Research Article

Pairwise Balanced Design of Order $6n + 4$ and 2-Fold System of Order $3n + 2$

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In the Steiner triple system, Bose (1939) constructed the $2-(v, 3, 1)$ design for $v = 6n + 3$ and later on Skolem (1958) constructed the same for $v = 6n + 1$. In the literature we found a pairwise balanced design (PBD) for $v = 6n + 5$. We also found the 2-fold triple system of the orders $3n$ and $3n + 1$. In this paper, we construct a PBD for $v = 6n + 4$ and a 2-fold system of the order $3n + 2$. The second construction completes the 2-fold system for all $n \in \mathbb{N}$.

1. Introduction

A Latin square of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \dots, n\}$, such that each row and each column of the array contain each of the symbols in $\{1, 2, \dots, n\}$ exactly once. A latin square is said to be idempotent if cell (i, i) contains symbol i for $1 \leq i \leq n$. A latin square of order $2n$ is said to be half-idempotent if for $1 \leq i \leq n$ cells (i, i) and $(n + i, n + i)$ contain the symbol i . A latin square is said to be commutative if cells (i, j) and (j, i) contain the same symbol, for all $1 \leq i, j \leq n$.

A quasi-group of order n is a pair $(Q, \circ)$, where Q is a set of size n and " $\circ$ " is a binary operation on Q such that for every pair of elements $a, b \in Q$ the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. As far as we are concerned a quasi-group is just a latin square with a headline and a sideline.

A $t-(v, k, \lambda)$ design is an ordered pair (X, B) where X is a v set of points and B , called a block set, is of k subsets of X with the property that every t subset of X is contained in exactly λ blocks [1]. A $t-(v, k, 1)$ design is defined as a Steiner system and denoted by $S(t, k, v)$. A Steiner triple system (STS) $2-(v, 3, 1)$ is an ordered pair (S, B) , where S is a finite set of points or symbols, and B is a set of 3-element subsets of S called triples, such that each pair of

distinct elements of S occurs together in exactly one triple of B . The order of a Steiner triple system (S, B) is the size of the set S , denoted by $|S| = v$.

Theorem 1.1 (see [2, Theorem 1.1.3]). *A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod{6}$.*

The Bose Construction ($v \equiv 3 \pmod{6}$), see [2, 3])

Let $v = 6n + 3$ and let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n + 1$, where $Q = \{1, 2, \dots, 2n + 1\}$. Let $S = Q \times \{1, 2, 3\}$, and define B to contain the following two types of triples.

Type 1. For $1 \leq i \leq 2n + 1$, $\{(i, 1), (i, 2), (i, 3)\} \in B$ (see Figure 2).

Type 2. For $1 \leq i \leq 2n + 1$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in B$ (see Figure 2).

Then (S, B) is a Steiner triple system of order $6n + 3$.

The Skolem Construction ($v \equiv 1 \pmod{6}$), see [2, 4])

Let $v = 6n + 1$ and let $(Q, \circ)$ be a half idempotent commutative quasi-group of order $2n$, where $Q = \{1, 2, \dots, 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define B as follows.

Type 1. For $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in B$.

Type 2. For $1 \leq i \leq n$, $\{\infty, (n + i, 1), (i, 2)\}, \{\infty, (n + i, 2), (i, 3)\}, \{\infty, (n + i, 3), (i, 1)\} \in B$.

Type 3. For $1 \leq i \leq 2n$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in B$ (see Figure 3).

Then (S, B) is a Steiner triple system of order $6n + 1$.

For t a positive integer, a t -wise balanced design D is an ordered pair (S, B) , where S is a finite nonempty set (of points) and B is a finite nonempty multiset of subsets of S (called blocks), such that every t subset of S is contained in a constant number $\lambda > 0$ of blocks. If $v = |S|$ and K is the set of sizes of the blocks, then we call D a $t - (v, K, \lambda)$ design. If all blocks of D have the same size k (i.e., $K = \{k\}$), then D is called a t design or a $t - (v, k, \lambda)$ design. A pairwise balanced design of order v with block sizes from K is a pair (S, B) , B is a family of subsets (blocks) of S that satisfy (1) if $B_i \in B$, then $|B_i| \in K$ and (2) every pair of distinct elements of S occurs in exactly λ blocks of B .

The $6n + 5$ Construction (see [2])

Let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n + 1$, where $Q = \{1, 2, \dots, 2n + 1\}$ and let α be the permutation $(1)(2, 3, \dots, 2n + 1)$. Let $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$ and let B contain the following blocks.

Type 1. $\{\infty_1, \infty_2, (1, 1), (1, 2), (1, 3)\}$.

Type 2. $\{\infty_1, (2i, 1), (2i, 2)\}, \{\infty_1, (2i, 3), ((2i)\alpha, 1)\}, \{\infty_1, ((2i)\alpha, 2), ((2i)\alpha, 3)\}, \{\infty_2, (2i, 1), ((2i)\alpha^{-1}, 3)\},$ for $1 \leq i \leq n$.

Type 3. $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), ((i \circ j)\alpha, 1)\}$ for $1 \leq i \leq 2n+1$.

Then (S, B) is a PBD($6n+5$) with exactly one block of size 5 and the rest of size 3.

A λ -fold triple system is a pair of (S, B) , where S is a finite set and B is a collection of 3-element subsets of S called triples such that each pair of distinct elements of S belongs to exactly λ triples of B .

Theorem 1.2 (see [2, Theorem 2.3.7]). *The spectrum of 2-fold triple systems is precisely the set of all $v \equiv 0$ or $1 \pmod{3}$.*

3n Construction (see [2])

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \dots, n\}$. Let $S = Q \times \{1, 2, 3\}$. We denote B which contains the following two types of triples.

Type 1. $\{(x, 1), (x, 2), (x, 3)\}$ occurs exactly twice in B for all $x \in Q$.

Type 2. For $x \neq y$, the six triples, $\{(x, 1), (y, 1), (x \circ y, 2)\}, \{(y, 1), (x, 1), (y \circ x, 2)\}, \{(x, 2), (y, 2), (x \circ y, 3)\}, \{(y, 2), (x, 2), (y \circ x, 3)\}, \{(x, 3), (y, 3), (x \circ y, 1)\}, \{(y, 3), (x, 3), (y \circ x, 1)\} \in B$.

Then (S, B) is a 2-fold triple system of order $3n$.

3n + 1 Construction (see [2, 4])

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \dots, n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We denote B which contains the following two types of triples.

Type 1. The four triples $\{\infty, (x, 1), (x, 2)\}, \{\infty, (x, 2), (x, 3)\}, \{\infty, (x, 1), (x, 3)\}, \{(x, 1), (x, 2), (x, 3)\} \in B$ for all $x \in Q$

Type 2. If $x \neq y$, the six triples $\{(x, 1), (y, 1), (x \circ y, 2)\}, \{(y, 1), (x, 1), (y \circ x, 2)\}, \{(x, 2), (y, 2), (x \circ y, 3)\}, \{(y, 2), (x, 2), (y \circ x, 3)\}, \{(x, 3), (y, 3), (x \circ y, 1)\}, \{(y, 3), (x, 3), (y \circ x, 1)\} \in B$.

Then (S, B) is a 2-fold triple system of order $3n+1$.

2. Main Results

6n + 4 Construction

Let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n+1$, where $Q = \{1, 2, \dots, 2n+1\}$. Now we construct (S, B) where $S = \{\infty\} \cup \{Q \times \{1, 2, 3\}\}$ and B contains the following blocks.

Type 1. $\{\infty, (i, 1), (i, 2), (i, 3)\} \in B$, for $1 \leq i \leq 2n+1$.

Figure 1: Graphical representation of $(6n + 4)$ construction.

Type 2. $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in B$ for $1 \leq i < j \leq 2n + 1$.

Theorem 2.1. *The ordered pair (S, B) is a pairwise balanced design $2 - (6n + 4, \{4, 3\}, 1)$.*

Proof. The number of points v is $6n + 4$ (see Figure 1). The blocks of Type 1 are of length 4, whereas the blocks of Type 2 are of length 3. It is clear from the construction that any pair of points of S occurs in exactly one block. Hence (S, B) is a PBD $2 - (6n + 4, \{4, 3\}, 1)$.

The number of blocks of the PBD $2 - (6n + 4, \{4, 3\}, 1)$ are $(2n + 1) + (3 \cdot {}^{2n+1}C_2) = (2n + 1)(3n + 1)$. $\square$

For example, $n = 2$, that is, $v = 6 \cdot 2 + 4 = 16$, the quasi-group of order 5 is

$$\begin{array}{c|ccccc}
 o & 1 & 2 & 3 & 4 & 5 \\
 \hline
 1 & 1 & 5 & 2 & 3 & 4 \\
 2 & 5 & 2 & 4 & 1 & 3 \\
 3 & 2 & 4 & 3 & 5 & 1 \\
 4 & 3 & 1 & 5 & 4 & 2 \\
 5 & 4 & 3 & 1 & 2 & 5
 \end{array} \tag{2.1}$$

Let $S = \{\infty, 1, 2, \dots, 15\}$ and B contains the following blocks.

Type 1. $\{\infty, 1, 6, 11\}, \{\infty, 2, 7, 12\}, \{\infty, 3, 8, 13\}, \{\infty, 4, 9, 14\}, \{\infty, 5, 10, 15\} \in B$.

Type 2. $\{1, 2, 10\}, \{1, 3, 7\}, \{1, 4, 8\}, \{1, 5, 9\}, \{2, 3, 9\}, \{2, 4, 6\}, \{2, 5, 8\}, \{3, 4, 10\}, \{3, 5, 6\}, \{4, 5, 7\}, \{6, 7, 15\}, \{6, 8, 12\}, \{6, 9, 13\}, \{6, 10, 14\}, \{7, 8, 14\}, \{7, 9, 11\}, \{7, 10, 13\}, \{8, 9, 15\}, \{8, 10, 11\}, \{9, 10, 12\}, \{11, 12, 5\}, \{11, 13, 2\}, \{11, 14, 4\}, \{11, 15, 4\}, \{12, 13, 4\}, \{12, 14, 1\}, \{12, 15, 3\}, \{13, 14, 5\}, \{13, 15, 1\}, \{14, 15, 2\} \in B$.

So (S, B) is a PBD $2 - (16, \{4, 3\}, 1)$ with a total of 35 blocks in which $2 \cdot 2 + 1 (=5)$ blocks of size 4 and $3 \cdot {}^5C_2 (=30)$ blocks of size 3.

Figure 2: Graphical representation of $(3n + 2)$ construction (Type 1 and Type 2).

Figure 3: Graphical representation of $(3n + 2)$ construction (Type 3).

$3n + 2$ Construction

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \dots, n\}$. Let $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$. We define B as follows.

Type 1. $\{\infty_1, \infty_2, (1, 1), (1, 2), (1, 3)\}$ occurs exactly twice in B .

Type 2. For $1 \leq x \leq n$, $\{\infty_1, (x, 1), (x, 2)\}, \{\infty_1, (x, 2), (x, 3)\}, \{\infty_1, (x, 1), (x, 3)\}, \{\infty_2, (x, 1), (x, 2)\}, \{\infty_2, (x, 2), (x, 3)\}, \{\infty_2, (x, 1), (x, 3)\} \in B$ for all $x \in Q$.

Type 3. If $x \neq y$, the six triples $\{(x, 1), (y, 1), (x \circ y, 2)\}, \{(y, 1), (x, 1), (y \circ x, 2)\}, \{(x, 2), (y, 2), (x \circ y, 3)\}, \{(y, 2), (x, 2), (y \circ x, 3)\}, \{(x, 3), (y, 3), (x \circ y, 1)\}, \{(y, 3), (x, 3), (y \circ x, 1)\} \in B$.

Theorem 2.2. *The ordered pair (S, B) is a 2-fold system of order $3n + 2$.*

Proof. The number of points is $v = 3n + 2$. The Type 1 of new (S, B) construction consists of block length 5. The Type 2 and Type 3 of the above construction consist of length 3. Any pair of points of S occurs in exactly two blocks. Hence the construction (S, B) is a 2-fold system. Also the ordered pair (S, B) holds $2 - (3n + 2, \{3, 5\}, 2)$ design. $\square$

For example, $n = 4$, the quasi-group of order 4 is

$$\begin{array}{c|cccc} o & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 3 & 4 & 2 \\ 2 & 4 & 2 & 1 & 3 \\ 3 & 2 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 & 4 \end{array} \quad (2.2)$$

Let $S = \{\infty_1, \infty_2, 1, 2, \dots, 12\}$ and B contains the following blocks.

Type 1. $\{\infty_1, \infty_2, 1, 5, 9\}, \{\infty_1, \infty_2, 1, 5, 9\} \in B$.

Type 2. $\{\infty_1, 2, 6\}, \{\infty_1, 2, 10\}, \{\infty_1, 6, 10\}, \{\infty_1, 3, 7\}, \{\infty_1, 7, 11\}, \{\infty_1, 3, 11\}, \{\infty_1, 4, 8\}, \{\infty_1, 8, 12\}, \{\infty_1, 4, 12\}, \{\infty_2, 2, 6\}, \{\infty_2, 2, 10\}, \{\infty_2, 6, 10\}, \{\infty_2, 3, 7\}, \{\infty_2, 7, 11\}, \{\infty_2, 3, 11\}, \{\infty_2, 4, 8\}, \{\infty_2, 8, 12\}, \{\infty_2, 4, 12\} \in B$.

Type 3. $\{1, 2, 7\}, \{2, 1, 8\}, \{1, 3, 8\}, \{3, 1, 6\}, \{1, 4, 6\}, \{4, 1, 7\}, \{2, 3, 5\}, \{3, 2, 8\}, \{2, 4, 7\}, \{4, 2, 5\}, \{3, 4, 5\}, \{4, 3, 6\}, \{5, 6, 11\}, \{6, 5, 12\}, \{5, 7, 12\}, \{7, 5, 10\}, \{5, 8, 10\}, \{8, 5, 11\}, \{6, 7, 9\}, \{7, 6, 12\}, \{6, 8, 11\}, \{8, 6, 9\}, \{7, 8, 9\}, \{8, 7, 10\}, \{9, 10, 3\}, \{10, 9, 4\}, \{9, 11, 4\}, \{11, 9, 2\}, \{9, 12, 2\}, \{12, 9, 3\}, \{10, 11, 1\}, \{11, 10, 4\}, \{10, 12, 3\}, \{12, 10, 1\}, \{11, 12, 1\}, \{12, 11, 2\} \in B$.

So (S, B) is a PBD $2 - (14, \{5, 3\}, 1)$ with 2 blocks of size 5 and the rest of the blocks of size 3.

3. Conclusion

There already exist STS for $v = 6n + 1$ and $v = 6n + 3$ and PBD for $v = 6n + 5$. In this paper, we construct PBD for $v = 6n + 4$. The PBD for $v = 6n$ and, $v = 6n + 2$ is still now open. The 2-fold

triple system exists for $v \equiv 0$ or $1 \pmod{3}$ and, in this paper, we construct 2-fold system exists for $3n + 2$. These complete that the 2-fold system holds for all natural numbers n .

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