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Research Article

Taylor's Meshless Petrov-Galerkin Method for the Numerical Solution of Burger's Equation by Radial Basis Functions

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During the last two decades, there has been a considerable interest in developing efficient radial basis functions (RBFs) algorithms for solving partial differential equations (PDEs). In this paper, we introduce the Petrov-Galerkin method for the numerical solution of the one-dimensional nonlinear Burger equation. In this method, the trial space is generated by the multiquadric (MQ) RBF and the test space is generated by the compactly supported RBF. In the time discretization of the equation, the Taylor series expansion is used. This method is applied on some test experiments, and the numerical results have been compared with the exact solutions. The L_2, L_∞, and root-mean-square (RMS) errors in the solutions show the efficiency and the accuracy of the method.

1. Introduction

Burger's equation serves as a useful model for many interesting problems in applied mathematics and related sciences. It is in the models of effectively certain problems of a fluid flow nature, wherein either shocks or viscous dissipation is a significant factor.

The first steady-state solution of Burger's equation were given by Bateman [1] in 1915. However, the equation gets its name from the extensive research of Burgers [2] beginning in 1939. This equation has a large variety of applications in modeling of water in unsaturated soil, dynamics of soil water, statistics of flow problems, mixing and turbulent diffusion, cosmology, and seismology [3–5].

In one-dimension case, Burger's equation is given by

$$u_t + uu_x - \mu u_{xx} = 0, \quad x \in \Omega = [a,b],\ t \geqslant t_0, \tag{1.1}$$

where $\mu > 0$ is a viscosity constant and the subscripts x and t denote the space and time differentiations, respectively. The initial condition and the boundary conditions are given by

$$u(x, t_0) = u_0(x), \tag{1.2}$$

$$u(a, t) = g_1(t), \qquad u(b, t) = g_2(t). \tag{1.3}$$

Various numerical methods have been introduced to solve Burger's equation, such as cubic splines and finite differences [6], compact differencing method [7, 8], finite element method (FEM) [9], the Tau method [10], and the method of lines [11].

Many of these methods need the mesh generation and provide the solution of the problem on mesh points. But in recent years, other methods, namely, meshless methods, have been provided, which do not need to generate the mesh. In a meshless method, a set of scattered nodes is used instead of meshing the domain of the problem. A powerful tool for the scattered data interpolation problem is the RBFs. In the last two decades, the development of the RBFs as a truly meshless method for approximating the solutions of PDEs has drown the attention of many researchers in science and engineering. The initial development was due to the pioneering work of Kansa [12, 13] who directly collocated the RBFs for the approximated solution of the equations.

To date, useful meshless methods have been provided for the nonlinear PDEs, wherein the RBFs are used, such as the collocation method for solving Burger's equation [14], Korteweg-de Vries equation [15, 16], and sine-Gordon equation [17].

In this paper, we present the other meshless method, which is called the meshless local Petrov-Galerkin (MLPG) method. This method was first introduced by Atluri and Zhu [18]. The MLPG method is very general, can be based on the symmetric or unsymmetric local weak forms of the PDEs, and uses a variety of interpolation method for test and trial functions. Hence, we use the local unsymmetric weak form of the problem, and the MQ RBF and the compactly supported RBF are chosen as the trial and the test functions, respectively.

In classical methods, using low order time integration for transient problems may cause loss in the accuracy of the numerical schemes. On the other hand, increasing the order of time discretization can yield better results. The main idea behind the present time integration is to use more time derivatives in the Taylor series expansion. Recently, Dağ et al. [19] have developed the Taylor-Galerkin and the Taylor-collocation methods for the numerical solution of Burger's equation by B-splines. The objective of the present paper is to obtain the numerical solution of Burger's equation by combining the MLPG method with the aforesaid time discretization of the Taylor series expansion that according to the best of our knowledge, this is the first demonstration of the application of it.

The paper is organized as follows. A brief knowledge of the RBFs interpolation is given in Section 2. In Section 3, the time discretization of Burger's equation is introduced, and in Section 4, the MLPG method is presented. In Section 5, we give several numerical experiments to demonstrate the accuracy and efficiency of our meshless numerical scheme, and Section 6 contains some conclusion.

2. Radial Basis Function Interpolation

In order to explain multivariate scattered data interpolation by RBFs, suppose a data vector $f|_X = (f(x_1), \ldots, f(x_N))^T \in \mathbb{R}^d$ of function values, sampled from an unknown function f :

$\mathbb{R}^d \to \mathbb{R}$ at a scattered finite point set $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, $d \geq 1$, is given. Scattered data interpolation requires computing a suitable interpolant $S : \mathbb{R}^d \to \mathbb{R}$ satisfying $S|_X = f|_X$, that is,

$$S(x_j) = f(x_j), \quad j = 1, \ldots, N. \tag{2.1}$$

To this end, the RBF interpolation scheme works with a fixed radial function $\varphi : [0, \infty) \to \mathbb{R}$, and the interpolant S in (2.1) is assumed to have the form

$$S(x) = \sum_{j=1}^{N} \lambda_j \varphi(\|x - x_j\|) + \sum_{i=1}^{M} c_i p_i(x), \quad x \in \mathbb{R}^d, \tag{2.2}$$

where $\|\cdot\|$ denotes the Euclidean distances and $p_1, \ldots, p_M$ form a basis for the $M = \binom{m-1+d}{m-1}$-dimensional linear space Π^d_{m-1} of polynomials of total degree less than or equal to $m - 1$ in d variables.

Since enforcing the interpolation conditions in (2.1) leads to a system of N linear equations in the $N + M$ unknowns λ_j and c_i, one usually adds the M additional conditions

$$\sum_{j=1}^{N} \lambda_j p_i(x_j) = 0, \quad i = 1, \ldots, M. \tag{2.3}$$

In general, solving the interpolation problem based on the extended expansion (2.2), now amounts of solving a system to linear equations of the form:

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} = \begin{pmatrix} f|_X \\ 0 \end{pmatrix}, \tag{2.4}$$

where the pieces are given by $A_{ij} = \varphi(\|x_i - x_j\|)$, $i, j = 1, \ldots, N$, $P_{ik} = p_k(x_i)$, $i = 1, \ldots, N$, $k = 1, \ldots, M$, $\Lambda = [\lambda_1, \ldots, \lambda_N]^T$, $C = [c_1, \ldots, c_M]^T$, and 0 is a zero vector of length M.

From [20, 21], we know that we have a unique interpolant $S(x)$ of f if $\varphi(r)$ is a conditional positive definite RBF of order m. The commonly used RBFs are MQ, Gaussian, and compactly supported RBFs, see Table 1. The MQ has been found to provide the most accurate approximation in most of the applications of the RBFs [22]. For the MQ RBF, $\varphi(r) = \sqrt{r^2 + c^2}$, we have $m = 1$. In this case and for the one-dimensional space, the linear basis function is given by $p(x) = 1$. Therefore, in one-dimensional and for the MQ approximation scheme, (2.2) yields the following expansion:

$$S(x) = \sum_{j=1}^{N} \lambda_j \varphi(\|x - x_j\|) + c_1, \quad x \in \mathbb{R}, \tag{2.5}$$

and (2.3) yields

$$\sum_{j=1}^{N} \lambda_j = 0. \tag{2.6}$$

Table 1: Different radial basis functions.

Radial basis functions	$\phi(r)$	Parameters	Order
Gaussian	$\exp(-cr^2)$	$c > 0$	$m \geq 0$
Multiquadrics	$(-1)^{\lceil\beta\rceil}(c^2 + r^2)^{\beta}$	$\beta > 0,\ \beta \notin \mathbb{N}$	$m \geq \lceil\beta\rceil$
Inverse multiquadrics	$(c^2 + r^2)^{\beta}$	$\beta < 0$	$m \geq 0$
Polyharmonic splines	$(-1)^{\lceil\beta/2\rceil} r^{\beta}$	$\beta > 0, \beta \notin 2\mathbb{N}$	$m \geq \lceil\beta/2\rceil$
Polyharmonic splines	$(-1)^{k+1} r^{2k} \log(r)$	$k \in \mathbb{N}$	$m \geq k + 1$
Compactly supported	$(1 - r)_+^4 (1 + 4r)$	$d \leq 3$	$m \geq 0$

From this equation, we obtain

$$\lambda_1 = -\sum_{j=2}^{N} \lambda_j. \tag{2.7}$$

Substituting (2.7) into (2.5) yields

$$S(x) = \sum_{j=2}^{N} \lambda_j \left(\varphi\left(\left\|x - x_j\right\|\right) - \varphi(\|x - x_1\|)\right) + c_1, \quad x \in \mathbb{R}, \tag{2.8}$$

or in the compact form, we obtain

$$S(x) = \sum_{j=1}^{N} \lambda_j g_j(x), \quad x \in \mathbb{R}, \tag{2.9}$$

where the basis functions $g_j(x)$ are given by

$$g_1(x) = 1, \qquad g_j(x) = \varphi\left(\left\|x - x_j\right\|\right) - \varphi(\|x - x_1\|), \quad j = 2, \ldots, N. \tag{2.10}$$

For more details about RBFs, see [23].

The variable ε in the MQ RBF, $\varphi(r) = \sqrt{1 + (\varepsilon r)^2}$, wherein c is replaced by $1/\varepsilon$, is known as the shape parameter controls the shape of the functions. The shape parameter ε influences the solution profoundly. Since the exact value of ε cannot be calculated, empirical studies have been made by researchers in the past to obtain a suitable value for the shape parameter ε. There are many methods for choosing the optimal value of ε, such as the brute force method [24] and the leave-one-out cross-validation (LOOCV) algorithm [25]. Some other methods for finding the optimal shape parameter were produced by Kansa and Hon [26], Hardy [27], and Franke [28].

In this paper, we use the value of the shape parameter that is suggested by Hardy [27]. Hardy suggested the value

$$c = 0.815d, \quad d = \frac{\sum_{i=1}^{N} d_i}{N}, \tag{2.11}$$

where d_i is the distance from the ith center to the nearest neighbor and N is the number of centers.

3. Time Discretization

For the time discretization of (1.1), we use the Taylor series expansion. In this approach, the term $u_t^n = u_t(x, t_n)$, $t_{n+1} = t_n + \Delta t$, is arranged with the help of the Taylor series expansion as

$$u_t^n = \frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2} u_{tt}^n + O\left(\Delta t^2\right). \tag{3.1}$$

Differentiating (1.1) with respect to time, u_{tt}^n may be written by

$$u_{tt}^n = \left(-u^n u_x^n + \mu u_{xx}^n\right)_t = -u^n \left(u_t^n\right)_x - u_x^n u_t^n + \mu \left(u_t^n\right)_{xx}. \tag{3.2}$$

For the time derivative u_t^n in (3.2), using forward difference formula, u_{tt}^n can be rewritten as

$$\Delta t u_{tt}^n = -u^n \left(u_x^{n+1} - u_x^n\right) - u_x^n \left(u^{n+1} - u^n\right) + \mu \left(u_{xx}^{n+1} - u_{xx}^n\right). \tag{3.3}$$

Substituting (3.3) into (3.1) and using the resulting expression in (1.1) yield the following time-discretized form of Burger's equation:

$$2u^{n+1} + \Delta t u^n u_x^{n+1} + \Delta t u_x^n u^{n+1} - \mu \Delta t u_{xx}^{n+1} = 2u^n + \mu \Delta t u_{xx}^n. \tag{3.4}$$

4. The Meshless Local Petrov-Galerkin Method

The MLPG approach was first proposed by Atluri and Zhu [18] for solving linear potential problems, by using domain discretization technique. The MLPG approach uses either a local symmetric weak form, or an unsymmetric weak form of the governing equation over the local subdomain.

In the application of a local Petrov-Galerkin scheme to the Burger's equation, (3.4) is multiplied with the weight function w and the resulting equation is integrated over the local subdomain such as Ω_s, which is a small region taken for each node in the global domain Ω. Therefore, the local weak form,

$$\int_{\Omega_s^i} \left(2u^{n+1} + \Delta t u^n u_x^{n+1} + \Delta t u_x^n u^{n+1} - \mu \Delta t u_{xx}^{n+1}\right) w \, dx = \int_{\Omega_s^i} \left(2u^n + \mu \Delta t u_{xx}^n\right) w \, dx, \tag{4.1}$$

is obtained which requires u to be twice differentiable and u_x to be continuous. If not, u_{xx} will be infinity where u_x is discontinuous and (4.1) makes no sense. On the other hand, there is no continuity requirement on the weight function w in (4.1). Thus, the requirements on u and w are unsymmetric, in order to be admissible in (4.1). Hence, we denote (4.1) as a local unsymmetric weak form (LUWF). The weight function w is often referred to as the test function. In (4.1), Ω_s^i is a local subdomain associated with the point i.

In this paper, the compactly supported RBFs are chosen as the test functions. In this case, Ω_s^i is support of the compactly supported RBFs. The compactly supported RBFs are generally expressed in the form $\psi(r) = (1-r)_+^n p(r)$ (Wu [29] and Wendland [30]), and

$$(1-r)_+^n = \begin{cases} (1-r)^n, & 0 \leqslant r < 1, \\ 0, & r \geqslant 1, \end{cases} \tag{4.2}$$

where $p(r)$ is a prescribed polynomial. By replacing r with r/δ for $\delta > 0$, the basis function has support on $[0, \delta]$. In the numerical results for compactly supported RBF, the Wendland function is used as follow:

$$\psi(r) = (1-r)_+^8 \left(32r^3 + 25r^2 + 8r + 1\right). \tag{4.3}$$

In LUWF (4.1), the collocation approach is used to impose both the essential as well as natural boundary conditions. The 10-point Gauss quadrature rule [31] is used for the numerical integration of (4.1).

5. Numerical Experiments

Three test experiments are studied to investigate the robustness and the accuracy of the proposed method. The L_2, L_∞, and RMS errors, which are defined by

$$\begin{aligned} L_2 &= \left\| u^N - u \right\|_2 = \sqrt{h \sum_{i=1}^{N} \left(u_j^N - u_j \right)^2}, \\ L_\infty &= \left\| u^N - u \right\|_\infty = \max_j \left| u_j^N - u_j \right|, \\ \text{RMS} &= \sqrt{\frac{\sum_{i=1}^{N} \left(u_j^N - u_j \right)^2}{N}}, \end{aligned} \tag{5.1}$$

are used to measure the accuracy wherein u^N is the approximate solution and u is the exact solution of (1.1).

The computations associated with the experiments discussed above were performed in Maple 13 on a PC with a CPU of 2.4 GHZ.

Experiment 1. In this experiment, we consider the shock propagation solution of Burger's equation [19] as a numerical experiment. This solution is given by

$$u(x,t) = \frac{x}{t\left(1 + \sqrt{t/t_*}\, \exp\left(x^2/4\mu t\right)\right)}, \quad t \geqslant 1,\ t_* = \exp\left(\frac{1}{8\mu}\right). \tag{5.2}$$

Initial condition of the problem is obtained from (5.2) at time $t = 1$, and boundary conditions in (1.3) can be obtained from the exact solution. Propagation of the shock is studied with

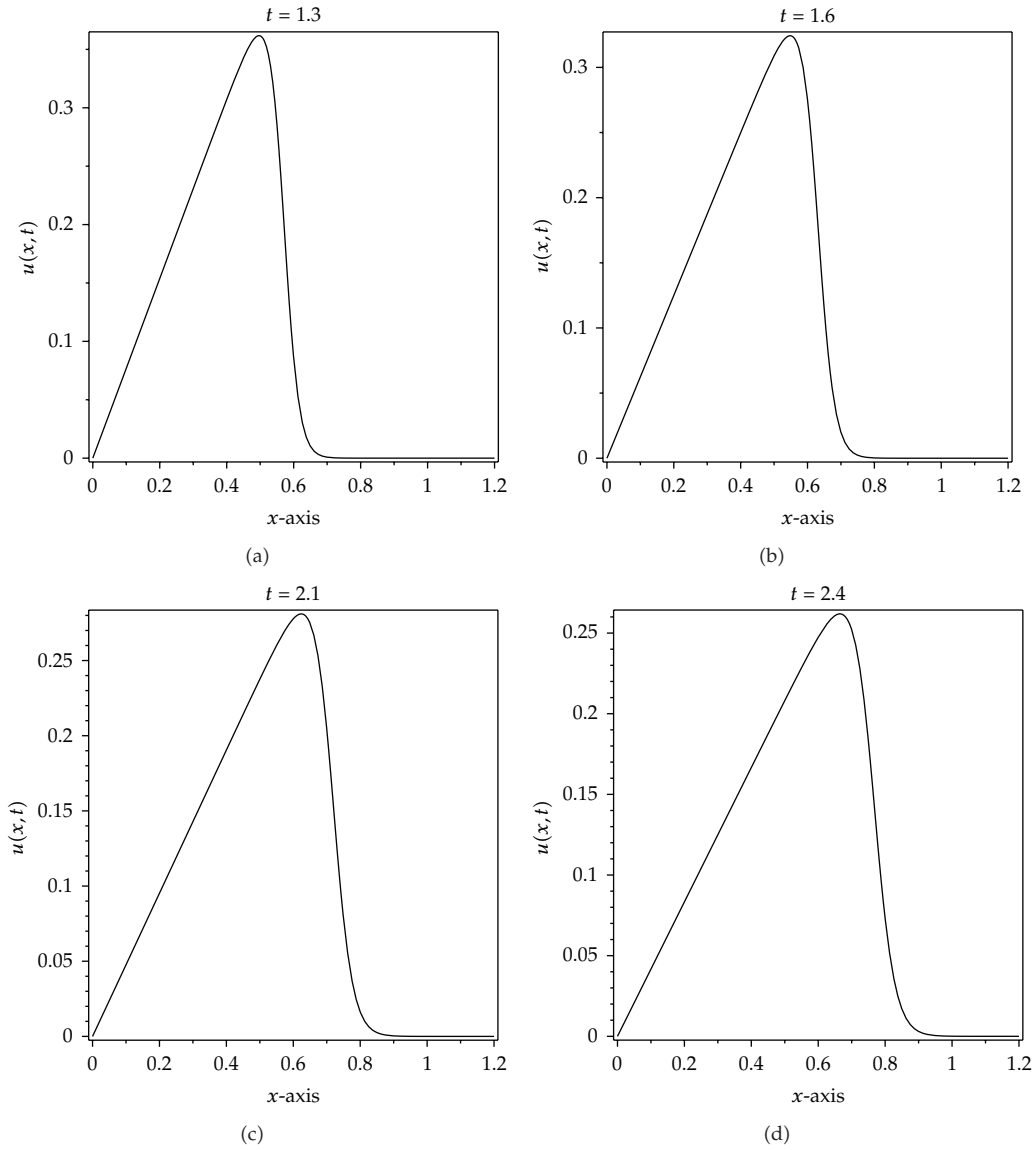

Figure 1: Solution profiles for $\mu = 0.005$, $\Delta t = 0.01$, and $N = 30$ at different times of Experiment 1.

the parameters $\mu = 0.005$ and $\Delta t = 0.01$ over the solution domain $[0, 1.2]$. The computational results are listed in Tables 2, 3, and 4. We also plot the profiles of the solutions at $t = 1.3, 1.6, 2.1$, and 2.4 in Figure 1. Figure 2 represents the percentage absolute error at different time levels. Figure 3 represents the percentage absolute error for various values of N at the time $t = 1.7$ of Experiment 1.

According to Table 3, we observe that first the value of errors decrease, but for $N = 40$ these values increase. When we add center points in order to improve the accuracy, the condition number of the interpolation matrix grows and then the problem becomes ill-conditioned. The ill-conditioning grows due to the decrease in the distance between points, and not to the increase in the number of center points [32].

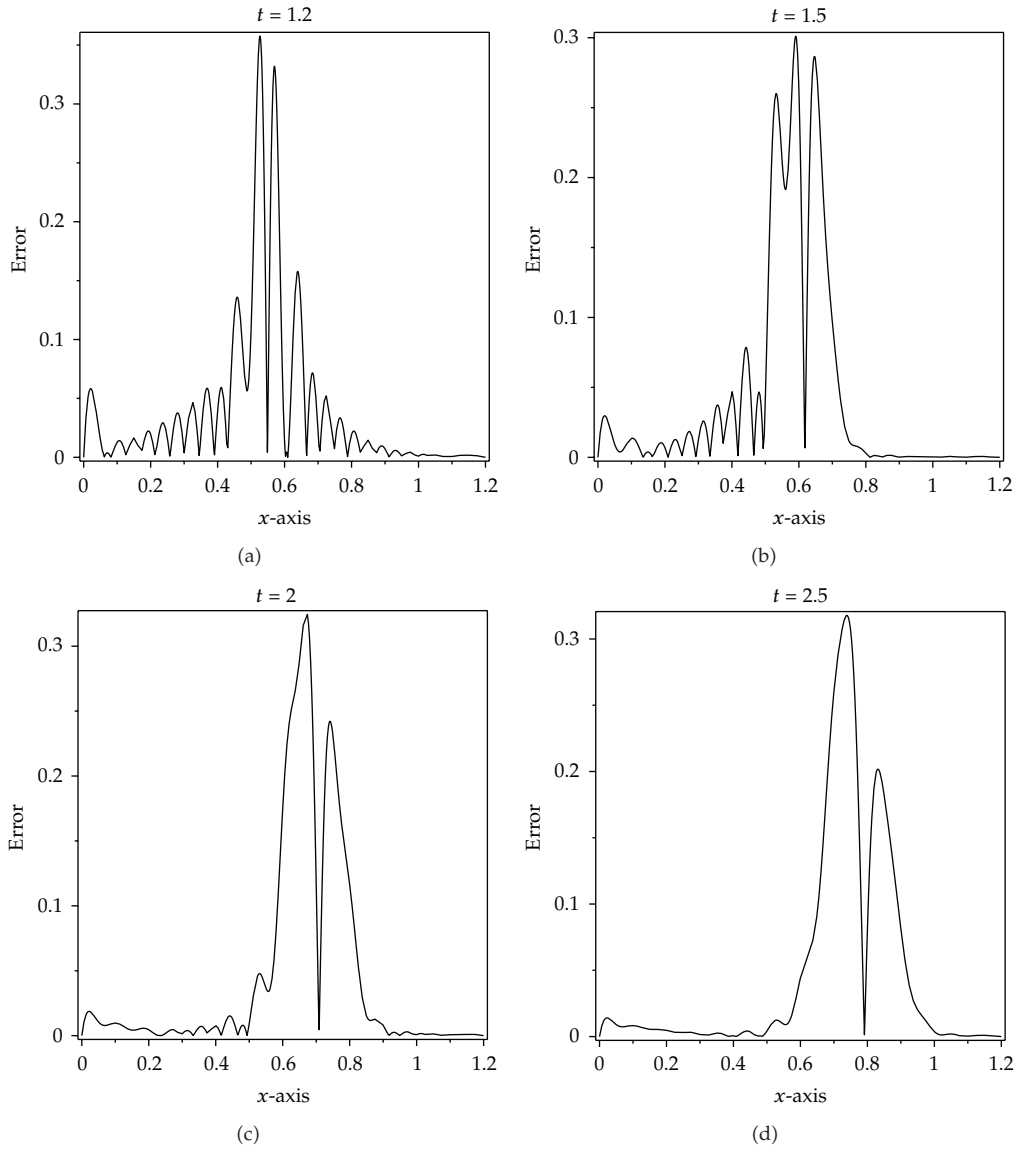

Figure 2: The percentage absolute error for $\mu = 0.005$, $\Delta t = 0.01$, and $N = 30$ at different times of Experiment 1.

Experiment 2. In this experiment, we consider Burger's equation (1.1) with exact solution [33],

$$u(x,t) = \frac{\alpha + \beta + (\beta - \alpha)\exp(\alpha/\beta(x - \beta t - \lambda))}{1 + \exp(\alpha/\beta(x - \beta t - \lambda))}, \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0, \tag{5.3}$$

where the parameters α, β, and λ are arbitrary constants. Initial condition of the problem is obtained from (5.3) at time $t = 0$, and boundary conditions in (1.3) can be obtained from

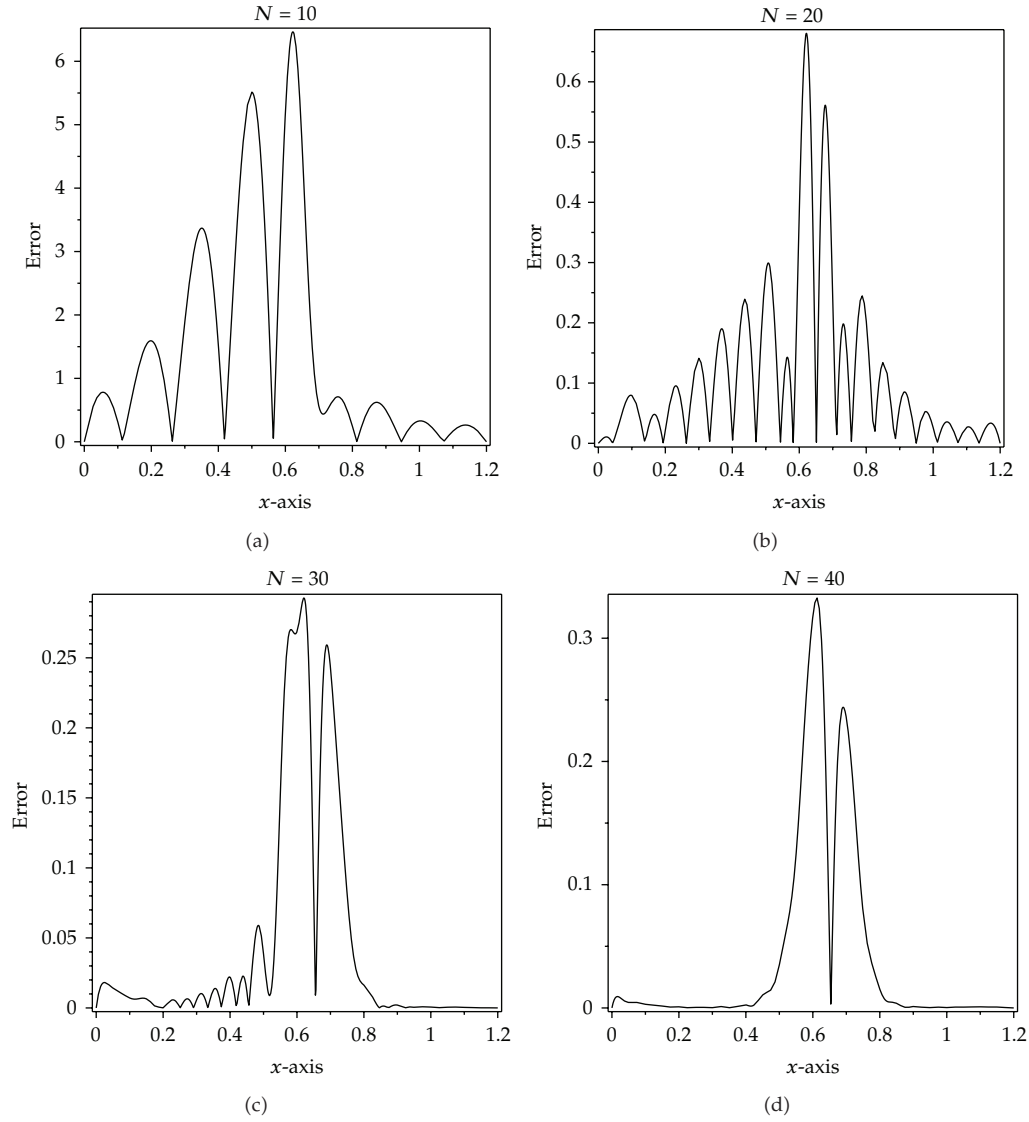

Figure 3: The percentage absolute error for $\mu = 0.005, \Delta t = 0.01$, and various values of N at $t = 1.7$ of Experiment 1.

the exact solution. We study this experiment with parameters $\alpha = 0.4$, $\beta = 0.6$, $\lambda = 0.125$, and $\mu = 1$. The computational results are listed in Tables 5, 6, and 7. We also plot profiles of the percentage absolute error at different time levels in Figure 4.

Experiment 3. In this experiment, we study (1.1) with the exact solution [34],

$$u(x,t) = \frac{0.1e^{-A} + 0.5e^{-B} + e^{-C}}{e^{-A} + e^{-B} + e^{-C}}, \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0, \tag{5.4}$$

Table 2: Comparison of results for $\mu = 0.005$, $\Delta t = 0.01$, and $N = 30$ at $t = 2$ of Experiment 1.

x	Exact solution	Approximation solution	Absolute error
0.1	0.0499996616	0.0499026831	0.0000969784
0.2	0.0999985674	0.0999517703	0.0000467971
0.3	0.1499924999	0.1500053758	0.0000128759
0.4	0.1999424670	0.2000195386	0.0000770716
0.5	0.2493193396	0.2492266449	0.0000926947
0.6	0.2877130559	0.2859612890	0.0017517669
0.7	0.1665563722	0.1653680358	0.0011883364
0.8	0.0083625462	0.0095237900	0.0178863362
0.9	0.0001370197	0.0002155982	0.0000785785
1.0	0.0000013175	−0.000007072	0.0000083903
1.1	$7.605374e-9$	−0.000008593	0.0000086008

Table 3: Errors for $\mu = 0.005$, $\Delta t = 0.01$, and various values of N at $t = 1.7$ of Experiment 1.

N	L_2	L_∞	RMS
10	$1.848057055e-2$	$3.92566434e-2$	$1.600464358e-2$
20	$2.067810738e-3$	$5.87098190e-3$	$1.839848088e-3$
30	$9.922837806e-4$	$2.93619882e-4$	$8.906019490e-4$
40	$1.000834848e-3$	$3.30824890e-3$	$9.021403401e-4$

Table 4: Errors at different times for $\mu = 0.005$, $\Delta t = 0.01$, and $N = 30$ of Experiment 1.

t	L_2	L_∞	RMS
1.1	$6.6890931e-4$	$2.4490467e-3$	$6.0036448e-4$
1.3	$7.9319372e-4$	$2.4701581e-3$	$7.1191315e-4$
1.5	$9.1972268e-4$	$2.6308646e-3$	$8.2547637e-4$
1.7	$9.9228378e-4$	$2.9361988e-3$	$8.9060194e-4$
1.9	$1.0294363e-3$	$2.9247284e-3$	$9.2394740e-4$
2.1	$1.0417382e-3$	$3.1386679e-3$	$9.3498871e-4$
2.3	$1.0406753e-3$	$3.2574158e-3$	$9.3403467e-4$
2.5	$1.0390624e-3$	$3.1575840e-3$	$9.3258707e-4$

Table 5: Errors at different times for $\mu = 1$, $\Delta t = 0.01$, and $N = 11$ of Experiment 2.

t	L_2	L_∞	RMS
0.3	$7.429115182e-5$	$1.396733e-4$	$7.083383397e-5$
0.5	$7.382220393e-5$	$1.382501e-4$	$7.038670970e-5$
0.7	$7.309629350e-5$	$1.362456e-4$	$6.969458126e-5$
0.9	$7.239658204e-5$	$1.340466e-4$	$6.902743256e-5$
1.1	$7.176128443e-5$	$1.317095e-4$	$6.842170006e-5$
1.3	$7.119752320e-5$	$1.292508e-4$	$6.788417482e-5$

with

$$A = \frac{0.05}{\mu}(x - 0.5 + 4.95t), \qquad B = \frac{0.25}{\mu}(x - 0.5 + 0.75t),$$
$$C = \frac{0.5}{\mu}(x - 0.5 + 0.375t). \tag{5.5}$$

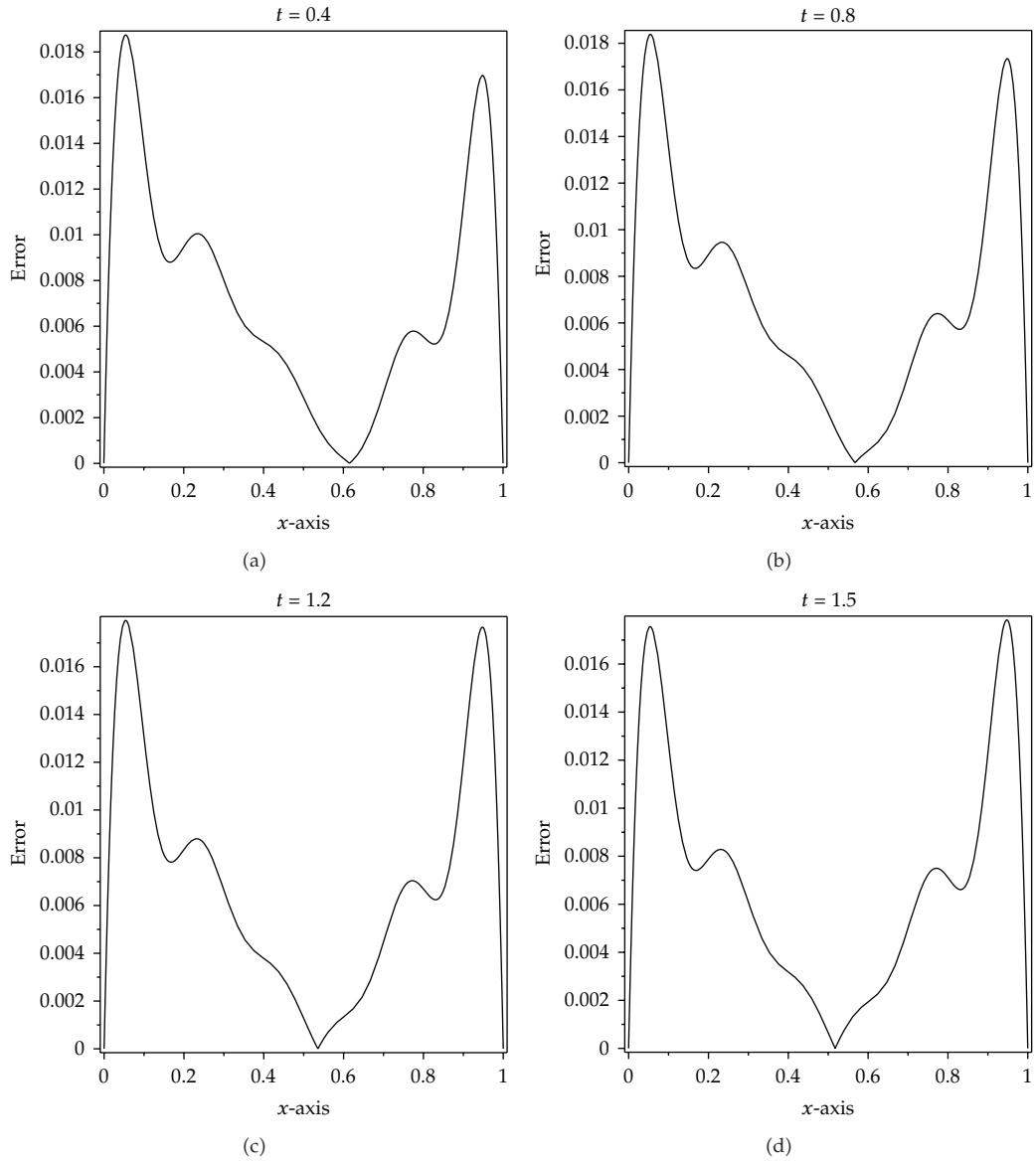

Figure 4: The percentage absolute error for $\mu = 1$, $\Delta t = 0.01$, and $N = 11$ at different times of Experiment 2.

The initial function $u_0(x)$ in (1.2) is obtained from (5.4) at time $t = 0$. The boundary functions $g_1(t)$ and $g_2(t)$ in (1.3) can be obtained from the exact solution. The computational results are listed in Tables 8, 9, and 10. Figure 5 represents the L_2 and L_∞ norm errors for $t = 0.7$ with various values of μ in Experiment 3.

6. Conclusion

In this paper, we present a new meshless method, the MLPG method, for solving Burger's equation. The Wendland compactly supported RBF is used as the test function in

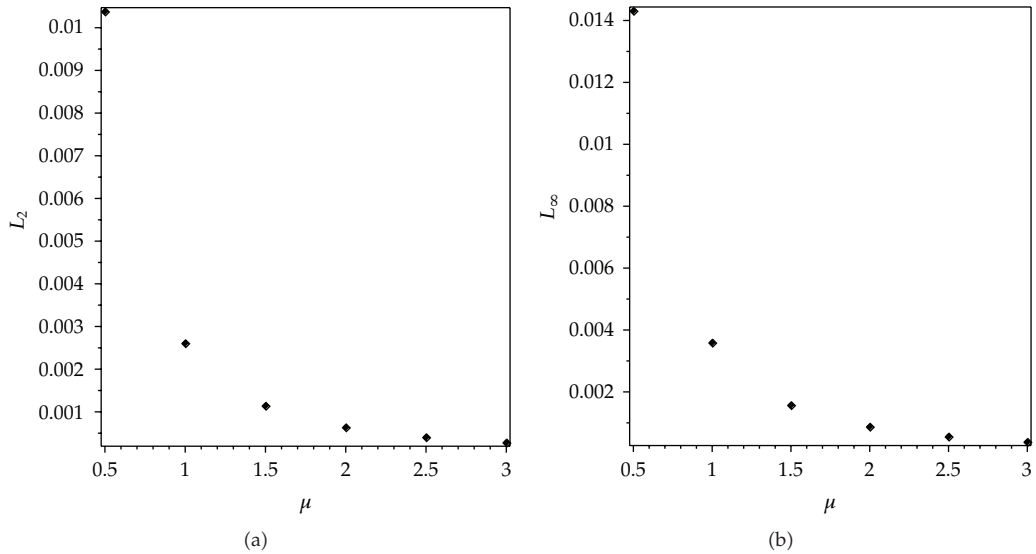

Figure 5: L_2 and L_∞ norm errors for $t = 0.7$ with various values of μ of Experiment 3.

Table 6: Comparison of results for $N = 11$ at $t = 1$, of Experiment 2.

x	Exact solution	Approximation solution	Absolute error
0.1	0.6497412010	0.6498740944	0.0003289340
0.2	0.6418463275	0.6419327944	0.0000864669
0.3	0.6339183523	0.6339885286	0.0000701763
0.4	0.6259634450	0.6260053930	0.0000419480
0.5	0.6179878599	0.6180048280	0.0000169681
0.6	0.6099979174	0.6099885933	0.0000093241
0.7	0.6019999834	0.6019588683	0.0000411151
0.8	0.5940004500	0.5939363938	0.0000640562
0.9	0.5860057138	0.5858875983	0.0001181155

Table 7: Errors at different times for $\mu = 1$, $\Delta t = 0.01$, and various values of N of Experiment 2.

N	t	L_2	L_∞	RMS
10	0.3	$1.493019395e-2$	$2.8379857e-2$	$1.416402564e-2$
	0.5	$2.091619459e-2$	$3.7525608e-3$	$1.984284447e-3$
	0.7	$2.477933450e-2$	$4.3387323e-2$	$2.350774078e-2$
20	0.3	$2.017363647e-5$	$3.8560643e-5$	$2.017363647e-5$
	0.5	$1.989079023e-5$	$3.7836201e-5$	$9.387144170e-5$
	0.7	$1.917941817e-5$	$3.6967421e-5$	$1.869378446e-5$
30	0.3	$8.834348123e-6$	$1.4274934e-5$	$8.685861109e-6$
	0.5	$7.413312585e-6$	$1.3824312e-5$	$7.288710222e-6$
	0.7	$6.383168865e-6$	$1.3315404e-5$	$6.275881075e-6$

the local weak form, and in time discretization, the Taylor series expansion is used. Results of numerical experiments indicate that it is an accurate and efficient numerical scheme. The proposed method is a truly meshless method, which does not require domain elements in the

Table 8: Errors at different times for $\mu = 1$, $\Delta t = 0.01$, and $N = 20$ of Experiment 3.

t	L_2	L_∞	RMS
0.1	$1.617490520e-3$	$2.1872586e-3$	$5.765347406e-3$
0.2	$2.191361199e-3$	$3.0036034e-3$	$2.135874694e-3$
0.3	$2.408978384e-3$	$3.3120934e-3$	$2.347981689e-3$
0.4	$2.502277200e-3$	$3.4436135e-3$	$2.438918126e-3$
0.5	$2.551492086e-3$	$3.5124404e-3$	$2.486886863e-3$
0.6	$2.584324486e-3$	$3.5580186e-3$	$2.518887928e-3$
0.7	$2.610306047e-3$	$3.5939218e-3$	$2.544211622e-3$
0.8	$2.632685403e-3$	$3.6247780e-3$	$2.566024320e-3$
0.9	$2.652545794e-3$	$3.6521317e-3$	$2.585381834e-3$
1	$2.670221101e-3$	$3.6764621e-3$	$2.602609592e-3$

Table 9: Comparison of results for $\mu = 1$, $\Delta t = 0.01$, and $N = 20$ at $t = 0.7$ of Experiment 3.

x	Exact solution	Approximation solution	Absolute error
0.1	0.6341806242	0.6331762537	0.0010043705
0.2	0.6272831040	0.6252326570	0.0020504470
0.3	0.6203506694	0.6174954438	0.0028552256
0.4	0.6133869804	0.6099911083	0.0033958721
0.5	0.6063957668	0.6027394214	0.0036563454
0.6	0.5993808174	0.5957602017	0.0036206157
0.7	0.5923459753	0.5890736196	0.0032723557
0.8	0.5852951233	0.5827004257	0.0025946976
0.9	0.5782321803	0.5766664052	0.0015657751

Table 10: Errors at different times for various values of parameter μ with $N = 20$ of Experiment 3.

μ	t	L_2	L_∞	RMS
0.5	0.2	$6.585671191e-3$	$8.9254155e-3$	$6.418918272e-3$
	0.4	$9.092700100e-3$	$1.2483140e-2$	$8.862467791e-3$
	0.6	$1.011921028e-2$	$1.3932070e-2$	$9.862986150e-3$
	0.8	$1.054927663e-2$	$1.4562501e-2$	$1.028216298e-2$
1.5	0.2	$1.057692710e-3$	$1.4529447e-3$	$1.030911333e-3$
	0.4	$1.118214487e-3$	$1.5382537e-3$	$1.089900663e-3$
	0.6	$1.136741332e-3$	$1.5638079e-3$	$1.107958398e-3$
	0.8	$1.152271622e-3$	$1.5851543e-3$	$1.123095453e-3$
3	0.2	$2764808197e-4$	$3.793052e-4$	$2.694801689e-4$
	0.4	$2.792642480e-4$	$3.831386e-4$	$2.721931193e-4$
	0.6	$2.815388964e-4$	$3.862552e-4$	$2.744101723e-4$
	0.8	$2.837383595e-4$	$3.892683e-4$	$2.765539437e-4$

interpolation. In the weak form, integrals are evaluated over the local subdomain instead of the global domain. This method can be extended to solve Burger's equation in the higher-order dimensions because the MQ RBF can easily be extended to higher dimensions. Also, because of the infinite differentiability of the RBFs, we can use the MLPG method for the PDEs with higher-order derivatives with respect to x, for example, the Korteweg-de Vries equation.

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