Research Article

On a Class of Meromorphic Functions Related to Cho-Kwon-Srivastava Operator

F. Ghanim¹ and M. Darus²

¹ Faculty of Management, Multimedia University, Selangor Darul Ehsan, 63100 Cyberjaya, Malaysia

² School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Selangor Darul Ehsan, 43600 Bangi, Malaysia

Correspondence should be addressed to M. Darus, maslina@ukm.my

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We introduce new classes, $\Sigma_{n,\lambda}^{*,\mu,k}(\alpha, \beta, \rho)$ and $\Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho)$, of meromorphic functions defined by means of the Hadamard product of Cho-Kwon-Srivastava operator, and we define here a similar transformation by means of an operator given by Ghanim and Darus, in $U = \{z : 0 < |z| < 1\}$, and investigate a number of inclusion relationships of these classes. We also derive some interesting properties of these classes.

1. Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of Σ consisting of all meromorphic functions, which are, respectively, starlike of order β and convex of order β in U .

For functions $f_j(z)$ ($j = 1; 2$) defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad (1.2)$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \quad (1.3)$$

Let $P_n(\rho)$ be the class of functions $p(z)$ analytic in U satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1 - \rho} \right| d\theta \leq n\pi, \quad (1.4)$$

where $z = re^{i\theta}$, $n \geq 2$, and $0 \leq \rho < 1$. This class has been introduced in [1]. We note that $P_n(0) = P_n$ and $P_2(\rho) = P(\rho)$, see ([2, 3]), the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (1.6), we can write $p \in P_n(\rho)$ as

$$p(z) = \left(\frac{n}{4} + \frac{1}{2} \right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4} \right) p_2(z), \quad (1.5)$$

where $p_i(z) \in P(\rho)$, $i = 1, 2$ and $z \in U$.

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| z^n, \quad (1.6)$$

for $\beta \neq 0, -1, -2, \dots$, and $\alpha \in \mathbb{C}/\{0\}$, where $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ is the Pochhammer symbol. We note that

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} {}_2F_1(1, \alpha, \beta; z), \quad (1.7)$$

where

$${}_2F_1(b, \alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(b)_n (\alpha)_n}{(\beta)_n} \frac{z^n}{n!} \quad (1.8)$$

is the well-known Gaussian hypergeometric function.

Let us put

$$q_{\lambda, \mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} z^n, \quad (\lambda > 0, \mu \geq 0). \quad (1.9)$$

Corresponding to the functions $\tilde{\phi}(\alpha, \beta; z)$ and $q_{\lambda, \mu}(z)$, and using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L(\alpha, \beta, \lambda, \mu)$ on Σ by

$$\begin{aligned} L(\alpha, \beta, \lambda, \mu)f(z) &= \left(f(z) * \tilde{\phi}(\alpha, \beta; z) * q_{\lambda, \mu}(z) \right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} a_n z^n. \end{aligned} \quad (1.10)$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [4, 5], Liu [6], Liu and Srivastava [7–9], and Cho and Kim [10].

For a function $f \in L(\alpha, \beta, \lambda, \mu)f(z)$, we define

$$I_{\alpha, \beta, \lambda}^{\mu, 0} = L(\alpha, \beta, \lambda, \mu)f(z) \quad (1.11)$$

and for $k = 1, 2, 3, \dots$,

$$\begin{aligned} I_{\alpha, \beta, \lambda}^{\mu, k}f(z) &= z \left(I^{k-1} L(\alpha, \beta, \lambda, \mu)f(z) \right)' + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} a_n z^n. \end{aligned} \quad (1.12)$$

Note that if $n = \beta$, $k = 0$, the operator $I_{\alpha, n, \lambda}^{\mu, 0}$ has been introduced by Cho et al. [11] for $\mu \in \mathbb{N}_0 = \mathbb{N} \cup 0$. It was known that the definition of the operator $I_{\alpha, n, \lambda}^{\mu, 0}$ was motivated essentially by the Choi-Saigo-Srivastava operator [12] for analytic functions, which includes a simpler integral operator studied earlier by Noor [13] and others (cf. [14–16]). Note also the operator $I_{\alpha, \beta}^{0, k}$ has been recently introduced and studied by Ghanim and Darus, [17], [18] and [19], respectively. To our best knowledge, the recent work regarding operator $I_{\alpha, n, \lambda}^{\mu, 0}$ was charmingly studied by Piejko and Sokół [20]. Moreover, the operator $I_{\alpha, \beta, \lambda}^{\mu, k}$ was defined and studied by Ghanim and Darus [21]. In the same direction, we will study the operator $I_{\alpha, \beta, \lambda}^{\mu, k}$ given in (1.12).

Now, it follows from (1.10) and (1.12) that

$$z \left(I_{\alpha, \beta, \lambda}^{\mu, k}f(z) \right)' = \alpha I_{\alpha+1, \beta, \lambda}^{\mu, k}f(z) - (\alpha+1) I_{\alpha, \beta, \lambda}^{\mu, k}f(z). \quad (1.13)$$

In the present paper, we will use of the operator $I_{\alpha, \beta, \lambda}^{\mu, k}f(z)$ and introduce some new classes of meromorphic functions.

Definition 1.1. Let $f \in \Sigma$. Then $f \in \Sigma_{n, \lambda}^{*, \mu, k}(\alpha, \beta, \rho)$ if and only if

$$-(1-\alpha)z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k}f(z) \right)' - \alpha z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k}f(z) \right)' \in P_n(\rho), \quad (1.14)$$

where $\alpha > 0$, $n \geq 2$, $0 \leq \rho < 1$, and $z \in U$.

Definition 1.2. Let $f \in \Sigma$. Then $f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho)$ if and only if

$$(1 - \alpha)z \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' - \alpha z \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' \in P_n(\rho), \quad (1.15)$$

where $\alpha > 0$, $n \geq 2$, $0 \leq \rho < 1$, and $z \in U$.

2. Preliminary Results

Lemma 2.1 (see [22]). *If $p(z)$ is analytic in U with $p(0) = 1$, and if α is a complex number satisfying $\Re(\alpha) \geq 0$, then*

$$\Re\{p(z) + \alpha zp'(z)\} > \beta \quad (0 \leq \beta < 1) \quad (2.1)$$

implies

$$\Re p(z) > \beta + (1 - \beta)(2\gamma - 1), \quad (2.2)$$

where γ is given by

$$\gamma = \gamma(\alpha) = \int_0^1 \left(1 + t^{\Re \alpha}\right)^{-1} dt, \quad (2.3)$$

which is an increasing function of $\Re(\alpha)$ and $1/2 \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2 (see [23]). *If $p(z)$ is analytic in U , $p(0) = 1$, and $\Re p(z) > 1/2$, $z \in U$, then for any function F analytic in U , the function $p * F$ takes values in the convex hull of the image of U under F .*

Lemma 2.3 (see [24]). *Let $p(z) = 1 + b_1 z + b_2 z^2 + \dots \in P(\rho)$. Then*

$$\Re p(z) \geq 2\rho - 1 + \frac{2(1 - \rho)}{1 + |z|}. \quad (2.4)$$

3. Main Results

Theorem 3.1. *Let $f \in \Sigma_{n,\lambda}^{*,\mu,k}(\alpha, \beta, \rho)$. Then*

$$-z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' \in P_n(\rho_1), \quad (3.1)$$

where ρ_1 is given by

$$\rho_1 = \rho + (1 - \rho)(2\gamma - 1), \quad (3.2)$$

$$\gamma = \int_0^1 \left(1 + t^{\Re \alpha}\right)^{-1} dt. \quad (3.3)$$

Proof. Let

$$-z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' = p(z) = \left(\frac{n}{4} + \frac{1}{2} \right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4} \right) p_2(z). \quad (3.4)$$

Then $p(z)$ is analytic in U with $p(0) = 1$. Applying the identity (1.13) in (3.4) and differentiating the resulting equation with respect to z , we have

$$-(1 - \alpha)z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' - \alpha z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' = [p(z) + \alpha z p'(z)]. \quad (3.5)$$

Since $f \in \Sigma_{n, \lambda}^{*, \mu, k}(\alpha, \beta, \rho)$, so $\{p(z) + \alpha z p'(z)\} \in P_n(\rho)$ for $z \in U$. This implies that

$$\Re\{p_i(z) + \alpha z p'_i(z)\} > \rho, \quad i = 1, 2. \quad (3.6)$$

Using Lemma 2.1, we see that $\Re\{p_i(z)\} > \rho_1$, where ρ_1 is given by (3.2).

Consequently, $p \in P_n(\rho_1)$ for $z \in U$, and the proof is complete. $\square$

Theorem 3.2. Let $f \in \Sigma_{n, \lambda}^{*, \mu, k}(0, \beta, \rho)$ for $z \in U$. Then $f \in \Sigma_{n, \lambda}^{*, \mu, k}(\alpha, \beta, \rho)$ for $|z| < \Re(\alpha)$, where

$$\Re(\alpha) = \frac{1}{\alpha + \sqrt{1 + \alpha^2}}. \quad (3.7)$$

Proof. Set

$$-z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' = (1 - \rho)h(z) + \rho, \quad h \in P_n. \quad (3.8)$$

Now proceeding as in Theorem 3.1, we have

$$\begin{aligned} & -(1 - \alpha)z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' - \alpha z^2 \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' - \rho \\ &= (1 - \rho)\{h(z) + \alpha z h'(z)\} \\ &= (1 - \rho) \left[\left(\frac{n}{4} + \frac{1}{2} \right) \{h_1(z) + \alpha z h'_1(z)\} + \left(\frac{1}{2} - \frac{n}{4} \right) \{h_2(z) + \alpha z h'_2(z)\} \right], \end{aligned} \quad (3.9)$$

where we have used (1.6) and $h_1, h_2 \in P$, $z \in U$. Using the following well-known estimate [25]

$$|zh'_i(z)| \leq \frac{2r}{1-r^2} \Re\{h_i(z)\} \quad (|z| = r < 1, i = 1, 2), \quad (3.10)$$

we have

$$\Re\{h_i(z) + \alpha zh'_i(z)\} \geq \Re\{h_i(z) + \alpha |zh'_i(z)|\} \geq \Re h_i(z) \left\{1 - \frac{2\alpha r}{1-r^2}\right\}. \quad (3.11)$$

The right-hand side of this inequality is positive if $r < R(\alpha)$, where $R(\alpha)$ is given by (3.7). Consequently, it follows from (3.9) that $f \in \Sigma_{n,\lambda}^{*,\mu,k}(\alpha, \beta, \rho)$ for $|z| < R(\alpha)$. Sharpness of this result follows by taking $h_i(z) = (1+z)/(1-z)$ in (3.9), $i = 1, 2$. $\square$

Theorem 3.3. Let $f \in \Sigma_{n,\lambda}^{*,\mu,k}(0, \beta, \rho)$, and let

$$F_\delta(f(z)) = \frac{\delta}{z^{\delta+1}} \int_0^z t^\delta f(t) dt \quad (\delta > 0, z \in U). \quad (3.12)$$

Then

$$-z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} F(f(z)) \right)' \in P_n(\rho_2), \quad (3.13)$$

where ρ_2 is given by

$$\rho_2 = \rho + (1-\rho)(2\gamma_1 - 1), \quad (3.14)$$

$$\gamma_1 = \int_0^1 \left(1 + t^{\Re(1/\delta)}\right)^{-1} dt. \quad (3.15)$$

Proof. Setting

$$-z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} F(f(z)) \right)' = p(z) = \left(\frac{n}{4} + \frac{1}{2} \right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4} \right) p_2(z), \quad (3.16)$$

then $p(z)$ is analytic in U with $p(0) = 1$. Using the following operator identity:

$$z \left(I_{\alpha,\beta,\lambda}^{\mu,k} F(f(z)) \right)' = \delta \left(I_{\alpha,\beta,\lambda}^{\mu,k} F(f(z)) \right) - (\delta + 1) \left(I_{\alpha,\beta,\lambda}^{\mu,k} F(f(z)) \right), \quad (3.17)$$

in (3.16), and differentiating the resulting equation with respect to z , we find that

$$-z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' = \left\{ p(z) + \frac{1}{\delta} z p'(z) \right\} \in P_n(\rho). \quad (3.18)$$

Using Lemma 2.1, we see that $-z^2(I_{\alpha,\beta,\lambda}^{\mu,k}F(f(z)))' \in P_n(\rho_2)$ for $z \in U$, where ρ_2 is given by (3.14).

Hence, the proof is complete. $\square$

Theorem 3.4. Let $\phi(z) \in \Sigma$ satisfy the following inequality:

$$\Re(z\phi(z)) > \frac{1}{2}, \quad z \in U. \quad (3.19)$$

If $f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho)$, then $\phi * f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho)$.

Proof. Let $G = \phi * f$. Then

$$\begin{aligned} & (1-\alpha)z\left(I_{\alpha,\beta,\lambda}^{\mu,k}G(z)\right) + \alpha z\left(I_{\alpha,\beta,\lambda}^{\mu,k}G(z)\right) \\ &= (1-\alpha)z\left(I_{\alpha,\beta,\lambda}^{\mu,k}(\phi * f)(z)\right) + \alpha z\left(I_{\alpha,\beta,\lambda}^{\mu,k}(\phi * f)(z)\right) \\ &= z\phi(z) * h(z), \quad h \in P_n(\rho) \\ &= \left(\frac{n}{4} + \frac{1}{2}\right)[(1-\rho)\{z\phi(z) * h_1(z) + \rho\}] \\ &\quad + \left(\frac{1}{2} - \frac{n}{4}\right)[(1-\rho)\{z\phi(z) * h_2(z) + \rho\}], \quad h_1, h_2 \in P. \end{aligned} \quad (3.20)$$

Since $\Re(z\phi(z)) > 1/2$ and by using Lemma 2.2, we can conclude that $G = \phi * f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho)$. $\square$

Theorem 3.5. Let $\phi(z) \in \Sigma$ satisfy the inequality (3.19) and $f \in \Sigma_{n,\lambda}^{*,\mu,k}(0, \beta, \rho)$. Then $\phi * f \in \Sigma_{n,\lambda}^{*,\mu,k}(0, \beta, \rho)$.

Proof. We have

$$-z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}(\phi * f)(z)\right)' = -z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)' * z\phi(z), \quad z \in U. \quad (3.21)$$

Now the remaining part of Theorem 3.5 follows by employing the techniques that we used in proving Theorem 3.4 above. $\square$

Theorem 3.6. Let $f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho_3)$ and $g \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho_4)$, and let $F = f * g$. Then $F \in \Sigma_n(\alpha, \beta, k, \rho_5)$, where

$$\rho_5 = 1 - 4(1-\rho_3)(1-\rho_4)\left[1 - \frac{1}{\alpha} \int_0^1 \frac{u^{(1/(1-\alpha))-1}}{1+u} du\right]. \quad (3.22)$$

This result is sharp.

Proof. Since $f \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho_3)$ and $g \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho_4)$, it follows that

$$\begin{aligned} S(z) &= (1-\alpha)z \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) + \alpha z \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) \in P_n(\rho_3), \\ S^*(z) &= (1-\alpha)z \left(I_{\alpha,\beta,\lambda}^{\mu,k} g(z) \right) + \alpha z \left(I_{\alpha,\beta,\lambda}^{\mu,k} g(z) \right) \in P_n(\rho_4), \end{aligned} \quad (3.23)$$

and so using identity (1.13) in the above equation, we have

$$\begin{aligned} I_{\alpha,\beta,\lambda}^{\mu,k} f(z) &= \frac{1}{\alpha} z^{-1-1/\alpha} \int_0^z t^{1/\alpha-1} S(t) dt, \\ I_{\alpha,\beta,\lambda}^{\mu,k} g(z) &= \frac{1}{\alpha} z^{-1-1/\alpha} \int_0^z t^{1/\alpha-1} S^*(t) dt. \end{aligned} \quad (3.24)$$

Using (3.24), we have

$$I_{\alpha,\beta,\lambda}^{\mu,k} f(z) = \frac{1}{\alpha} z^{-1-1/\alpha} \int_0^z t^{1/\alpha-1} Q(t) dt, \quad (3.25)$$

where

$$Q(z) = \left(\frac{n}{4} + \frac{1}{2} \right) q_1(z) + \left(\frac{1}{2} - \frac{n}{4} \right) q_2(z) = \frac{1}{\alpha} z^{-1-1/\alpha} \int_0^z t^{1/\alpha-1} (S * S^*)(t) dt. \quad (3.26)$$

Now

$$S(z) = \left(\frac{n}{4} + \frac{1}{2} \right) s_1(z) + \left(\frac{1}{2} - \frac{n}{4} \right) s_2(z), \quad (3.27)$$

$$S^*(z) = \left(\frac{n}{4} + \frac{1}{2} \right) s_1^*(z) + \left(\frac{1}{2} - \frac{n}{4} \right) s_2^*(z), \quad (3.28)$$

where $s_i \in P(\rho_3)$ and $s_i^* \in P(\rho_4)$, $i = 1, 2$. Since

$$P_i^*(z) = \frac{s_i^*(z) - \rho_4}{2(1 - \rho_4)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \quad i = 1, 2, \quad (3.29)$$

we obtain that $(s_i * p_i^*)(z) \in P(\rho_3)$, by using the Herglots formula. Thus

$$(s_i * s_i^*)(z) \in P(\rho_5) \quad (3.30)$$

with

$$\rho_5 = 1 - 2(1 - \rho_3)(1 - \rho_4). \quad (3.31)$$

Using (3.25), (3.26), (3.28), (3.31), and Lemma 2.3, we have

$$\begin{aligned}
 \Re q_i(z) &= \frac{1}{\alpha} \int_0^1 u^{1/\alpha-1} \Re \{ (s_i * s_i^*)(uz) \} du \geq \frac{1}{\alpha} \int_0^1 u^{1/\alpha-1} \left(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u|z|} \right) du \\
 &\geq \frac{1}{\alpha} \int_0^1 u^{1/\alpha-1} \left(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u} \right) du \\
 &= 1 - 4(1-\rho_3)(1-\rho_4) \left[1 - \frac{1}{\alpha} \int_0^1 \frac{u^{(1/(1-\alpha))-1}}{1+u} du \right].
 \end{aligned} \tag{3.32}$$

From this, we conclude that $F \in \Sigma_{n,\lambda}^{\mu,k}(\alpha, \beta, \rho_5)$, where ρ_5 is given by (3.22). We discuss the sharpness as follows: we take

$$\begin{aligned}
 S(z) &= \left(\frac{n}{4} + \frac{1}{2} \right) \frac{1 + (1-\rho_3)z}{1-z} + \left(\frac{1}{2} - \frac{n}{4} \right) \frac{1 - (1-\rho_3)z}{1+z}, \\
 S^*(z) &= \left(\frac{n}{4} + \frac{1}{2} \right) \frac{1 + (1-\rho_4)z}{1-z} + \left(\frac{1}{2} - \frac{n}{4} \right) \frac{1 - (1-\rho_4)z}{1+z}.
 \end{aligned} \tag{3.33}$$

Since

$$\left(\frac{1 + (1-\rho_3)z}{1-z} \right) * \left(\frac{1 + (1-\rho_4)z}{1-z} \right) = 1 - 4(1-\rho_3)(1-\rho_4) + \frac{4(1-\rho_3)(1-\rho_4)}{1-z}, \tag{3.34}$$

it follows from (3.26) that

$$\begin{aligned}
 q_i(z) &= \frac{1}{\alpha} \int_0^1 u^{1/\alpha-1} \left\{ 1 - 4(1-\rho_3)(1-\rho_4) + \frac{4(1-\rho_3)(1-\rho_4)}{1-z} \right\} du \\
 &\longrightarrow 1 - 4(1-\rho_3)(1-\rho_4) \left\{ 1 - \frac{1}{\alpha} \int_0^1 \frac{u^{(1/(1-\alpha))-1}}{1+u} du \right\}
 \end{aligned} \tag{3.35}$$

as $z \rightarrow -1$. This completes the proof. $\square$

Note 1. Some other works related to Cho-Kwon-Srivastava operator can also be referred to [26–29].

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