$, we obtain

$$\overline{g}(\overline{\nabla}_X Y, N) = \cos^2 \overline{g}(\overline{\nabla}_X Y, N) + \overline{g}(A_{FQ_1 TQ_1 Y} X - BD^l(X, FQ_1 Y), N), \quad (3.38)$$

which can be written as

$$\sin^2 \theta \overline{g}(\overline{\nabla}_X Y, N) = \overline{g}(A_{FQ_1 TQ_1 Y} X - BD^l(X, FQ_1 Y), N). \quad (3.39)$$

Thus, our assertion follows from (3.39). $\square$

Theorem 3.10. *Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$. Then the distribution $\text{Rad } TM$ defines totally geodesic foliation in M if and only if $h^*(\xi_1, TQ_1 X) - A_{FQ_1 X} \xi_1$ has no component in $\text{Rad } TM$ for any $\xi_1 \in \Gamma(\text{Rad } TM)$ and $X \in \Gamma(D^\theta)$.*

Proof. For any $\xi_1, \xi_2 \in \Gamma(\text{Rad } TM)$ and $X \in \Gamma(D^\theta)$, from (2.7), (2.9), (2.12), and (3.12), we obtain

$$g(\nabla_{\xi_1} \xi_2, X) = -\overline{g}(\phi \xi_2, h^*(\xi_1, TQ_1 X) - A_{FQ_1 X} \xi_1). \quad (3.40)$$

Hence our assertion follows from (3.40). $\square$

The necessary and sufficient conditions under which the induced connection ∇ on a hemi-slant lightlike submanifold immersed in indefinite Kenmotsu manifold to be metric connection is given by the following result.

Theorem 3.11. *Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifolds $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if*

$$TQ_1 A_{FQ_2 Y} X - BD^s(X, FQ_2 Y) = 0, \quad (3.41)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\text{Rad } TM)$.

Proof. For $X \in \Gamma(TM)$ and $Y \in \Gamma(\text{Rad } TM)$, from (3.1), we have

$$\overline{\nabla}_X Y = -\phi \overline{\nabla}_X \phi Y - (\overline{\nabla}_X \phi) \phi Y. \quad (3.42)$$

Making use of (2.3), (2.7), and (2.9) in the above equation, we arrive at

$$\nabla_X Y + h^l(X, Y) + h^s(X, Y) = -\phi \left(-A_{FQ_2 Y} X + \nabla_X^l FQ_2 Y + D^s(X, FQ_2 Y) \right) + g(\phi X, \phi Y) V. \quad (3.43)$$

Using (3.9) and (3.10) in (3.43) and comparing the tangential components of the resulting equation, we obtain

$$\nabla_X Y = TQ_1 A_{FQ_2 Y} X - B \nabla_X^l FQ_2 Y - BD^s(X, FQ_2 Y), \quad (3.44)$$

from which our assertion follows. $\square$

Theorem 3.12. *Let M be a hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then T is never parallel on M .*

Proof. For $X \in \Gamma(TM)$, from (2.2) and (2.7), we have

$$\nabla_X V + h^l(X, V) + h^s(X, V) = -X + \eta(X) V. \quad (3.45)$$

Comparing the tangential Components of the above equation, we get

$$\nabla_X V = -X + \eta(X) V. \quad (3.46)$$

Since $TV = 0$, so differentiating $TV = 0$ covariantly with respect to X we obtain

$$(\nabla_X T) V + T(\nabla_X V) = 0. \quad (3.47)$$

Using (3.46) in the above equation, we get

$$(\nabla_X T) V = TX, \quad (3.48)$$

from which our assertion follows. $\square$

Similar to the notion of curvature invariant submanifolds given by Atceken and Kilic [13], we define curvature invariant lightlike submanifolds as follows.

Definition 3.13. Let M be a $2q$ -lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then M is said to be curvature invariant lightlike submanifold of $\overline{M}$ if the covariant derivatives of the second fundamental forms h^l and h^s of M satisfy

$$(\overline{\nabla}_X h^l)(Y, Z) - (\overline{\nabla}_Y h^l)(X, Z) = 0, \quad (\overline{\nabla}_X h^s)(Y, Z) - (\overline{\nabla}_Y h^s)(X, Z) = 0, \quad (3.49)$$

for any $X, Y \in \Gamma(TM)$.

Theorem 3.14. *There does not exist any curvature-invariant proper hemi-slant lightlike submanifold of an indefinite Kenmotsu space form $\overline{M}(c)$ with $c \neq -1$.*

Proof. Using (2.15) and (2.16), for any $X, Y \in \Gamma(S(TM))$ and $Z \in \Gamma(\text{Rad } TM)$, we obtain

$$(\overline{\nabla}_X h^s)(Y, Z) - (\overline{\nabla}_Y h^s)(X, Z) = 0, \quad (3.50)$$

$$(\overline{\nabla}_X h^l)(Y, Z) - (\overline{\nabla}_Y h^l)(X, Z) = -\frac{c+1}{2} \{g(TQ_1 X, Y) FQ_2 Z\}. \quad (3.51)$$

Thus, our assertion follows from (3.50) and (3.51). $\square$

4. Minimal Hemi-Slant Submanifolds

In this section, we study minimal hemi-slant lightlike submanifolds immersed in an indefinite Kenmotsu manifold. First we recall the following definition of minimal lightlike submanifolds of a semi-Riemannian manifold $\overline{M}$ given by Bejan and Duggal [14].

Definition 4.1. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be minimal if

- (i) $h^s = 0$ on $\text{Rad } TM$ and
- (ii) $\text{trace } h = 0$,

where trace is written with respect to $S(TM)$.

In [14], it has been shown that the above definition is independent of $S(TM)$ and $S(TM^\perp)$, but it depends on the choice of transversal bundle $\text{tr}(TM)$.

For the existence of proper minimal hemi-slant lightlike submanifolds, we give the following example.

Example 4.2. Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space of signature $(-, -, +, +, +, +, +, +, +)$ with respect to canonical basis $(\partial x_1, \partial y_1, \partial x_2, \partial y_2, \partial x_3, \partial y_3, \partial x_4, \partial y_4, \partial z)$.

Consider an almost contact structure ϕ on R_2^9 defined by $\phi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, z) = (-y_1, x_1, -y_2, x_2, -x_4 \cos \alpha - y_3 \sin \alpha, -y_4 \cos \alpha + x_3 \sin \alpha, x_3 \cos \alpha + y_4 \sin \alpha, y_3 \cos \alpha - x_4 \sin \alpha, 0)$.

For some $\alpha \in (0, \pi/2)$. Let M be a submanifold of (R_2^9, ϕ) defined by the following equations:

$$\begin{aligned} x^1 &= u_1, & x^2 &= u_2, & x^3 &= u_3, & x^4 &= u_4, \\ y^1 &= u_2, & y^2 &= u_1, & y^3 &= \sin u_3 \sinh u_4, & y^4 &= \cos u_3 \cosh u_4. \end{aligned} \quad (4.1)$$

Then the tangent bundle TM of M is spanned by

$$\begin{aligned} \xi_1 &= e^{-z}(\partial x_1 + \partial y_2), & \xi_2 &= e^{-z}(\partial x_2 + \partial y_1), \\ X_1 &= e^{-z}(\partial x_3 + \cos u_3 \sinh u_4 \partial y_3 - \sin u_3 \cosh u_4 \partial y_4), \\ X_2 &= e^{-z}(\partial x_4 + \sin u_3 \cosh u_4 \partial y_3 + \cos u_3 \sinh u_4 \partial y_4), & V &= \partial z. \end{aligned} \quad (4.2)$$

Suppose that the distribution $\text{Rad } TM = \text{span}\{\xi_1, \xi_2\}$. Then M is a 2-lightlike submanifold of $\overline{M}$. If we take $S(TM) = \text{span}\{X_1, X_2 \perp \{V\}\}$, then $S(TM)$ is Riemannian vector subbundle and one can easily see that $S(TM)$ is a slant distribution with slant angle α . A direct calculation shows that the screen transversal bundle is spanned by

$$\begin{aligned} W_1 &= e^{-z}(-\cosh u_4 \sinh u_4 \partial x_3 + \cos u_3 \cosh u_4 \partial y_3 - \sin u_3 \cos u_3 \partial x_4 + \sin u_3 \sinh u_4 \partial y_4), \\ W_2 &= e^{-z}(\cos u_3 \sin u_3 \partial x_3 + \sin u_3 \sinh u_4 \partial y_3 - \cosh u_4 \sinh u_4 \partial x_4 + \cos u_3 \cosh u_4 \partial y_4), \end{aligned} \quad (4.3)$$

and $\text{ltr}(TM)$ is spanned by

$$\begin{aligned} N_1 &= -\frac{e^{-z}}{2}(-\partial y_2 + \partial x_1), \\ N_2 &= \frac{e^{-z}}{2}(-\partial y_1 + \partial x_2). \end{aligned} \quad (4.4)$$

For a vector field X tangent to M , from Gauss formula and after calculations, we get

$$\begin{aligned} h^l &= 0, & h^s(X, \xi_1) &= 0, & h^s(X, \xi_2) &= 0, \\ h^s(X_1, X_1) &= \frac{-e^{-z}(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_2, \\ h^s(X_2, X_2) &= \frac{e^{-z}(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_2, \\ h^s(X_1, X_2) &= \frac{e^{-z}(\sinh^2 u_4 + \cos^2 u_3)}{\cosh^4 u_4 - \sin^4 u_3} W_1. \end{aligned} \quad (4.5)$$

Hence, M is a minimal proper hemi-slant lightlike submanifold of R_2^9 , and the induced connection on M is a metric connection. We note that M is neither totally geodesic nor totally umbilical.

Now, we prove a lemma which will be helpful for the study of minimal hemi-slant lightlike submanifolds.

Lemma 4.3. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ such that $\dim(D^\theta) = \dim(S(TM^\perp))$. If $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal basis of $\Gamma(D^\theta)$, then $\{\csc \theta Fe_1, \dots, \csc \theta Fe_m\}$ is an orthonormal basis of $S(TM^\perp)$.*

Proof. Suppose that $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal basis of D^θ . From (3.25) together with the fact that D^θ is Riemannian, we conclude that

$$\overline{g}(\csc \theta Fe_i, \csc \theta Fe_j) = \delta_{ij}, \quad (4.6)$$

where $i, j = 1, 2, 3, \dots, m$. Hence our assertion follows from (4.6). $\square$

The existence of minimal hemi-slant lightlike submanifolds in an indefinite Kenmotsu manifold is given by the following two theorems.

Theorem 4.4. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then M is minimal if and only if*

$$\begin{aligned} \text{trace } A_{\xi_j}^* \Big|_{S(TM)} &= 0, & \text{trace } A_{W_\alpha} \Big|_{S(TM)} &= 0, \\ g(D^l(X, W), Y) &= 0, \end{aligned} \quad (4.7)$$

for any $X, Y \in \Gamma(\text{Rad } TM)$ and $W \in \Gamma(S(TM^\perp))$, where $\{\xi_j\}_{j=1}^r$ is a basis of $\text{Rad } TM$ and $\{W_\alpha\}_{\alpha=1}^m$ is a basis of $S(TM^\perp)$.

Proof. Using $\overline{\nabla}_V V = 0$ and (2.2), we obtain $h^l(V, V) = h^s(V, V) = 0$. It is known that $h^l = 0$ on $\text{Rad } TM$ [14, Proposition 4.1]. If we take $\{e_1, e_2, \dots, e_m\}$ as an orthonormal basis of D^θ , then M is minimal if and only if

$$\sum_{k=1}^m h(e_k, e_k) = 0 \quad (4.8)$$

and $h^s = 0$ on $\text{Rad } TM$. From (2.10) and (2.14), we have

$$\sum_{k=1}^m h(e_k, e_k) = \sum_{k=1}^m \frac{1}{r} \sum_{j=1}^r g(A_{\xi_j}^* e_k, e_k) N_j + \frac{1}{m} \sum_{\alpha=1}^m g(A_{W_\alpha} e_k, e_k) W_\alpha. \quad (4.9)$$

On the other hand, from (2.10), we infer that $h^s = 0$ on $\text{Rad } TM$ if

$$\overline{g}(D^l(X, W), Y) = 0, \quad (4.10)$$

for $X, Y \in \Gamma(\text{Rad } TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus, our assertion follows from (4.9) and (4.10). $\square$

Theorem 4.5. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ such that $\dim(D^\theta) = \dim(S(TM^\perp))$. Then M is minimal if and only if*

$$\begin{aligned} \text{trace } A_{\xi_i}^* \Big|_{S(TM)} &= 0, & \text{trace } A_{We_i} \Big|_{S(TM)} &= 0, \\ g(D^l(X, Fe_i), Y) &= 0. \end{aligned} \quad (4.11)$$

for $X, Y \in \Gamma(\text{Rad } TM)$, where $\{\xi_k\}_{k=1}^r$ is a basis of $\text{Rad } TM$ and $\{e_j\}_{j=1}^m$ is a basis of D^θ .

Proof. From $\overline{\nabla}_V V = 0$ and (2.2), we obtain $h^l(V, V) = h^s(V, V) = 0$. Recall that $h^l = 0$ on $\text{Rad } TM$ [14, Proposition 4.1]. In view of Lemma 4.3, we observe that $\{\csc \theta Fe_1, \dots, \csc \theta Fe_m\}$ is an orthonormal basis of $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$ and for some function A_i , $i \in (1, 2, \dots, m)$, we can write

$$h^s(X, X) = \sum_{i=1}^m A_i \csc \theta Fe_i, \quad (4.12)$$

from which we have

$$h^s(X, X) = \sum_{i=1}^m \csc \theta g(A_{Fe_i} X, X) Fe_i, \quad (4.13)$$

for any $X \in \Gamma(S(TM))$. Thus our assertion follows from the above equation and Theorem 4.4. $\square$

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