Research Article

Inference on $\Pr(X < Y)$ in the Two-Parameter Weibull Model Based on Records

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Received 2 April 2012; Accepted 21 June 2012

Academic Editors: S. Lototsky, V. Makis, M. Montero, and P. Neal

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We consider the stress-strength reliability based on record values from the Weibull distribution. The Bayes estimator based on squared error loss and the maximum likelihood estimator are derived and their bias and mean squared error performance are studied. Likelihood-based confidence intervals as well as some bootstrap intervals are developed. We derived also the highest posterior density interval. Simulation studies are conducted to investigate and compare the performance of the estimators and intervals.

1. Introduction

Chandler [1] introduced and studied some properties of record values. Since then a considerable amount of the literature is devoted to the study of records. Ahsanullah [2] and Arnold et al. [3] provided a detailed account of theory of records and the inference issues associated with records. In this paper, we will consider confidence interval estimation of the stress-strength reliability based on record data when the underlying distribution is Weibull. The Weibull model is quite popular in analyzing reliability and life-testing data. Its flexibility and capability of modeling various forms of failure mechanisms and hazard function types gave him an important role in reliability literature; see Johnson et al. [4] and Murthy et al. [5]. The probability density function (pdf) and the cumulative distribution function (cdf) of the Weibull distribution $W(\theta, \beta)$ are given by

$$\begin{aligned} f(x) &= \theta \beta x^{\beta-1} e^{-\theta x^\beta}, \quad x > 0, \theta > 0, \beta > 0 \\ F(x) &= 1 - e^{-\theta x^\beta}. \end{aligned} \tag{1.1}$$

Let $X_1, X_2, \dots$ be an infinite sequence of *iid* random variables. An observation X_j is called a record if its value is greater than all previous observations, that is $X_j > X_i$ for every $i < j$. We want to estimate the stress-strength reliability $\Pr(X < Y)$ using the data on records. The stress-strength reliability $\Pr(X < Y)$ arises in life-testing experiments when X and Y represent the lifetimes of two devices, and it gives the probability that the device with life time X fails before the other. As an example, Hall [6] studied a situation where the breakdown voltage Y of a capacitor must exceed the voltage output X of a power supply in order for the component to work properly. Weerahandi and Johnson [7] considered another example on rocket motors. This probability has other interpretations in other disciplines, for example, in medical sciences; it is used as a measure of treatment effectiveness in studies involving comparison between control and treatment groups. Various other examples may be found in Kotz et al. [8].

Estimation of the stress-strength reliability based on record data was considered by Baklizi [9] for the exponential distribution with record values and by Baklizi [10] for the generalized exponential distribution. Kundu and Gupta [11] investigated this problem for simple random samples from the Weibull distribution. In Section 2 we derive the maximum likelihood estimator and the associated large sample intervals. Bayesian procedures based on records are derived in Section 3. Simulations to investigate the performance of the asymptotic inference procedures and to compare them with the bootstrap intervals are described in Section 4. The results and conclusions are given in Section 5.

2. Likelihood Inference

We shall consider the case when the shape parameters are equal. Let $X \sim W(\theta_1, \beta)$ and $Y \sim W(\theta_2, \beta)$ be independent random variables. Let $R = \Pr(X < Y)$ be the stress strength reliability. Let $\underline{r} = r_0, \dots, r_n$ be a set of records from $W(\theta_1, \beta)$, and let $\underline{s} = s_0, \dots, s_m$ be an independent set of records from $W(\theta_2, \beta)$. The likelihood functions are given by [3] as follows:

$$\begin{aligned} L_1(\beta, \theta_1 \mid \underline{r}) &= f(r_n) \prod_{i=0}^{n-1} \frac{f(r_i)}{(1 - F(r_i))}, \\ L_2(\beta, \theta_2 \mid \underline{s}) &= g(s_m) \prod_{i=0}^{m-1} \frac{g(s_i)}{(1 - G(s_i))}, \end{aligned} \quad (2.1)$$

where f and F are the pdf and cdf of $X \sim W(\theta_1, \beta)$ and g and G are the pdf and cdf of $Y \sim W(\theta_2, \beta)$. The likelihood function of $(\beta, \theta_1, \theta_2)$ based on $\underline{r}, \underline{s}$ is given by

$$L(\beta, \theta_1, \theta_2 \mid \underline{r}, \underline{s}) = (\theta_1 \beta)^{n+1} e^{-\theta_1 r_n^\beta} \prod_{i=0}^n r_i^{\beta-1} (\theta_2 \beta)^{m+1} e^{-\theta_2 s_m^\beta} \prod_{i=0}^m s_i^{\beta-1}. \quad (2.2)$$

Taking the natural logarithm we get the log-likelihood function

$$\begin{aligned} l(\beta, \theta_1, \theta_2) &= (n + m + 2) \ln \beta + (n + 1) \ln \theta_1 + (m + 1) \ln \theta_2 - \theta_1 r_n^\beta - \theta_2 s_m^\beta \\ &\quad + (\beta - 1) \left[\sum_{i=1}^n \ln r_i + \sum_{i=1}^m \ln s_i \right]. \end{aligned} \quad (2.3)$$

The first partial derivatives are given by

$$\frac{\partial l(\beta, \theta_1, \theta_2)}{\partial \beta} = \frac{(n+m+2)}{\beta} - \theta_1 r_n^\beta \ln r_n - \theta_2 s_m^\beta \ln s_m + \left[\sum_{i=1}^n \ln r_i + \sum_{i=1}^m \ln s_i \right], \quad (2.4)$$

$$\frac{\partial l(\beta, \theta_1, \theta_2)}{\partial \theta_1} = \frac{(n+1)}{\theta_1} - r_n^\beta, \quad (2.5)$$

$$\frac{\partial l(\beta, \theta_1, \theta_2)}{\partial \theta_2} = \frac{(m+1)}{\theta_2} - s_m^\beta.$$

Equating these partial derivatives to zero and solving simultaneously we obtain

$$\begin{aligned} \hat{\beta} &= \frac{(n+m+2)}{(n+1)\ln r_n + (m+1)\ln s_m - [\sum_{i=1}^n \ln r_i + \sum_{i=1}^m \ln s_i]}, \\ \hat{\theta}_1 &= \frac{n+1}{r_n^{\hat{\beta}}}, \\ \hat{\theta}_2 &= \frac{m+1}{s_m^{\hat{\beta}}}. \end{aligned} \quad (2.6)$$

Hence the MLE of R is given by $\hat{R} = \hat{\theta}_2 / \hat{\theta}_1 + \hat{\theta}_2$. The study of the exact distribution of $\hat{R}$ is apparently rather complicated, so we will consider the asymptotic distribution. We need the asymptotic joint distribution of $\hat{\beta}$, $\hat{\theta}_1$ and $\hat{\theta}_2$. The second partial derivatives of the log-likelihood function are given by

$$\begin{aligned} \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \beta^2} &= -\frac{(n+m+2)}{\beta^2} - \theta_1 r_n^\beta (\ln r_n)^2 - \theta_2 s_m^\beta (\ln s_m)^2, \\ \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \theta_1^2} &= -\frac{(n+1)}{\theta_1^2}, \\ \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \theta_2^2} &= -\frac{(m+1)}{\theta_2^2}, \\ \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \beta \partial \theta_1} &= -r_n^\beta \ln r_n, \\ \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \beta \partial \theta_2} &= -s_m^\beta \ln s_m, \\ \frac{\partial^2 l(\beta, \theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} &= 0. \end{aligned} \quad (2.7)$$

The Fisher information matrix is given by

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = - \begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \theta_1}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \theta_1}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \theta_2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_2^2}\right) \end{pmatrix}. \quad (2.8)$$

Now we will find the entries of the information matrix. We need to find $E(r_n^\beta \ln r_n)$, $E(r_n^\beta (\ln r_n)^2)$, $E(s_m^\beta \ln s_m)$ and $E(s_m^\beta (\ln s_m)^2)$. Using result (2.4.3) of Arnold et al. [3] we have $U = \theta_1 r_n^\beta \sim \Gamma(n+1, 1)$, and it follows that

$$\begin{aligned} E(r_n^\beta \ln r_n) &= \frac{1}{\beta \theta_1} E\left(\theta_1 r_n^\beta (\ln \theta_1 r_n^\beta - \ln \theta_1)\right) = \frac{1}{\beta \theta_1} [E(U \ln U) - \ln \theta_1 E(U)] \\ &= \frac{(n+1)}{\beta \theta_1} [\psi(n+2) - \ln \theta_1], \end{aligned} \quad (2.9)$$

$$\begin{aligned} E(r_n^\beta (\ln r_n)^2) &= \frac{1}{\theta_1 \beta^2} E\left(\theta_1 r_n^\beta (\ln \theta_1 r_n^\beta - \ln \theta_1)^2\right) \\ &= \frac{1}{\theta_1 \beta^2} E\left(\theta_1 r_n^\beta \left((\ln \theta_1 r_n^\beta)^2 - 2 \ln \theta_1 \ln \theta_1 r_n^\beta + (\ln \theta_1)^2\right)\right) \\ &= \frac{1}{\theta_1 \beta^2} [E(U (\ln U)^2) - 2 \ln \theta_1 E(U \ln U) + (\ln \theta_1)^2 E(U)] \\ &= \frac{(n+1)}{\theta_1 \beta^2} [\psi(n+2)^2 + \psi'(n+2) - 2 \ln \theta_1 \psi(n+2) + (\ln \theta_1)^2]. \end{aligned} \quad (2.10)$$

Similarly, since $V = \theta_2 s_m^\beta \sim \Gamma(m+1, 1)$, we have

$$\begin{aligned} E(s_m^\beta \ln s_m) &= \frac{(m+1)}{\beta \theta_2} [\psi(m+2) - \ln \theta_2], \\ E(s_m^\beta (\ln s_m)^2) &= \frac{(m+1)}{\theta_2 \beta^2} [\psi(m+2)^2 + \psi'(m+2) - 2 \ln \theta_2 \psi(m+2) + (\ln \theta_2)^2]. \end{aligned} \quad (2.11)$$

Let $R = \theta_1 / (\theta_1 + \theta_2) = h(\theta_1, \theta_2)$, the maximum likelihood estimator of R is given by $\hat{R} = \hat{\theta}_1 / (\hat{\theta}_1 + \hat{\theta}_2) = h(\hat{\theta}_1, \hat{\theta}_2)$, and it follows that

$$\sqrt{n}(\hat{R} - R) = \sqrt{n}[h(\hat{\theta}_1, \hat{\theta}_2) - h(\theta_1, \theta_2)] \xrightarrow{D} N(0, \eta^2), \quad (2.12)$$

as $n \rightarrow \infty, m \rightarrow \infty$ such that $m/n \rightarrow p, 0 < p < 1$. The asymptotic variance is given by

$$\begin{aligned} & \eta^2 \left(\frac{\partial h(\theta_1, \theta_2)}{\partial \theta_1} \right)^2 \text{var}(\hat{\theta}_1) + \left(\frac{\partial h(\theta_1, \theta_2)}{\partial \theta_2} \right)^2 \frac{\text{var}(\hat{\theta}_2)}{p} \\ & + 2 \text{cov}(\hat{\theta}_1, \hat{\theta}_2) \left(\frac{\partial h(\theta_1, \theta_2)}{\partial \theta_1} \right) \left(\frac{\partial h(\theta_1, \theta_2)}{\partial \theta_2} \right). \end{aligned} \quad (2.13)$$

A $(1 - \alpha)\%$ confidence interval for R based on this asymptotic result is given by

$$\left\{ \hat{R} - z_{1-\alpha/2} \hat{\eta}, \hat{R} + z_{1-\alpha/2} \hat{\eta} \right\}, \quad (2.14)$$

where $\hat{\eta}$ is obtained by substituting m/n for p and the MLEs of θ_1 and θ_2 in the asymptotic standard deviation η . Another interval can be obtained by using the matrix of minus the second partial derivatives of the log-likelihood function. This matrix can be used in place of the Fisher information matrix to obtain $\hat{\eta}$ as an estimator of the variance of $\hat{R}$. The new interval is given by

$$\left\{ \hat{R} - z_{1-\alpha/2} \hat{\eta}, \hat{R} + z_{1-\alpha/2} \hat{\eta} \right\}. \quad (2.15)$$

Many authors suggested that parameter transformation may improve the performance of intervals based on the asymptotic normality of the maximum likelihood estimator. For parameters representing probabilities, the logit transformation seems appropriate. Let $\lambda = \ln(R/1 - R)$, and the maximum likelihood of λ is given by $\hat{\lambda} = \ln(\hat{R}/1 - \hat{R})$. The asymptotic variance of $\sqrt{n}(\hat{\lambda} - \lambda)$ is given by $\eta^2(d\lambda/dR)^2$. This variance can be estimated by substituting the maximum likelihood estimator instead of the parameter. A $(1 - \alpha)\%$ confidence interval for λ is given by $(\hat{\lambda}_L, \hat{\lambda}_U)$, where $\hat{\lambda}_L = \hat{\lambda} - z_{1-\alpha/2} \hat{\eta} d\lambda/dR|_{R=\hat{R}}$ and $\hat{\lambda}_U = \hat{\lambda} + z_{1-\alpha/2} \hat{\eta} d\lambda/dR|_{R=\hat{R}}$, $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution. Therefore A $(1 - \alpha)\%$ confidence interval for R is given by

$$\left(\frac{e^{\hat{\lambda}_L}}{1 + e^{\hat{\lambda}_L}}, \frac{e^{\hat{\lambda}_U}}{1 + e^{\hat{\lambda}_U}} \right). \quad (2.16)$$

3. Bayesian Inference

As in Kundu and Gupta [11], we will use the conjugate gamma priors for the scale parameters θ_1 and θ_2 and a squared error loss function. The prior density of θ_j therefore is given by

$$\pi_j(\theta_j | a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j-1} e^{-b_j \theta_j}, \quad j = 1, 2, \quad (3.1)$$

where $a_j > 0, b_j > 0$ are the parameters of the prior distributions. Assuming that θ_1 and θ_2 are independent, the joint prior distribution of θ_1 and θ_2 is given by $\pi(\theta_1, \theta_2) = \prod_{j=1}^2 \pi_j(\theta_j)$.

Assume that the prior distribution of β , denoted by $\zeta(\beta)$, has a support on $(0, \infty)$. Recall that the likelihood function of $(\beta, \theta_1, \theta_2)$ based on $\underline{r} = r_0, \dots, r_n$ and $\underline{s} = s_0, \dots, s_m$ is given by

$$L(\beta, \theta_1, \theta_2 \mid \underline{r}, \underline{s}) = (\theta_1 \beta)^{n+1} e^{-\theta_1 r_n^\beta} \prod_{i=0}^n r_i^{\beta-1} (\theta_2 \beta)^{m+1} e^{-\theta_2 s_m^\beta} \prod_{i=0}^m s_i^{\beta-1}. \quad (3.2)$$

The joint posterior density function of $(\beta, \theta_1, \theta_2)$ therefore is given by;

$$\pi^*(\beta, \theta_1, \theta_2) = \frac{\pi(\theta_1, \theta_2) \zeta(\beta) L(\beta, \theta_1, \theta_2 \mid \underline{r}, \underline{s})}{\iint \int_0^\infty \pi(\theta_1, \theta_2) \zeta(\beta) L(\beta, \theta_1, \theta_2 \mid \underline{r}, \underline{s}) d\beta d\theta_1 d\theta_2}, \quad (3.3)$$

where the numerator is given by

$$\beta^{n+m+2} \zeta(\beta) \frac{b_1^{a_1}}{\Gamma(a_1)} \theta_1^{n+a_1} e^{-\theta_1 (r_n^\beta + b_1)} \prod_{i=0}^n r_i^{\beta-1} \frac{b_2^{a_2}}{\Gamma(a_2)} \theta_2^{m+a_2} e^{-\theta_2 (s_m^\beta + b_2)} \prod_{i=0}^m s_i^{\beta-1}. \quad (3.4)$$

This expression for $\pi^*(\beta, \theta_1, \theta_2)$ is difficult or even impossible to find in closed form. A simulation technique is needed. It is clear that the conditional posterior distributions of θ_1 and θ_2 given β are given by

$$\pi_1^*(\theta_1 \mid \beta) = \frac{(r_n^\beta + b_1)^{n+a_1+1}}{\Gamma(n+a_1+1)} \theta_1^{n+a_1} e^{-\theta_1 (r_n^\beta + b_1)}, \quad (3.5)$$

$$\pi_2^*(\theta_2 \mid \beta) = \frac{(s_m^\beta + b_2)^{m+a_2+1}}{\Gamma(m+a_2+1)} \theta_2^{m+a_2} e^{-\theta_2 (s_m^\beta + b_2)}. \quad (3.6)$$

Note that given β , θ_1 , and θ_2 are independent. From the joint posterior distribution of θ_1 , θ_2 , and β we obtain

$$\zeta^*(\beta) \propto \zeta(\beta) \beta^{n+m+2} \left(\prod_{i=0}^n r_i \prod_{i=0}^m s_i \right)^\beta (r_n^\beta + b_1)^{-(n+a_1+1)} (s_m^\beta + b_2)^{-(m+a_2+1)}. \quad (3.7)$$

Taking $a_1 = a_2 = 0$ and $b_1 = b_2 = 0$ in the prior distributions we obtain

$$\zeta^*(\beta) \propto \zeta(\beta) \beta^{n+m+2} \left(r_n^{-(n+1)} s_m^{-(m+1)} \prod_{i=0}^n r_i \prod_{i=0}^m s_i \right)^\beta. \quad (3.8)$$

If we can take $\zeta(\beta) = e^{-\beta}$, we obtain

$$\zeta^*(\beta) \propto \beta^{n+m+2} \left(e^{-1} r_n^{-(n+1)} s_m^{-(m+1)} \prod_{i=0}^n r_i \prod_{i=0}^m s_i \right)^\beta. \quad (3.9)$$

Table 1: Simulated biases and mean squared errors of the estimators, $\theta_1 = 1$.

	Bias $\hat{R}$	Bias $\tilde{R}$	MSE $\hat{R}$	MSE $\tilde{R}$	Bias $\hat{R}$	Bias $\tilde{R}$	MSE $\hat{R}$	MSE $\tilde{R}$
θ_2	$(n, m) = (6, 6)$				$(n, m) = (8, 8)$			
1	-0.004	-0.004	0.026	0.020	0.000	0.000	0.018	0.016
3	-0.020	0.011	0.015	0.012	-0.019	0.003	0.011	0.009
5	-0.021	0.027	0.010	0.008	-0.017	0.015	0.008	0.006
7	-0.018	0.029	0.006	0.007	-0.013	0.018	0.005	0.005
	$(n, m) = (10, 10)$				$(n, m) = (15, 15)$			
9	-0.017	0.031	0.004	0.006	-0.013	0.022	0.004	0.004
1	0.001	0.000	0.014	0.013	0.000	-0.000	0.009	0.009
3	-0.012	0.004	0.010	0.008	-0.009	-0.003	0.006	0.006
5	-0.015	0.007	0.006	0.005	-0.009	0.002	0.004	0.004
7	-0.013	0.012	0.004	0.004	-0.010	0.003	0.003	0.003
9	-0.012	0.014	0.003	0.003	-0.008	0.005	0.002	0.002

It follows that the posterior distribution of β is

$$\Gamma\left(n + m + 3, \ln\left(e^{-1} r_n^{-(n+1)} s_m^{-(m+1)} \prod_{i=0}^n r_i \prod_{i=0}^m s_i\right)^{-1}\right). \quad (3.10)$$

We can use the following Monte Carlo method to find approximate point estimates.

- (1) Generate β_1 from $\zeta^*(\beta)$ given by (3.10).
- (2) Generate $\theta_{1,1}$ and $\theta_{2,1}$ from $\pi_1^*(\theta_1 | \beta_1)$ and $\pi_2^*(\theta_2 | \beta_1)$ given by (3.5) and (3.6), respectively, and calculate $R_1 = \theta_{1,1}/\theta_{1,1} + \theta_{2,1}$.
- (3) Repeat steps (1) and (2) M times to get $R_1, \dots, R_M$.
- (4) Calculate the approximate Bayes estimator and the approximate posterior variance;

$$\begin{aligned} \hat{E}(R | \underline{r}, \underline{s}) &= \sum_{i=1}^M \frac{R_i}{M} = \bar{R}, \\ \hat{V}(R | \underline{r}, \underline{s}) &= \sum_{i=1}^M \frac{(R_i - \bar{R})^2}{M}. \end{aligned} \quad (3.11)$$

Approximate highest posterior density (HPD) intervals for the stress-strength reliability R may be found using the algorithm of Chen and Shao [12].

4. A Simulation Study

We conducted simulations to compare the performance of the point and interval estimators developed in this paper. In our simulations we also included some bootstrap intervals [13], namely, the percentile interval and the bootstrap- t interval based on the logit transformation

Table 2: Simulated error probabilities and expected width of the intervals, $\theta_1 = 1$.

Interval		L	U	T	W	L	U	T	W
θ_2		(n, m) = (6,6)				(n, m) = (8,8)			
1	ANE	0.079	0.080	0.159	0.492	0.065	0.062	0.127	0.443
1	ANO	0.079	0.080	0.159	0.492	0.065	0.062	0.127	0.443
1	ANT	0.043	0.046	0.088	0.460	0.040	0.037	0.076	0.418
1	HPD	0.027	0.065	0.092	0.545	0.023	0.068	0.091	0.507
1	Perc	0.063	0.063	0.126	0.562	0.053	0.048	0.101	0.498
1	BOOT	0.025	0.031	0.056	0.506	0.026	0.026	0.052	0.456
3	ANE	0.170	0.017	0.187	0.398	0.135	0.015	0.150	0.359
3	ANO	0.170	0.017	0.187	0.398	0.135	0.015	0.150	0.359
3	ANT	0.039	0.027	0.065	0.398	0.048	0.025	0.073	0.358
3	HPD	0.057	0.050	0.107	0.434	0.052	0.043	0.094	0.395
3	Perc	0.164	0.009	0.172	0.408	0.140	0.005	0.145	0.366
3	BOOT	0.029	0.015	0.043	0.435	0.036	0.015	0.051	0.387
5	ANE	0.211	0.009	0.220	0.313	0.186	0.008	0.194	0.287
5	ANO	0.211	0.009	0.220	0.313	0.186	0.008	0.194	0.287
5	ANT	0.051	0.023	0.074	0.339	0.049	0.025	0.073	0.304
5	HPD	0.037	0.042	0.079	0.364	0.051	0.037	0.088	0.322
5	Perc	0.200	0.004	0.204	0.296	0.178	0.004	0.181	0.273
5	BOOT	0.035	0.018	0.053	0.370	0.031	0.020	0.050	0.329
7	ANE	0.224	0.005	0.229	0.263	0.200	0.003	0.203	0.243
7	ANO	0.224	0.005	0.229	0.263	0.200	0.003	0.203	0.243
7	ANT	0.044	0.017	0.060	0.302	0.047	0.022	0.069	0.270
7	HPD	0.044	0.036	0.079	0.312	0.055	0.037	0.092	0.274
7	Perc	0.209	0.002	0.211	0.237	0.187	0.001	0.188	0.223
7	BOOT	0.033	0.014	0.047	0.331	0.030	0.020	0.050	0.293
9	ANE	0.261	0.003	0.264	0.222	0.212	0.004	0.215	0.208
9	ANO	0.261	0.003	0.264	0.222	0.212	0.004	0.215	0.208
9	ANT	0.049	0.013	0.062	0.271	0.035	0.017	0.051	0.242
9	HPD	0.038	0.036	0.074	0.278	0.048	0.033	0.080	0.244
9	Perc	0.237	0.001	0.237	0.195	0.188	0.003	0.191	0.186
9	BOOT	0.031	0.010	0.041	0.298	0.023	0.017	0.040	0.264
θ_2		(n, m) = (10,10)				(n, m) = (15,15)			
1	ANE	0.053	0.059	0.111	0.404	0.048	0.049	0.096	0.339
1	ANO	0.050	0.058	0.108	0.404	0.048	0.049	0.096	0.340
1	ANT	0.032	0.042	0.074	0.385	0.038	0.036	0.074	0.328
1	HPD	0.015	0.068	0.083	0.475	0.009	0.066	0.074	0.415
1	Perc	0.043	0.053	0.095	0.449	0.045	0.041	0.086	0.368
1	BOOT	0.020	0.028	0.047	0.415	0.021	0.027	0.048	0.348
3	ANE	0.119	0.020	0.139	0.330	0.097	0.015	0.112	0.281
3	ANO	0.113	0.020	0.133	0.333	0.095	0.015	0.110	0.283
3	ANT	0.041	0.031	0.072	0.331	0.041	0.022	0.063	0.281
3	HPD	0.044	0.049	0.093	0.370	0.028	0.036	0.063	0.321
3	Perc	0.115	0.015	0.130	0.339	0.096	0.009	0.105	0.286
3	BOOT	0.028	0.024	0.051	0.355	0.023	0.020	0.043	0.296

Table 2: Continued.

	Interval	L	U	T	W	L	U	T	W
θ_2		(n, m) = (10,10)				(n, m) = (15,15)			
5	ANE	0.152	0.007	0.159	0.263	0.125	0.009	0.134	0.228
5	ANO	0.147	0.006	0.153	0.267	0.120	0.009	0.129	0.231
5	ANT	0.045	0.019	0.063	0.279	0.038	0.022	0.060	0.237
5	HPD	0.055	0.034	0.089	0.294	0.040	0.034	0.074	0.255
5	Perc	0.143	0.003	0.145	0.255	0.116	0.007	0.123	0.223
5	BOOT	0.031	0.016	0.047	0.300	0.025	0.022	0.047	0.250
7	ANE	0.168	0.004	0.172	0.220	0.147	0.006	0.152	0.190
7	ANO	0.162	0.003	0.165	0.224	0.142	0.006	0.148	0.193
7	ANT	0.042	0.021	0.062	0.244	0.040	0.022	0.061	0.204
7	HPD	0.048	0.032	0.080	0.250	0.048	0.032	0.080	0.211
7	Perc	0.147	0.001	0.148	0.207	0.135	0.004	0.139	0.182
7	BOOT	0.027	0.022	0.049	0.263	0.026	0.029	0.055	0.216
9	ANE	0.201	0.004	0.204	0.188	0.143	0.003	0.146	0.166
9	ANO	0.198	0.003	0.201	0.192	0.139	0.003	0.141	0.168
9	ANT	0.039	0.016	0.055	0.218	0.033	0.020	0.053	0.183
9	HPD	0.051	0.031	0.081	0.219	0.053	0.023	0.075	0.183
9	Perc	0.176	0.002	0.178	0.174	0.124	0.002	0.126	0.156
9	BOOT	0.026	0.017	0.043	0.235	0.025	0.027	0.052	0.194

L: lower error probability, U: upper error probability, T: total error probability, W: expected width of the interval.

of R . We used $n = m = 6, 8, 10, 15$. We fixed $\theta_1 = 1$ and used $\theta_2 = 1, 3, 5, 7, 9$. The confidence level taken is $(1 - \alpha) = 0.95$. In each simulation run we generated 2000 samples of records from the distributions of X and Y . For each pair of samples we calculated the following intervals:

- (1) ANE: the interval based on the asymptotic normality of the MLE with variance estimate based on the Fisher information matrix, given by (2.14),
- (2) ANO: the interval based on the asymptotic normality of the MLE with variance estimate based on the observed information matrix, given by (2.15),
- (3) ANT: the interval based on the asymptotic normality of the MLE of the transformed parameter with variance estimate based on the observed information matrix, given by (2.16),
- (4) HPD: the approximate highest posterior density interval of Chen and Shao [12],
- (5) Perc: the percentile interval, [13], and
- (6) Boot: the bootstrap- t interval based on the logit transformation of the stress-strength reliability parameter, [13].

The biases and mean squared error of the Bayes and the maximum likelihood estimator are simulated and the results are given in Table 1. The lower (L), upper (U), and total (T) error rates and the expected widths (W) of the intervals are approximated using the results of the 2000 simulation replications. For the percentile and bootstrap- t intervals we used 1000 bootstrap resamples. The results of our simulations are given in Table 2.

5. Conclusions

Results concerning the performance of the Bayes estimator and the maximum likelihood estimator are given in Table 1. It appears that the MLE is negatively biased while the Bayes estimator is positively biased. However, the biases are very small, especially for large sample sizes. The mean squared errors of the estimators are very close, especially for larger samples. However, it appears that the MLE has higher MSE for values of R near 0.5 and small sample sizes. The situation is reversed for values of R close to the extremes.

The simulation results in Table 2 show that the widths of all intervals are maximized when $R = 0.5$, and they become narrower as the true value of R approaches the extremes. As expected, increasing the sample sizes also results in shorter intervals. The performance of the intervals (ANO) and (ANE) based on the asymptotic normality of the MLE are very similar. They tend to be anticonservative and are short. However, for the transformed intervals, the situation completely changed, and the results are very encouraging. This is true for (ANT) interval and the bootstrap- t interval based on the transformed parameter. The performance of bootstrap- t intervals for the original parameter is also investigated, but the results are very poor and not included in Table 2. The Bayes interval (HPD) appears to have better performance than (ANO), and (ANE). However it is dominated by the intervals based on the transformed parameter (ANT) and (Boot). The same is true for the percentile interval which tends to be anticonservative. Concerning the symmetry of lower (L) and upper (U) error rates, it appears that the (ANE), (ANO) and (Perc) intervals are highly asymmetric, especially for extreme values of the stress-strength reliability. On the other hand, the (ANT), (Boot), and (HPD) tend to be symmetric even for small samples. In conclusion, we would recommend the use of the bootstrap- t interval (Boot) based on the transformed parameter followed by (ANT) for all sample sizes.

References

- [1] K. N. Chandler, "The distribution and frequency of record values," *Journal of the Royal Statistical Society B*, vol. 14, pp. 220–228, 1952.
- [2] M. Ahsanullah, *Record Values: Theory and Applications*, University Press of America, Lanham, Md, USA, 2004.
- [3] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *Records*, Wiley, 1998.
- [4] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, vol. 1, John Wiley & Sons, New York, NY, USA, 1994.
- [5] D. N. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, Wiley Series in Probability and Statistics, Wiley, 2004.
- [6] I. J. Hall, "Approximate one-sided tolerance limit for the difference or sum of two independent random variates," *Journal of Quality Technology*, vol. 16, no. 1, pp. 15–19, 1984.
- [7] S. Weerahandi and R. A. Johnson, "Testing reliability in a stress-strength model when X and Y are normally distributed," *Technometrics*, vol. 34, no. 1, pp. 83–91, 1992.
- [8] S. Kotz, Y. Lumelskii, and M. Pensky, *The Stress-Strength Model and Its Generalizations: Theory and Applications*, World Scientific, 2003.
- [9] A. Baklizi, "Estimation of $\Pr(X < Y)$ using record values in the one and two parameter exponential distributions," *Communications in Statistics. Theory and Methods*, vol. 37, no. 5, pp. 692–698, 2008.
- [10] A. Baklizi, "Likelihood and Bayesian estimation of $\Pr(X < Y)$ using lower record values from the generalized exponential distribution," *Computational Statistics and Data Analysis*, vol. 52, no. 7, pp. 3468–3473, 2008.
- [11] D. Kundu and R. D. Gupta, "Estimation of $P[Y < X]$ for Weibull distributions," *IEEE Transactions on Reliability*, vol. 55, no. 2, pp. 270–280, 2006.

- [12] M. H. Chen and Q. M. Shao, "Monte carlo estimation of bayesian credible and HPD interval," *Journal of Computational and Graphical Statistics*, vol. 8, no. 1, pp. 69–92, 1999.
- [13] B. Efron and R. J. Tibshirani, *An Introduction To the Bootstrap*, Chapman & Hall, 1993.