

Research Article

Mimicking General Relativity with Newtonian Dynamics

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The aim of this paper is twofolded. (1) Showing that Newtonian mechanics of point particles in static potentials admits an alternative description in terms of effective riemannian *spacetimes*. (2) Using the above geometrization scheme to investigate aspects of the gravitational field as it appears in the Einstein theory. It is shown that the mechanical (3+1) effective metrics are quite similar to Gordon's metric, as it is suggested by the well-known optical-mechanical analogy. Some special potentials are worked out.

1. Introduction

The last years have witnessed a great interest in the study of analog gravity [1–14]. The underlying idea is to investigate kinematical aspects of Einstein's theory using physical situations where a kind of geometrization procedure can also be applied. The main motivation to the relativist is the fact that it is possible to assign an effective spacetime to describe propagation features of different physical systems. The generality behind these effective metric techniques is being investigated in diverse contexts such as hydrodynamics, optics inside media, nonlinear field theory, superfluids, and quantum condensates (see [15, 16] for a complete review).

Although the first analog models appeared in the early days of general relativity [17] it was only in the eighties that they called the attention of the community as a whole. The main reason for this is Unruh's seminal paper on "experimental black hole evaporation" [18]. Unruh conjectured that analogue models could help us to understand not only classical aspects of GR but also some features of quantum field theory in curved spacetimes. In particular, Unruh showed that an analogue black hole may present some properties of

gravitational ones as far as the quantum thermal radiation is concerned. Since then, it became a common practice to use analog gravity as a tool to mimic different aspects of the gravitational theory: artificial black holes, Hawking radiation, emergent gravity, quantum field theory in curved backgrounds, and others.

In this paper, I adopt a different perspective concerning “artificial” gravity. I analyze if effective spacetime techniques are suitable to describe some aspects of Newtonian mechanics of point particles and vice versa. As it is known, the relation between curved geometries and particle dynamics was studied at least since the time of Jacobi [19]. After him, this geometrization scheme was to be the concern of Liouville, Lipschitz, Thomson, Tait, and Hertz (see [19, 20] for a detailed discussion). With the advent of tensor calculus, it became clear that there existed a map between the trajectories of certain mechanical systems in configuration space and the geodesics of a curved manifold. This fact was explored to solve mechanical problems of holonomic constraints by Ricci, Levi-Civita and in the relativistic era by Synge [21, 22], Lanczos [23], Lichnerowicz [24], and others. Nevertheless, despite the intense use of geometrical techniques in the context of dynamics, it seems that the relation between mechanics and geometry was not clearly appreciated in the literature of analog gravity (see, nevertheless, [25–27]). This is, perhaps, because many of the above maps did not take the time coordinate into account in the geometrization scheme.

It will be shown that Newton’s mechanics of point particles in static potentials may provide a very simple analog model of gravitation. At first this statement seems to be suspicious once we will obtain an alternative description of Newtonian trajectories in terms of pseudo-Riemannian spacetimes without any reference to relativity theory. Nevertheless, using the well-known optical-mechanical analogy as a starting point we will see that, indeed, it is possible to give an effective spacetime description of the motion instead of using the traditional description in terms of forces. We will see that the resulting analog model works as a counterpart of Gordon’s metric in the context of optics, while the trajectory of the particle is mapped into null geodesics of an effective spacetime with metric $\hat{g}_{\mu\nu}$. Finally, we explore some specific potentials and explicitly calculate the effective metrics.

This new geometrized scenario of mechanics may give us some interesting hints to the study of future models because of three main reasons. (1) It introduces curved spacetimes from a very simple and well understood physics. (2) It may extend our laboratory perspectives to measure effective gravitation using, for instance, electronic optics. (3) It may provide an interesting and smooth transition to the issue of quantization since we are working with the mechanics of particles instead of fields.

2. The Optical-Mechanical Analogy Revisited

The well-known optical-mechanical analogy (OMA) has been discussed often and from many different points of view [25, 27]. The analogy may be understood as a formal tool which maps mechanical systems into optical ones and vice versa. In this sense it implies that the experience and insights developed in one area may be extrapolated to solve problems in the other. Although the OMA is quite familiar, I briefly discuss its main points to fix notation and to make the paper self-contained.

Let us start by considering a particle with mass “ m ” in a static potential $V(x^k)$. We are interested in the subset of particle trajectories with a fixed energy E_0 according to Newton’s mechanics. It is instructive to begin with an arbitrary spatial coordinate system x^k ($k = 1, 2, 3$)

and three-dimensional euclidean space with metric $g_{ik}(x)$. Also, instead of using the usual time coordinate “ t ” as a parameter I use the arclength “ l ” along the curve, where

$$dl^2 = g_{ik} \dot{x}^i \dot{x}^k dt^2. \quad (2.1)$$

Defining the normalized tangent vector u^i to the curve it is straightforward to show that Newton’s second law can be cast in the compact form

$$(pu^i)_{;k} u^k = g^{ij} \frac{\partial p}{\partial x^j}, \quad (2.2)$$

where “ $;$ ” is defined as the three-dimensional covariant derivative, and the modulus of the momentum is related to the potential by the usual energy relation

$$p = \sqrt{2m(E_0 - V)}. \quad (2.3)$$

It is important to note here that, because the modulus “ p ” is automatically known if x^k is given, we just need as initial conditions the position x_0^k and the direction of propagation u_0^k . If we specify only the initial position, the solutions of (2.2) characterize all possible trajectories passing through x_0^k with fixed momentum $p(x_0^k)$.

A similar situation appears in the context of geometrical optics. According to Fermat’s principle (see, for instance [28]), the ray equation inside an heterogeneous isotropic material with index of refraction $n(x^k)$ is given by the extrema of the optical length $\lambda \equiv ct$

$$\lambda = \int n dl = \int n \sqrt{g_{ik} \dot{x}^i \dot{x}^k} dt, \quad (2.4)$$

where “ c ” is the velocity of light in vacuum. By varying the action and again parametrizing the paths with the arclength “ l ” it follows

$$(nu^i)_{;k} u^k = g^{ij} \frac{\partial n}{\partial x^j}. \quad (2.5)$$

In the same way, because the velocity of light is automatically given in terms of the index of refraction, one only needs to specify position and direction to obtain a given trajectory. By comparing (2.5) with (2.2) one immediately recognizes the similarity between the equations. Thus, there exists a clear analogy between Newtonian trajectories for point particles in static conservative fields and the light rays inside isotropic heterogeneous media.

To complete the analogy we divide (2.2) by $p_0 \equiv \sqrt{2m|E_0|}$ to express it in terms of an adimensional quantity $\hat{p} \equiv p/p_0$. Because the index of refraction is also an adimensional quantity, one may define the map

$$n \mapsto \hat{p} = \sqrt{\pm \left(1 - \frac{V}{E_0}\right)}, \quad (2.6)$$

where “+” stands if $E_0 > 0$ and “-” if $E_0 < 0$. In this way, the content of the OMA can be formulated as follows.

- (i) To a given optical trajectory inside an isotropic material with index of refraction “ n ,” it is possible to assign a static potential “ V ” given by relation (2.6) such that it mimics this trajectory in the context of Newtonian mechanics or vice versa.

It is important to note that the parallelism extends only to the geometrical form of the extremal curve. As was noted by Lanczos [23], the evolution of the systems in time is not equivalent. We will explore this fact later.

2.1. The Relation between Paths and Three-Dimensional Effective Geodesics

It is a well-known fact in optics that Fermat’s principle may be mapped in the problem of finding geodesics of a three-dimensional curved manifold [23, 29]. If one interprets the optical length $d\lambda \equiv n dl$ as an infinitesimal element of arc, it is immediate to see that the extremal of the action (2.4) is completely equivalent to the geodesics of the riemannian geometry given by the metric

$$\hat{g}_{ij} \equiv n^2 g_{ij}. \quad (2.7)$$

In other words, the light path is such that, if it is parametrized by the optical length λ it satisfies

$$\frac{d^2 x^k}{d\lambda^2} + \hat{\Gamma}^k_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0, \quad (2.8)$$

where the effective Christoffel symbol is such that

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} + n^{-1} (\delta_i^k n_{,j} + \delta_j^k n_{,i} - n^k g_{ij}). \quad (2.9)$$

Although this is a purely formal property of the action principle, one can convince himself/herself by explicitly substituting the identity $\lambda = n dl$ in (2.8). We obtain, after a straightforward calculation and definition (2.9), the equation

$$n^2 \frac{d}{dl} \left(\frac{1}{n} u^k \right) + n \Gamma^k_{ij} u^i u^k + 2 n_{,j} u^j u^k = g^{kj} n_{,j}. \quad (2.10)$$

By calculating explicitly the first term one finally obtains

$$\frac{d}{dl} (n u^k) + n \Gamma^k_{ij} u^i u^k = g^{kj} n_{,j}, \quad (2.11)$$

which is no more than the ray equation (2.5). The identification trajectories/geodesics appears often in analog models of gravitation based on optics and is currently being explored

in diverse contexts such as in the architecture of metamaterials, cloaking devices, negative refraction structures, and perfect magnifying lenses (see [30–36] and references therein).

Less discussed in the literature of analogue gravity is the geometrical version of the mechanical counterpart of Fermat's principle. Nevertheless, the similarity between (2.2) and (2.5) is not fortuitous. Newton's equation as well is given in terms of an action principle of the form (2.4) which is Jacobi's principle. One only needs to replace n by \hat{p} in (2.4) to obtain (2.2). Thus, there exists also in mechanics a geometrical interpretation of the motion as described in its configuration space. The trajectories (2.2) may be alternatively obtained from the geodesics of the effective riemannian space given by the metric

$$\hat{g}_{ij} = \hat{p}^2 g_{ij}. \quad (2.12)$$

Note however that, although the four-dimensional generalization of (2.7) made by Gordon [17] became the paradigm of analog gravity, the same did not happen with its mechanical version. The aim of the next section is to explore this last possibility.

3. Gordon's Metric and Newtonian Trajectories

Gordon was the first to develop effective metric techniques in the context of analog models. He was interested in trying to describe dielectric media by an effective metric while at the same time using the gravitational field in Einstein ansatz to mimic a dielectric medium. Gordon showed that the trajectories of light inside a dielectric medium was such that they could be mapped in null geodesics of a four-dimensional pseudo-Riemannian metric given by

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \left(\frac{c^{-2}}{\mu\epsilon} - 1 \right) v_\mu v_\nu, \quad (3.1)$$

where v_μ is a normalized timelike vector with respect to Minkowski's background metric, that is, $\eta^{\mu\nu} v_\mu v_\nu = 1$ and represents the motion of the dielectric medium. In the simplest case where the medium is at rest in laboratory's frame, we have $v^\mu = \delta_0^\mu$ and Gordon's metric acquire the simpler form

$$\hat{g}_{\mu\nu} = \text{diag}(n^{-2}, -g_{ij}). \quad (3.2)$$

Now, we are going to show that there exists an entirely equivalent situation in Newtonian mechanics of point particles. We will see that Gordon's metric provides the apparatus to describe Newtonian motions by means of the optical-mechanical analogy. Although it is possible to give a formal demonstration of our proposition we are going to adopt a simpler route by showing that Newton's second law in the form (2.2) is completely equivalent to a null geodesic on the four-dimensional curved spacetime given by the metric (3.2).

We first note that the energy relation (2.3) can be put in the suggestive form

$$\hat{p}^{-2} - g_{ij} \hat{p}^i \hat{p}^j = 0, \quad (3.3)$$

where the first term has the index of refraction form given by (2.6) and $\hat{p}^i \equiv dx^i/d\lambda$. Our next aim is to write this equation as a quadratic form in a four-dimensional riemannian manifold. We thus define an effective spacetime \widehat{M} with coordinates $x^\mu \equiv (\lambda, x^1, x^2, x^3)$ with λ given by (2.4) and (2.6) (note that the coordinate λ has the dimensions of length).

We are going to parametrize the trajectory of the particle in \widehat{M} in terms of λ , that is, $x^\mu(\lambda)$. Defining the tangent four-vector \hat{p}^μ as

$$\hat{p}^\mu(\lambda) \equiv \frac{dx^\mu}{d\lambda}, \quad (3.4)$$

we see that (3.3) is equivalent to the expression

$$\hat{g}_{\mu\nu}\hat{p}^\mu\hat{p}^\nu = 0, \quad (3.5)$$

with the effective metric $\hat{g}_{\mu\nu}$ given by the Gordon metric (3.2) and the index of refraction given by the optical-mechanical relation (2.6). Relation (3.5) is the mechanical analogue of the light cone in relativity.

Note that, for the effective metric to make sense it has to carry a hyperbolic (Lorentzian) signature. This is in complete agreement with the fact that the modulus of the three-dimensional momentum “ p ” is real, and thus the potential is such that $V(x) \leq E_0$. We thus succeeded to write the energy relation of Newtonian particles with fixed energy in terms of a “dispersion relation” in a pseudo-Riemannian geometry. Nevertheless, contrarily to our relativistic intuition, the trajectories of massive particles are not mapped into timelike geodesics, but are tangent to effective null curves.

To conclude our four-dimensional geometrization we are going to show that, indeed, the null geodesics of Gordon’s metric coincide with Newton’s equation in the form (2.2). To make our calculations simpler, we note that null geodesics are insensitive to conformal transformations. We thus make the conformal map by the substitution

$$\hat{g}_{\mu\nu} \mapsto \hat{p}^2 \hat{g}_{\mu\nu}, \quad (3.6)$$

to obtain a new manifold with metric given by

$$\hat{g}_{\mu\nu} = \text{diag} \left[1, \mp \left(1 - \frac{V}{E_0} \right) g_{ij} \right], \quad (3.7)$$

where “ $-$ ” stands if $E_0 > 0$ and “ $+$ ” if $E_0 < 0$. The four-dimensional geodesics of this geometry is given by

$$\frac{d^2 x^\mu}{d\lambda^2} + \hat{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0. \quad (3.8)$$

We first note that, because the potential $V(x^k)$ depends only on position by hypothesis we have

$$\hat{\Gamma}^0_{\mu\nu} = \hat{\Gamma}^k_{\mu 0} = 0. \quad (3.9)$$

We thus obtain that, because $x^0 \equiv \lambda$, the “0” component of the equation is identically satisfied. Also, because we have $\hat{\Gamma}^k_{\mu 0} = 0$, the spatial components are immediately reduced to an equation of the form (2.8), which is identically satisfied as a consequence of Jacobi principle (one can convince herself/himself of this fact by explicitly substituting $d\lambda = \hat{p}v dt$ in the geodesic equation to obtain Newton’s equations). We thus arrive at the following conclusion.

The trajectories of particles with given energy E_0 in Newton’s mechanics of static potentials are such that they can be derived from the null geodesics of a curved four-dimensional spacetime given by Gordon’s metric up to a conformal transformation.

We thus define the effective four-dimensional element of distance as

$$d\hat{s}^2 \equiv d\lambda^2 \mp \left(1 - \frac{V}{E_0}\right) g_{ij} dx^i dx^j. \quad (3.10)$$

Trajectories are simply given by the condition $d\hat{s}^2 = 0$. Note that the effective geometry is dependent on the total energy of the system. This means that we have to deal with different geometries to describe motions with different energies. Note also that, in general, the *mechanical* Riemann tensor $\hat{R}_{\alpha\beta\mu\nu}$ associated to the effective metric will be nonzero. At this point it is appropriate to use some words of Barcelo et al. in [15]: “It is quite remarkable that even though the underlying *particle* “dynamics is Newtonian, nonrelativistic, and takes place in flat space plus time”, the trajectories “are governed by a curved (3 + 1) dimensional Lorentzian (pseudo-Riemannian) spacetime geometry. For practitioners of general relativity this observation describes a very simple and concrete physical model for certain classes of Lorentzian spacetimes, including black holes. It is also potentially of interest to practitioners of *particle* “mechanics in that it provides a simple concrete introduction to Lorentzian differential geometric techniques.”

4. Effective Geometries from Newtonian Potentials

We now turn to some explicit situations where our geometrization procedure may be applied. Our aim is to show that some mechanical properties of the systems may as well be described in terms of an effective geometrical language.

4.1. The Geometry of Harmonic Oscillations

We start with a one-dimensional oscillator with potential energy $V = kx^2/2$. The (1 + 1) dimensional effective line element (3.10) is of the form

$$d\hat{s}^2 \equiv d\lambda^2 - \left[1 - \left(\frac{x}{a}\right)^2\right] dx^2, \quad (4.1)$$

where “ a ” is the amplitude of the oscillator. Note that this metric is hyperbolic only if $|x| < a$. Thus, this effective spacetime is such that there exists two *geometrical* boundaries given by the conditions $x = \pm a$. This is an immediate consequence of the fact that we are considering

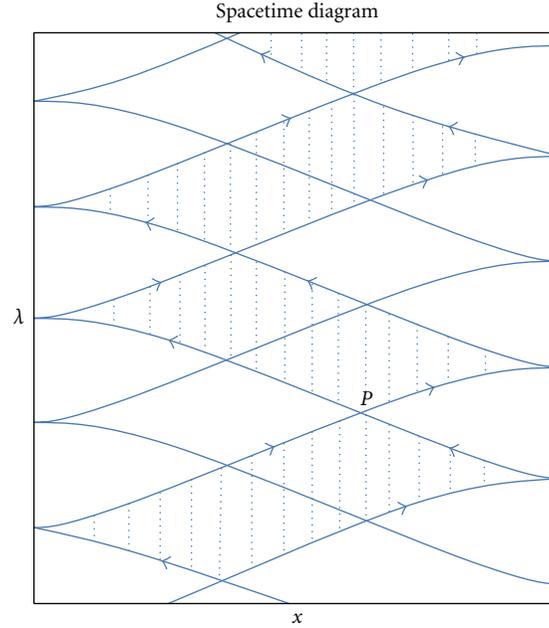


Figure 1: Harmonic oscillator space-time diagram.

only trajectories such that the total energy E_0 is fixed, and thus the motion is not defined anywhere.

The null trajectories may be obtained by imposing the condition $d\hat{s} = 0$, that is,

$$\frac{dx}{d\lambda} = \pm \left[1 - \left(\frac{x}{a} \right)^2 \right]^{-1/2}. \quad (4.2)$$

In this simple case, it is possible to integrate this equation to obtain the implicit relation

$$\lambda = \pm \frac{1}{2} \left\{ x \sqrt{1 - \left(\frac{x}{a} \right)^2} + a \text{Arctan} \left(\frac{x}{a} \right) \right\} + \text{const.} \quad (4.3)$$

A diagram for the trajectories is given in Figure 1. The set of all possible null curves $x(\lambda)$ given by the implicit relation (4.3) characterizes the space-time diagram for the oscillator. Note however that, although the curves exhibit a periodic behavior in terms of λ the derivative $dx/d\lambda$ is not well defined in the limit $x \rightarrow \pm a$. In this limit the effective volume $\sqrt{-\hat{g}} dx d\lambda = [1 - (x/a)] dx d\lambda$ also vanishes, and one can interpret this as a consequence of the fact that the effective spacetime only makes sense inside the boundaries given by $x = \pm a$.

At a given point of the diagram there exist only two admissible trajectories with fixed energy E_0 . These two curves determine at each point P a conoid structure in the same way as the propagation of light characterizes the light cone in relativity. We see from the figure that the intersection between the curves gives the mechanical analogue of the light cone at each spacetime point. The domain of causality of the point P is represented by the shaded regions.

The curves (4.3) are null geodesics with respect to the effective metric. Nevertheless, a careful inspection in (4.1) shows that this metric is flat. Thus, metric (4.1) is simply Minkowski metric in a curvilinear coordinate system. We are now going to investigate a more complicated situation where the Riemann tensor does not vanish.

4.2. The Geometry of Newtonian Gravitation

We now turn to the motion of point test particles with mass “ m ” in the Newtonian gravitational potential. As in the previous section, we are interested in the motions associated to a given total energy E_0 . According to our previous discussion, the null geodesics of the respective effective metric (3.10) are such that they reproduce ordinary Newtonian trajectories in the gravitational potential.

To be specific, let us consider the spherically symmetric potential $V(r) = -GMm/r$. We will consider only the case of bounded orbits, that is, $E_0 < 0$. The other situations will be considered elsewhere.

4.2.1. $E_0 < 0$ Effective Metrics

In this case we have a maximum admissible radii $r_H \equiv -GMm/E_0$ for the trajectories. One obtains for the effective metric, in spherical coordinates, the expression

$$d\hat{s}^2 = d\lambda^2 + \left(1 - \frac{r_H}{r}\right) [dr^2 + r^2 d\Omega^2]. \quad (4.4)$$

Analogously to the harmonic oscillator case, we obtain an effective metric that admits a hyperbolic (Lorentzian) signature only in a particular domain. In the present case this domain is given by the condition $0 < r < r_H$. This is because outside this region, that is, for $r > r_H$, the movement is not allowed for the considered energies $E_0 = -GMm/r_H$.

This fact has an interesting geometrical interpretation in our scheme. First, let us note that the square root of the determinant is given by

$$\sqrt{-\hat{g}} = r^2 \sin(\theta) \left(\frac{r_H}{r} - 1\right)^{3/2}. \quad (4.5)$$

This relation implies that the element of effective volume $dV_{\text{eff}} \equiv \sqrt{-\hat{g}} d^4x$ vanishes for the values $r = 0$ and $r = r_H$. This is a hint that the confinement of the motion between these values of the radial coordinate has a deep geometrical meaning.

In fact, the components of the Riemann tensor are

$$\begin{aligned} \hat{R}_{1212} &= -\frac{1}{2} \frac{r_H}{r_H - r}, \\ \hat{R}_{1313} &= -\frac{1}{2} \frac{r_H \sin^2(\theta)}{r_H - r}, \\ \hat{R}_{2323} &= -\frac{1}{4} \frac{r r_H \sin^2(\theta) (3r_H - 4r)}{r_H - r}. \end{aligned} \quad (4.6)$$

One immediately obtains for the Ricci scalar the expression

$$R = -\frac{3}{2} \frac{r_H^2}{r(r_H - r)^3}. \quad (4.7)$$

This object diverges for the values $r = 0$ and $r = r_H$ (note that $R < 0$). Thus, the effective (3+1) spacetime is delimited by two singularities in the effective geometry. The first one is quite similar to Schwarzschild singularity and admits a similar interpretation due to the strength of the gravitational field. It appears because the Newtonian gravitational potential also diverges at the origin. The second singularity has, nevertheless, a different and unexpected origin. It appears because we confined our attention only to the movement of particles with a fixed energy E_0 . As we mentioned before, all possible trajectories with this energy cannot scape from the region $r < r_H$. This fact appears in the geometrical description as a kind of geometrical barrier that do not allow the null geodesics to escape from the allowed region.

To develop some additional feeling on our geometrical description of dynamics, it is instructive to solve the well-known problem of finding the trajectories in the Newtonian gravitational potential in terms of the effective null geodesics (3.8). We recall that the following quantity is a constant of the motion:

$$\hat{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (4.8)$$

From the geodesic equation we obtain (for the sake of conciseness I use $dx/d\lambda \equiv \dot{x}$)

$$\ddot{r} + A(r)\dot{r}^2 + B(r)\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] = 0, \quad (4.9)$$

$$\ddot{\theta} + C(r)\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \quad (4.10)$$

$$\ddot{\phi} + C(r)\dot{\phi}\dot{r} + 2\cot g\theta\dot{\phi}\dot{\theta} = 0, \quad (4.11)$$

where

$$\begin{aligned} A(r) &\equiv \frac{1}{2} \frac{r_H/r}{r - r_H}, \\ B(r) &\equiv -\frac{1}{2} \frac{2r^2 - r_H r}{r - r_H}, \\ C(r) &\equiv \frac{2r - r_H}{r^2 - r_H r}. \end{aligned} \quad (4.12)$$

As usual, we assume that the orbit lies in the plane $\theta = \pi/2$. Thus, (4.10) ensures that, if $\theta = \pi/2$ and $\dot{\theta} = 0$ initially, the orbit remains always in this plane. Equation (4.11) may be integrated directly. We obtain that the following quantity is also conserved

$$(r^2 - r_H r)\dot{\phi} = \xi, \quad (4.13)$$

where ξ is a constant. It is straightforward to show that this equation implies the usual conservation of angular momentum. In fact, from (4.13), it follows, after a reparametrization that

$$\frac{d\phi}{dt} = \frac{l}{r^2}, \quad l = \left(-\frac{2E_0}{m}\right)^{1/2} \xi. \quad (4.14)$$

We finally turn to (4.9). As expected, this is not a simple equation. Nevertheless, we can bypass this difficulty by considering the constraint (4.8). We obtain

$$\dot{r}^2 = -\frac{1}{(1 - r_H/r)} \left[1 + \frac{(\xi/r)^2}{(1 - r_H/r)} \right]. \quad (4.15)$$

This is the equation of motion of the radial part written in terms of the parameter λ . By introducing the auxiliary variable

$$u = \frac{1}{r}, \quad (4.16)$$

we obtain

$$\xi^2 \left(\frac{du}{d\phi} \right)^2 = -(1 - r_H u + \xi^2 u^2). \quad (4.17)$$

Deriving with respect to ϕ we arrive at the familiar Newtonian equation for the orbit

$$\frac{d^2 u}{d\phi^2} + u - \frac{GM}{l^2} = 0, \quad (4.18)$$

which admits the solution

$$r(\phi) = \frac{l^2/GM}{1 + \epsilon \cos(\phi - \phi_0)}, \quad (4.19)$$

where $0 \leq \epsilon < 1$ because $r \leq r_H$ always. Thus, we succeeded to find the orbits around the field of a spherically object with mass "M" in terms of the null geodesics of the effective metric (4.4).

4.3. The Geometry of Electrostatic Potentials

The conservative potentials that appear in electromagnetism are, perhaps, the most interesting from the point of view of analog gravity in our context. This is not only because electromagnetic fields are simple to handle in laboratory sizes, but because many interesting configurations of static potentials occur in the advanced field of electronic optics. I will

concentrate here on simple electrostatic configurations. The potential is obtained by the familiar formula

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x', \quad (4.20)$$

while the potential energy is $V(\vec{x}) = q\phi$. The effective metric is given by the expression

$$\hat{g}_{\mu\nu} = \text{diag} \left[1, \mp \left(1 - \frac{q\phi}{E_0} \right) g_{ij} \right], \quad (4.21)$$

where q is the charge of the test charge. Our next task is to calculate this metric for a specific case and to use some of its geometrical properties to infer about mechanical properties of the system.

4.3.1. The Case of the Electric Dipole

One interesting situation appears in the geometrization of the static electric dipole. If the dipole moment \vec{p} is oriented in the z direction, we obtain, for large distances,

$$\phi(r, \theta) = \frac{p}{4\pi\epsilon_0} \frac{\cos(\theta)}{r^2}. \quad (4.22)$$

For the sake of simplicity let us consider $pq > 0$ and positive energies $E_0 > 0$. The effective line element reads

$$d\hat{s}^2 = d\lambda^2 + \left(\kappa^2 \frac{\cos(\theta)}{r^2} - 1 \right) \cdot \left[dr^2 + r^2 d\Omega^2 \right], \quad (4.23)$$

where $\kappa^2 \equiv pq/4\pi\epsilon_0 E_0$. Note that this effective spacetime only makes sense if

$$r^2 > \kappa^2 \cos(\theta). \quad (4.24)$$

This is an immediate consequence of the fact that the energy E_0 is fixed from the beginning and there are regions that are not allowed for the particle motion. In our scheme, this fact has an interesting geometrical interpretation. Note that the determinant of the metric

$$\sqrt{-\hat{g}} = r^2 \sin(\theta) \left(1 - \frac{\kappa^2 \cos(\theta)}{r^2} \right)^{3/2} \quad (4.25)$$

vanishes for some values of the radii. At first we could suspect that the coordinate system we are using is not well defined in this region. Nevertheless, we obtain for the scalar of curvature the expression

$$R = \frac{3\kappa^4}{2} \frac{(3 \cos(\theta)^2 + 1)}{(\kappa^2 \cos(\theta) - r^2)^3}, \quad (4.26)$$

which diverges in the abovementioned region. Thus, the description of charged particle motions (such as electrons or ions) in terms of null geodesics of an effective spacetime forces us to deal with a singular geometry in the case of the electric dipole. Nevertheless, the singularity is not so simple as it appears in Schwarzschild geometry, but it is determined by a compact bidimensional region given by the condition

$$r^2 = \kappa^2 \cos(\theta). \quad (4.27)$$

The null geodesics are not defined in the interior region. Note, however, that the entire region outside the singularity is well defined. Nevertheless, the geodesics are not so simple to be integrated as in the case of the solar system. Also, it is immediate to see that this geometry is asymptotically flat. Thus, for very large values of r , where the field of the dipole tends to vanish, the trajectories are described by the null geodesics of Minkowski spacetime.

As a final remark, I would like to point out that the geometrical framework may introduce techniques of general relativity and riemannian geometry in the realm of electron optics. We suspect that it is possible to envisage situations where electronic dispositives may be projected using these new tools. This last statement is strongly suggested by the recent achievements of transformation optics [31–35].

5. Conclusions

The relation between geometry and dynamics is an interesting problem that has been investigated from many different perspectives along the years. In this paper it is shown that Newtonian mechanics of point particles in static potentials admits a geometrical interpretation in terms of $(3 + 1)$ effective spacetimes in a similar way as it appears in the context of analog gravity models. The mechanical effective metric $\hat{g}_{\mu\nu}$ is given explicitly in terms of the potential energy $V(x)$ and the total energy E_0 and is similar to Gordon's metric in optics. From this last similarity, we inferred that mechanical systems may provide a very simple arena for the study of analogue gravity. This new arena may bring interesting theoretical and experimental challenges. I investigated some explicit examples, where the use of geometrical techniques were used to infer about some mechanical properties. Thus, it may be natural to speak about effective horizons, singularities and other structures that appear in general relativity in mechanical problems also. In particular it was considered the motion of particles in the gravitational field of a central mass and in the potential of an electric dipole. The null geodesics of the above geometries coincides with the usual trajectories as they're described in the Newtonian *ansatz*. It may be interesting to investigate if these geometrical picture can be sustained after first quantization.

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