Research Article

Bianchi Type—IX Barotropic Fluid Model with Time-Dependent Displacement Vector in Lyra Geometry

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Bianchi Type IX barotropic fluid cosmological model in the frame work of Lyra geometry is investigated. To get the deterministic model of the universe, it is assumed that shear (σ) is proportional to expansion (θ). This leads to $a = b^n$, where a and b are metric potentials and n is a constant. To get the results in terms of cosmic time t , we have also considered a special case $\gamma = 0$ (dust filled universe) and $n = 2$. We find that the model starts with a big bang initially and the displacement vector (β) is initially large but decreases due to lapse of time. The models $ds^2 = -(T^6/[3N^2/20 - ((1 - \gamma)/4(5\gamma + 7))T^8 - (\gamma/(5\gamma + 4))T^6])dT^2 + T^4dX^2 + T^2dY^2 + (T^2\sin^2Y + T^4\cos^2Y)dZ^2 - 2T^4\cos YdXdZ$ and $ds^2 = -d\tau^2 + [\sqrt{21/5}N \sin((2/\sqrt{7})\tau)]dx^2 + [\sqrt{21/5}N \sin((2/\sqrt{7})\tau)]^{1/2}dy^2 + ([\sqrt{21/5}N \sin((2/\sqrt{7})\tau)]^{1/2}\sin^2y + [\sqrt{21/5}N \sin((2/\sqrt{7})\tau)]\cos^2y)dz^2 - 2[\sqrt{21/5}N \sin((2/\sqrt{7})\tau)]\cos ydxdz$ have point-type singularity at $T = 0$ and $\tau = 0$, respectively. The physical and geometrical aspects of the models are also discussed.

1. Introduction

Bianchi Type IX space-time is the generalization of FRW model with positive curvature. Bianchi Types cosmological models create more interest in the study, because familiar models like Robertson-Walker model [1], the de-Sitter universe [2], Taub-NUT [3, 4] space times are of Bianchi Type IX space-time. The solutions [3, 4] allow expansion, rotation, and shear. Vaidya and Patel [5] have obtained the solution for spatially homogeneous Bianchi Type IX space time and have given a general scheme for the derivation of exact solutions of Einstein's field equations corresponding to a perfect fluid and pure radiation field. Bianchi Type IX space times are also studied by many research workers namely Krori et al. [6], Chakraborty and Nandy [7], Chakraborty [8], and Bali and Upadhaya [9].

By geometrizing gravitation, Einstein derived the field equations of general relativity. Weyl [10] developed a theory to geometrize gravitation and electromagnetism inspired by the idea of geometrizing gravitation of Einstein. But Weyl's theory was discarded due to nonintegrability of length of vector under parallel displacement. Lyra [11] modified Riemannian geometry by introducing a gauge function into the structureless manifold. This step removed the main obstacle of Weyl's theory [10] and made length of vector integrable under parallel displacement. Sen [12] investigated an analogue of Einstein's field equation by introducing a new scalar theory of gravitation. Halford [13] pointed out that constant displacement vector (ϕ_μ) in Lyra geometry plays the role of cosmological constant in General Relativity. A number of authors, namely, T. Singh and G. P. Singh [14], Rahman and Bera [15], Rahman et al. [16], Pradhan et al. [17–19], Bali and Chandnani [20, 21], Ram et al. [22], and Bali et al. [23], have investigated cosmological models for different Bianchi space time under different contexts in the frame work of Lyra geometry.

In this paper, we have investigated Bianchi Type IX barotropic fluid cosmological model in the frame work of Lyra geometry. To get the deterministic model, we have assumed that the shear (σ) is proportional to expansion (θ). We have also considered the dust distribution ($p = 0$) model to get the result in terms of cosmic time. We find that the model starts with a big bang initially and expansion decreases as time increases. The displacement vector is initially large but decreases due to lapse of time. The physical and geometrical aspects of the models are also discussed.

2. The Metric and Field Equations

We consider Bianchi Type IX metric in the form

$$ds^2 = -dt^2 + a^2 dx^2 + b^2 dy^2 + (b^2 \sin^2 y + a^2 \cos^2 y) dz^2 - 2a^2 \cos y dx dz, \quad (2.1)$$

where a and b are functions of t -alone.

The energy momentum tensor (T_i^j) for perfect fluid distribution is given by

$$T_i^j = (\rho + p)v_i v^j + p g_i^j. \quad (2.2)$$

The modified Einstein's field equation in normal gauge for Lyra's manifold obtained by Sen [12] is given by

$$R_i^j - \frac{1}{2} R g_i^j + \frac{3}{2} \phi_i \phi^j - \frac{3}{4} \phi_k \phi^k g_i^j = -T_i^j, \quad (2.3)$$

(in geometrized units where $8\pi G = 1$ and $c = 1$) where $v_i = (0, 0, 0, -1)$; $v_i v^i = -1$, $\phi_i = (0, 0, 0, \beta(t))$, p is the isotropic pressure, ρ the matter density, v^i the fluid flow vector, and β the gauge function.

The modified Einstein's field equation (2.3) for the metric (2.1) leads to

$$\frac{2b_{44}}{b} + \frac{b_4^2}{b^2} + \frac{1}{b^2} - \frac{3}{4} \frac{a^2}{b^4} + \frac{3}{4} \beta^2 = -p, \quad (2.4)$$

$$\frac{a_4 b_4}{ab} + \frac{b_{44}}{b} + \frac{a_{44}}{a} + \frac{a^2}{4b^4} + \frac{3}{4} \beta^2 = -p, \quad (2.5)$$

$$\frac{2a_4 b_4}{ab} + \frac{b_4^2}{b^2} + \frac{1}{b^2} - \frac{a^2}{4b^4} - \frac{3}{4} \beta^2 = \rho. \quad (2.6)$$

Equations (2.5) and (2.6) after using barotropic condition $p = \gamma\rho$ lead to

$$(2\gamma + 1) \frac{a_4 b_4}{ab} + \gamma \frac{b_4^2}{b^2} + \frac{\gamma}{b^2} + \frac{b_{44}}{b} + \frac{a_{44}}{a} + (1 - \gamma) \frac{a^2}{4b^4} + \frac{3}{4} (1 - \gamma) \beta^2 = 0. \quad (2.7)$$

The conservation equation $T_{i;j}^j = 0$ leads to

$$\frac{3}{2} \beta \beta_4 + \frac{3}{2} \beta^2 \left(\frac{a_4}{a} + \frac{2b_4}{b} \right) = 0, \quad (2.8)$$

which leads to

$$\beta = \frac{N}{ab^2}, \quad (2.9)$$

N being a constant of integration.

3. Solution of Field Equations

For deterministic model, we assume that the shear (σ) is proportional to the expansion (θ). This leads to

$$a = b^n, \quad (3.1)$$

where $\sigma = \sqrt{2/3}(a_4/a - b_4/b)$, $\theta = a_4/a + 2b_4/b$, n being a constant.

Equation (3.1) leads to

$$\frac{a_4}{a} = n \frac{b_4}{b}, \quad (3.2)$$

$$\frac{a_{44}}{a} = (n^2 - n) \left(\frac{b_4}{b} \right)^2 + n \frac{b_{44}}{b}. \quad (3.3)$$

Using (2.9)–(3.3) in (2.7), we have

$$2b_{44} + \frac{2(2\gamma n + n^2 + \gamma)}{(n+1)} \frac{b_4^2}{b} = -\frac{2\gamma}{b(n+1)} - \frac{(1-\gamma)b^{2n-3}}{2(n+1)} - \frac{3}{2} \frac{(1-\gamma)}{(n+1)} \frac{N^2}{b^{2n+3}}. \quad (3.4)$$

To get the simplified result, we assume $n = 2$, thus (3.4) leads to

$$2b_{44} + \frac{2(5\gamma + 4)}{3} \frac{b_4^2}{b} = -\frac{2\gamma}{3b} - \frac{(1-\gamma)}{6} b - \frac{(1-\gamma)}{2} \frac{N^2}{b^7}. \quad (3.5)$$

To find the solution of (3.5), we assume

$$b_4 = f(b). \quad (3.6)$$

Thus

$$b_{44} = f f', \quad (3.7)$$

where

$$f' = \frac{df}{db}. \quad (3.8)$$

Therefore, (3.5) leads to

$$\frac{df^2}{db} + \frac{2(5\gamma + 4)}{3b} f^2 = -\frac{2\gamma}{3b} - \frac{(1-\gamma)}{6} b - \frac{(1-\gamma)}{2} \frac{N^2}{b^7} \quad (3.9)$$

which again leads to

$$f^2 = -\frac{\gamma}{(5\gamma + 4)} - \frac{(1-\gamma)}{4(5\gamma + 7)} b^2 + \frac{3N^2}{20} b^{-6}, \quad (3.10)$$

where constant of integration has been assumed zero.

Equation (3.10) leads to

$$\left(\frac{db}{dt}\right)^2 = b^{-6} \left[\frac{3N^2}{20} - \frac{(1-\gamma)}{4(5\gamma + 7)} b^8 - \frac{\gamma}{5\gamma + 4} b^6 \right]. \quad (3.11)$$

Thus, the metric (2.1) can be written in the form

$$ds^2 = -\frac{T^6}{[3N^2/20 - ((1-\gamma)/4(5\gamma + 7))T^8 - (\gamma/(5\gamma + 4))T^6]} dT^2 + T^4 dX^2 + T^2 dY^2 \\ + (T^2 \sin^2 Y + T^4 \cos^2 Y) dZ^2 - 2T^4 \cos Y dX dZ, \quad (3.12)$$

where $T = b$, $x = X$, $y = Y$, $z = Z$, and cosmic time t is given by

$$t = \int \frac{T^3}{[3N^2/20 - ((1-\gamma)/4(5\gamma+7))T^8 - (\gamma/(5\gamma+4))T^6]^{1/2}} dT. \quad (3.13)$$

4. Some Physical and Geometrical Properties

The displacement vector (β) is given by (2.9) as

$$\beta = \frac{N}{ab^2} = \frac{N}{T^4}. \quad (4.1)$$

The expansion (θ) is given by

$$\theta = \frac{a_4}{a} + \frac{2b_4}{b}, \quad (4.2)$$

which leads to

$$\theta = \frac{4}{T^4} \left[\frac{3N^2}{20} - \frac{(1-\gamma)}{4(5\gamma+7)} T^8 - \frac{\gamma T^6}{5\gamma+4} \right]^{1/2}. \quad (4.3)$$

The shear (σ) is given by

$$\sigma^2 = \frac{2}{3} \left(\frac{a_4}{a} - \frac{b_4}{b} \right)^2, \quad (4.4)$$

which leads to

$$\sigma^2 = \frac{2}{3T^8} \left[\frac{3N^2}{20} - \frac{(1-\gamma)}{4(5\gamma+7)} T^8 - \frac{\gamma T^6}{5\gamma+4} \right]. \quad (4.5)$$

The matter density (ρ) is given by

$$\rho = \frac{5}{T^8} \left[\frac{3N^2}{20} - \frac{(1-\gamma)}{4(5\gamma+7)} T^8 - \frac{\gamma T^6}{5\gamma+4} \right] + \frac{1}{T^2} - \frac{1}{4} - \frac{3}{4} \frac{N^2}{T^8}, \quad (4.6)$$

which leads to

$$\rho = -\frac{5}{T^8} \left[\frac{(1-\gamma)}{4(5\gamma+7)} T^8 + \frac{\gamma T^6}{(5\gamma+4)} \right] + \frac{1}{T^2} - \frac{1}{4}, \quad (4.7)$$

which again leads to

$$\rho = \frac{4}{T^2(5\gamma + 4)} - \frac{3}{5\gamma + 7}, \quad (4.8)$$

and the isotropic pressure is given by

$$p = \gamma\rho = \gamma \left[\frac{4}{T^2(5\gamma + 4)} - \frac{3}{5\gamma + 7} \right]. \quad (4.9)$$

The spatial volume (V^3) is given by

$$V^3 = T^4. \quad (4.10)$$

5. Special Case: Dust Model ($p = 0$)

To get the model of dust filled universe, we assume that $n = 2$, and using $\gamma = 0$ in (3.5), we get

$$2b_{44} + \frac{8}{3} \frac{b_4^2}{b} = -\frac{1}{6}b - \frac{N^2}{2b^7}, \quad (5.1)$$

which leads to

$$\left(\frac{db}{dt} \right)^2 = \frac{3N^2}{20} \frac{1}{b^6} - \frac{1}{28}b^2, \quad (5.2)$$

which after integration leads to

$$b^2 = \left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right]^{1/2}, \quad (5.3)$$

where $t + \ell = \tau$, ℓ being constant of integration.

Thus, (2.1) takes the form

$$\begin{aligned} ds^2 = & -d\tau^2 + \left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right] dx^2 + \left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right]^{1/2} dy^2 \\ & + \left(\left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right]^{1/2} \sin^2 y + \left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right] \cos^2 y \right) dz^2 \\ & - 2 \left[\sqrt{\frac{21}{5}} N \sin\left(\frac{2}{\sqrt{7}}\tau\right) \right] \cos y dx dz. \end{aligned} \quad (5.4)$$

The displacement vector (β) is given by (2.9)

$$\beta = \frac{N}{ab^2} = \sqrt{\frac{5}{21}} \operatorname{cosec}\left(\frac{2}{\sqrt{7}}\tau\right). \quad (5.5)$$

The expansion (θ) is given by

$$\theta = \frac{a_4}{a} + \frac{2b_4}{b} = \frac{2}{\sqrt{7}} \cot\left(\frac{2}{\sqrt{7}}\tau\right). \quad (5.6)$$

The shear (σ) is given by

$$\sigma = \sqrt{\frac{2}{3}} \left(\frac{a_4}{a} - \frac{b_4}{b} \right). \quad (5.7)$$

Thus,

$$\sigma = \frac{1}{\sqrt{42}} \cot \tau. \quad (5.8)$$

The matter density (ρ) is given by (2.6)

$$\rho = \frac{1}{\left[\sqrt{21/5} N \sin\left(\left(2/\sqrt{7}\right)\tau\right) \right]^{1/2}} - \frac{3}{7}. \quad (5.9)$$

6. Discussion

The model (3.12) starts with a big bang at $T = 0$, and the expansion in the model decreases as T increases. The displacement vector (β) is initially large but decreases due to lapse of time. Since $\sigma/\theta \neq 0$, hence anisotropy is maintained throughout. The reality condition $\rho > 0$ implies that the model exists during the span of time given by

$$T < \sqrt{\frac{4(5\gamma + 7)}{3(5\gamma + 4)}}. \quad (6.1)$$

The model (3.12) has point type singularity at $T = 0$ (MacCallum [24]). The spatial volume increases as T increases.

The model (5.4) starts with a big bang at $\tau = 0$, and the expansion in the model decreases as τ increases. The displacement vector (β) is initially large but decreases due to

lapse of time. Since $\sigma/\theta \neq 0$, hence anisotropy is maintained throughout. The reality condition $\rho > 0$ implies that

$$\sin\left(\frac{2}{\sqrt{7}}\tau\right) < \frac{7\sqrt{35}}{9N\sqrt{3}}, \quad (6.2)$$

where $0 < N < 1$.

The model has point type singularity at $\tau = 0$. (MacCallum [24]).

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