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Research Article

Another Approach to the Extended Stokes' Problems for the Oldroyd-B Fluid

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The extended Stokes problems, which study the flow suddenly driven by relatively moving half-planes, are reexamined for the Oldroyd-B fluid. This topic has been studied (Liu, 2011) by applying the series expansion to calculate the inverse Laplace transform. The derived solution was correct but tough to perform the calculation due to the series expansion of infinite terms. Herein another approach, the contour integration, is applied to calculate the inversion. Moreover, the Heaviside unit step function is included into the boundary condition to ensure the consistence between boundary and initial conditions. Mathematical methods used herein can be applied to other fluids for the extended Stokes' problems.

1. Introduction

In 1995, Zeng and Weinbaum's paper studying the viscous flow driven by relatively moving half-planes is a new as well as pioneering work for theoretical fluid mechanics [1]. This problem is later named as the extended Stokes' problem. Different from the well-known traditional Stokes' problems in which the flow depends on only one spatial parameter [2, 3], the extended Stokes' problems possess two spatial dependences. The additional spatial dependence makes the problem more complex than the traditional one. To solve the velocity, one needs to perform two different integral transforms to the momentum equation with the help of boundary and initial conditions given. For the Newtonian fluid, the analytic solutions for a finite-depth and an infinite-depth cases have been provided [4, 5].

As for non-Newtonian fluids, the extended problem for the Oldroyd-B fluid was analyzed [6] by expanding the rheological parameters as well as the spatial and temporal variables in a series form. This expansion provides a way to carry out the inverse transform to pursue the exact velocity profile. As the final solution is also expressed in an infinite series

form, it is sometimes time-consuming to perform the calculation. To this end, the contour integration was adopted to perform the inverse calculation for the flow of a second-grade fluid [7]. Although the contour integration is a correct and universal tool for calculating the inverse Laplace transform, it is sometimes laborious, and even impossible, due to the complex mathematical structure of the transformed variable. Different fluid models usually have different transformed types. The variety of transformed types makes a fixed contour path impossible and thus leads to the difficulty of calculating the integration.

From the mathematical viewpoint, it seems easy to reduce the Oldroyd-B model to either the Maxwell fluid or the second-grade fluid by simply setting one parameter to be zero. However, it is not always true for the inversion calculation using the contour integration. For the second-grade fluid, the contour path shown in [7] succeeds in calculating the inversion while the same path cannot be applied to the Oldroyd-B fluid. More singularity points which should be excluded from the contour make the latter case much more complicated. This is why we study the topic again.

The organization of this paper is as follows. In Section 2, the fluid system including the momentum equation, boundary and initial conditions is elucidated. The transform integrals adopted are also introduced. Next, the detailed derivation is presented in Section 3, and concluding remarks are made in Section 4.

2. Problem Description

First, the constitutive equation, boundary and initial conditions considered are shown below (see [6] for details)

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2.1}$$

$$u(y=0,z>0)=g(t)H(t), \tag{2.2}$$

$$u(y=0,z<0)=-g(t)H(t), \tag{2.3}$$

$$u(y\longrightarrow\infty)=0, \tag{2.4}$$

$$u(z\longrightarrow\pm\infty)\quad\text{is finite}, \tag{2.5}$$

$$u(t=0)=0, \tag{2.6}$$

where u represents the velocity along the x direction, the fluid constants ν, λ, and λ_r are the kinematic viscosity, the relaxation time, and the retardation times, and

$$g(t)=\begin{cases}u_0,\\ u_0\cos(t\sigma+\theta)\end{cases} \tag{2.7}$$

are the plate boundary conditions for the first and second problems, respectively. The Heaviside unit step function $H(t)$ is defined as

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases} \tag{2.8}$$

The inclusion of the Heaviside unit step function is to avoid the inconsistence between (2.2) and (2.6) at the point $(y, t) = (0, 0)$ (also see [8] for reference). Due to the system being the odd function of z, one only needs to calculate the positive-z domain, and the boundary condition (2.3) is thus replaced by

$$u(z = 0) = 0. \tag{2.9}$$

Now applying the Laplace transform

$$\widehat{f}(s) \equiv \int_0^\infty f(t) \cdot e^{-st} dt, \tag{2.10}$$

and then the Fourier sine transform

$$\widetilde{f}(\omega) \equiv \int_0^\infty f(y) \sin(y\omega) dy, \tag{2.11}$$

to the present PDE system, the transformed velocity is solved to be (see [6] for details)

$$\widetilde{u} = \omega \left[\omega^2 + \frac{s(1+\lambda s)}{\nu(1+\lambda_r s)}\right]^{-1} \left[1 - \exp\left(-z\sqrt{\omega^2 + \frac{s(1+\lambda s)}{\nu(1+\lambda_r s)}}\right)\right] \cdot \widehat{g}(s), \tag{2.12}$$

where

$$\widehat{g}(s) = \begin{cases} \dfrac{u_0}{s}, \\ u_0\left(\dfrac{s\cos\theta}{s^2+\sigma^2} - \dfrac{\sigma\sin\theta}{s^2+\sigma^2}\right), \end{cases} \tag{2.13}$$

for two kinds of problems, respectively. Above methods and results have been elucidated in [6]. The existing paper employed a series expansion to (2.12) to calculate the inverse transformation and then derived the solution in a series form. However, this expansion may result in a possible divergence. In the next section, another approach using the contour integration will be introduced for the sake of overcoming this weakness.

3. Contour Integration

As mentioned above, a different approach for calculating the inverse Laplace transform is adopted to derive the solution. Firstly, the term in the square root of the exponential function of (2.12) is written

$$\omega^2 + \frac{s(1+\lambda s)}{\nu(1+\lambda_r s)} = \frac{\lambda}{\nu\lambda_r} \cdot \frac{(s-s_1)(s-s_2)}{(s-s_0)}, \tag{3.1}$$

where

$$s_0 = -\frac{1}{\lambda_r}, \tag{3.2}$$

$$s_1 = \frac{1}{2\lambda}\left[-\left(1+\omega^2\nu\lambda_r\right) + \sqrt{\left(1+\omega^2\nu\lambda_r\right)^2 - 4\omega^2\nu\lambda}\right], \tag{3.3}$$

$$s_2 = \frac{1}{2\lambda}\left[-\left(1+\omega^2\nu\lambda_r\right) - \sqrt{\left(1+\omega^2\nu\lambda_r\right)^2 - 4\omega^2\nu\lambda}\right]. \tag{3.4}$$

It is clear that the sign of $(1+\omega^2\nu\lambda_r)^2 - 4\omega^2\nu\lambda$ strongly dominate the pole positions inside the contour path which will be demonstrated later. With the help of (3.1), the inversion of (2.12) is shown

$$u = \frac{2}{\pi}\int_{\omega=0}^{\omega=\infty} d\omega\, \omega \sin(y\omega) \frac{1}{2\pi i}\int_{s=\kappa-i\infty}^{s=\kappa+i\infty} I \cdot ds, \tag{3.5}$$

where

$$I = \frac{\nu\lambda_r}{\lambda} \cdot \frac{(s-s_0)\widehat{g}(s)}{(s-s_1)(s-s_2)}\left[1 - \exp\left(-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s-s_0}}\right)\right] \cdot e^{st}. \tag{3.6}$$

Two cases for calculating (3.5) are discussed below.

3.1. The Case $\lambda_r \geq \lambda > 0$

For the case $\lambda_r \geq \lambda > 0$, the term in the square root of (3.3) and (3.4) can be rewritten as

$$\left(1-\omega^2\nu\lambda_r\right)^2 + 4\omega^2\nu(\lambda_r - \lambda), \tag{3.7}$$

which possess a nonnegative value implying that both of s_1 and s_2 are negative real numbers for all values of ω. Thus the relation between s_0, s_1, and s_2 is

$$s_1 \geq s_0 \geq s_2. \tag{3.8}$$

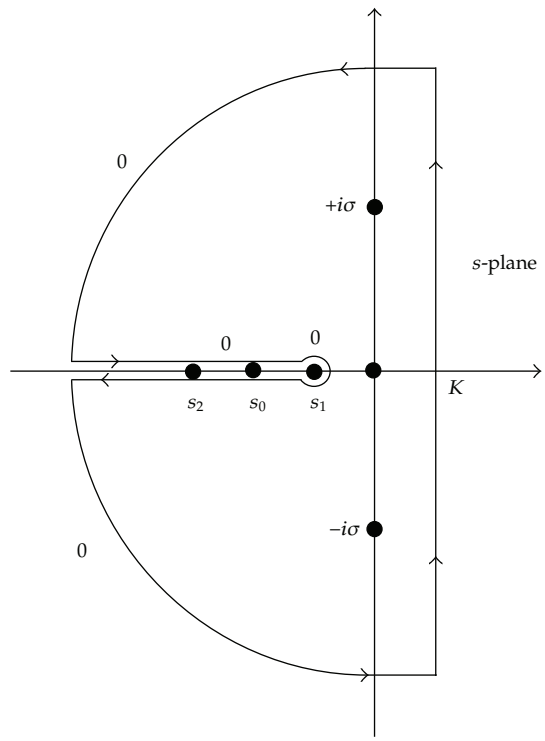

Figure 1: The contour path for the case $\lambda_r \geq \lambda > 0$. Single pole $s = 0$ is for the first problem and two poles $s = i\Omega$ and $s = -i\Omega$ are for the second problem.

Now the contour integration shown in Figure 1 is adopted to calculate the inner integral in (3.5). The result is

$$\begin{aligned}\int_{\kappa-i\infty}^{\kappa+i\infty} I \cdot ds = {} & 2i \int_{s_2}^{-\infty} \frac{\nu\lambda_r}{\lambda} \cdot \frac{(s-s_0)\widehat{g}(s)}{(s-s_1)(s-s_2)} \sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right) \cdot e^{st} ds \\ & + 2i \int_{s_1}^{s_0} \frac{\nu\lambda_r}{\lambda} \cdot \frac{(s-s_0)\widehat{g}(s)}{(s-s_1)(s-s_2)} \sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right) \cdot e^{st} ds \\ & + 2\pi i \cdot \mathcal{R},\end{aligned} \tag{3.9}$$

where $\mathcal{R}$ denotes the contribution from the poles. It is clear that only one pole $s = 0$ (note $\widehat{g}(s) = u_0/s$) exists for the first problem and the residue is

$$\mathcal{R} = \operatorname{Res}(0) = \frac{u_0 \nu\lambda_r s_0}{\lambda s_1 s_2}\left[\exp\left(-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{-s_1 s_2}{s_0}}\right) - 1\right]. \tag{3.10}$$

As for the second problem, there are two poles, $s = i\sigma$ and $s = -i\sigma$, and the result is

$$\begin{aligned}
\mathcal{R} &= \operatorname{Res}(+i\sigma) + \operatorname{Res}(-i\sigma) \\
&= \frac{u_0 \nu \lambda_r}{\lambda} \cdot \frac{r_0}{r_1 r_2} \cdot \Bigg\{ \cos(t\sigma + \theta + \phi_0 - \phi_1 - \phi_2) \\
&\qquad - \exp\left(-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{r_1 r_2}{r_0}} \cos\left(\frac{\phi_1 + \phi_2 - \phi_0}{2}\right)\right) \\
&\qquad \cdot \cos\Bigg[t\sigma + \theta + \phi_0 - \phi_1 - \phi_2 \\
&\qquad\qquad - z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{r_1 r_2}{r_0}} \sin\left(\frac{\phi_1 + \phi_2 - \phi_0}{2}\right)\Bigg]\Bigg\},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
r_0 &= \sqrt{\sigma^2 + \lambda_r^{-2}}, \\
r_1 &= \sqrt{\sigma^2 + \frac{1}{4\lambda^2}\left[(1 + \omega^2\nu\lambda_r) - \sqrt{(1 + \omega^2\nu\lambda_r)^2 - 4\omega^2\nu\lambda}\right]^2}, \\
r_2 &= \sqrt{\sigma^2 + \frac{1}{4\lambda^2}\left[(1 + \omega^2\nu\lambda_r) + \sqrt{(1 + \omega^2\nu\lambda_r)^2 - 4\omega^2\nu\lambda}\right]^2}, \\
\phi_0 &= \tan^{-1}(\sigma\lambda_r), \\
\phi_1 &= \tan^{-1}\left[\frac{2\lambda\sigma}{(1 + \omega^2\nu\lambda_r) - \sqrt{(1 + \omega^2\nu\lambda_r)^2 - 4\omega^2\nu\lambda}}\right], \\
\phi_2 &= \tan^{-1}\left[\frac{2\lambda\sigma}{(1 + \omega^2\nu\lambda_r) + \sqrt{(1 + \omega^2\nu\lambda_r)^2 - 4\omega^2\nu\lambda}}\right].
\end{aligned} \tag{3.12}$$

The definition sketch for (3.12) is displayed in Figure 2. It is remarked that r_1, r_2, ϕ_1, and ϕ_2 depend on ω. Substituting (3.9) into (3.6) with the help of (3.10) to (3.12), (3.5) can be calculated to obtain the exact solution of flow velocity u.

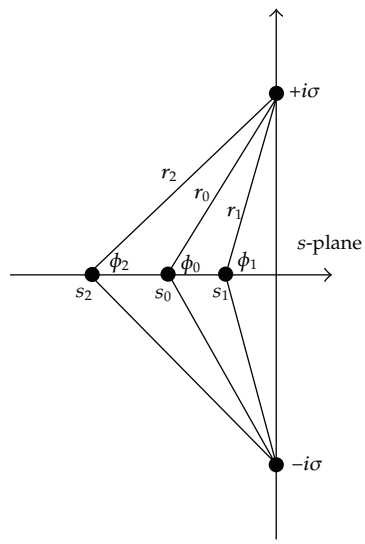

Figure 2: The relation between poles and s_i for the second problem for the case $\lambda_r \geq \lambda > 0$.

3.2. The Case $\lambda > \lambda_r > 0$

For this case, it is more complex than the previous case. Firstly the sign of ω has to be determined by setting the term in the square root of (3.3) and (3.4) to be zero. It reads

$$\left(1+\omega^2\nu\lambda_r\right)^2 - 4\omega^2\nu\lambda > 0, \quad \text{for } 0<\omega<\omega_1 \text{ or } \omega>\omega_2, \tag{3.13}$$

$$\left(1+\omega^2\nu\lambda_r\right)^2 - 4\omega^2\nu\lambda < 0, \quad \text{for } \omega_1<\omega<\omega_2, \tag{3.14}$$

where

$$\begin{aligned} \omega_1 &= \left(\frac{2\lambda-\lambda_r-2\sqrt{\lambda(\lambda-\lambda_r)}}{\nu\lambda_r^2}\right)^{0.5}, \\ \omega_2 &= \left(\frac{2\lambda-\lambda_r+2\sqrt{\lambda(\lambda-\lambda_r)}}{\nu\lambda_r^2}\right)^{0.5}. \end{aligned} \tag{3.15}$$

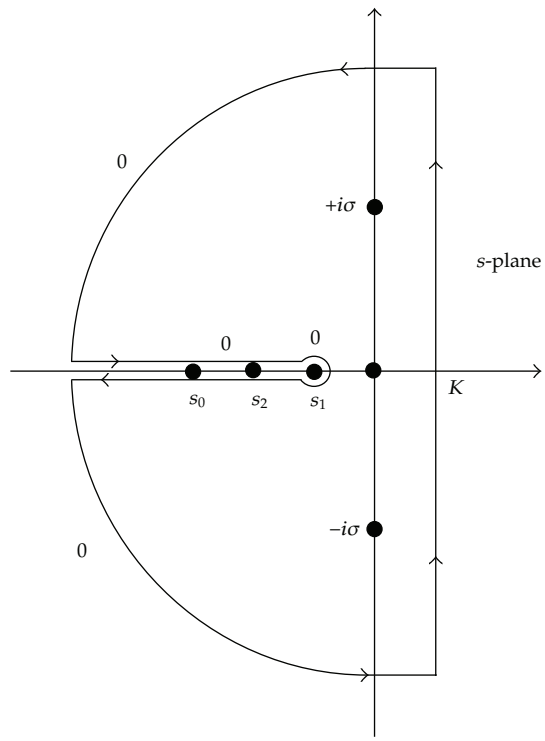

Figure 3: The contour path for the case $\lambda > \lambda_r > 0$ with the conditions $0 < \omega < \omega_1$ or $\omega > \omega_2$. Single pole $s = 0$ is for the first problem and two poles $s = i\Omega$ and $s = -i\Omega$ are for the second problem.

According to the sign of (3.13) and (3.14), the integration with respect to ω in (3.5) has to be divided into three parts, as shown below

$$\begin{aligned} u = {} & \frac{2}{\pi}\int_{\omega=0}^{\omega=\omega_1} d\omega\, \omega \sin(y\omega)\frac{1}{2\pi i}\int_{s=\kappa-i\infty}^{s=\kappa+i\infty} I\cdot ds \\ & + \frac{2}{\pi}\int_{\omega=\omega_1}^{\omega=\omega_2} d\omega\, \omega \sin(y\omega)\frac{1}{2\pi i}\int_{s=\kappa-i\infty}^{s=\kappa+i\infty} I\cdot ds \\ & + \frac{2}{\pi}\int_{\omega=\omega_2}^{\omega=\infty} d\omega\, \omega \sin(y\omega)\frac{1}{2\pi i}\int_{s=\kappa-i\infty}^{s=\kappa+i\infty} I\cdot ds. \end{aligned} \tag{3.16}$$

For the first and last integrals, the poles s_1 and s_2 are negative real numbers due to the positive sign of (3.13). Therefore, the inner integral can be calculated using the contour path shown in Figure 3:

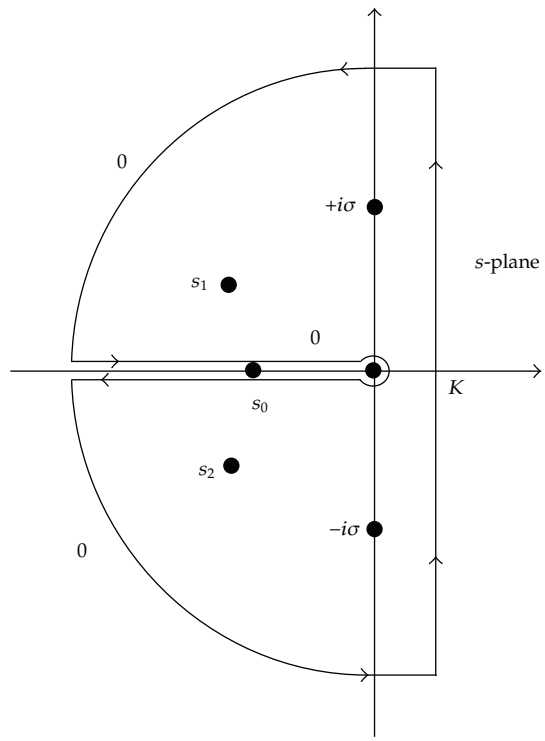

Figure 4: The contour path for the case $\lambda > \lambda_r > 0$ with the condition $\omega_1 < \omega < \omega_2$. The poles $s = i\Omega$ and $s = -i\Omega$ are for the second problem only.

$$\begin{aligned}\int_{\kappa-i\infty}^{\kappa+i\infty} I \cdot ds = {} & 2i\int_{s_0}^{-\infty} \frac{\nu\lambda_r}{\lambda} \cdot \frac{(s-s_0)\widehat{g}(s)}{(s-s_1)(s-s_2)} \sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right) \cdot e^{st} ds \\ & + 2i\int_{s_1}^{s_2} \frac{\nu\lambda_r}{\lambda} \cdot \frac{(s-s_0)\widehat{g}(s)}{(s-s_1)(s-s_2)} \sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right) \cdot e^{st} ds \\ & + 2\pi i \cdot \mathcal{R},\end{aligned} \tag{3.17}$$

where

$$s_1 > s_2 > s_0. \tag{3.18}$$

The residue $\mathcal{R}$ is evaluated to be equivalent to (3.10) and (3.11) for the first and second problems, respectively.

As for the second term in (3.16), where s_1 and s_2 are complex conjugates, the contour path shown in Figure 4 is used to calculate the inner integral. It is noted that the contour path goes around the origin instead of s_0. For the first problem, the inner integral is

$$\begin{aligned}\int_{\kappa-i\infty}^{\kappa+i\infty} I\cdot ds &= 2i\int_{s_0}^{-\infty} u_0\frac{\nu\lambda_r}{\lambda}\cdot\frac{(s-s_0)}{(s-s_1)(s-s_2)}\cdot\sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right)\cdot e^{st}ds\\ &\quad+2\pi i\frac{u_0\nu\lambda_r s_0}{\lambda s_1 s_2}\left[\exp\left(-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{s_1 s_2}{-s_0}}\right)-1\right]\\ &\quad+2\pi i\cdot\mathcal{R},\end{aligned}\tag{3.19}$$

where the second term in (3.19) is contributed by the integration around the origin. The residue

$$\mathcal{R} = \operatorname{Res}(s_1)+\operatorname{Res}(s_2),\tag{3.20}$$

is calculated to be zero.

For the second problem, we have

$$\begin{aligned}&\int_{\kappa-i\infty}^{\kappa+i\infty} I\cdot ds\\ &=2i\int_{s_0}^{-\infty}\frac{u_0(s\cos\theta-\sigma\sin\theta)}{s^2+\sigma^2}\cdot\frac{\nu\lambda_r}{\lambda}\cdot\frac{(s-s_0)}{(s-s_1)(s-s_2)}\cdot\sin\left(z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{(s-s_1)(s-s_2)}{s_0-s}}\right)\cdot e^{st}ds\\ &\quad+2\pi i\cdot\mathcal{R},\end{aligned}\tag{3.21}$$

where the residue is contributed from four poles. It reads

$$\begin{aligned}\mathcal{R} &= \operatorname{Res}(s_1)+\operatorname{Res}(s_2)+\operatorname{Res}(i\sigma)+\operatorname{Res}(-i\sigma)\\ &= \frac{u_0\nu\lambda_r}{\lambda}\cdot\frac{r_0}{r_3r_4}\cdot\left\{\cos(t\sigma+\theta+\phi_0-\phi_3-\phi_4)-\exp\left(-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{r_3r_4}{r_0}}\cos\left(\frac{\phi_3+\phi_4-\phi_0}{2}\right)\right)\right.\\ &\qquad\left.\cdot\cos\left[t\sigma+\theta+\phi_0-\phi_3-\phi_4-z\sqrt{\frac{\lambda}{\nu\lambda_r}}\sqrt{\frac{r_3r_4}{r_0}}\sin\left(\frac{\phi_3+\phi_4-\phi_0}{2}\right)\right]\right\},\end{aligned}\tag{3.22}$$

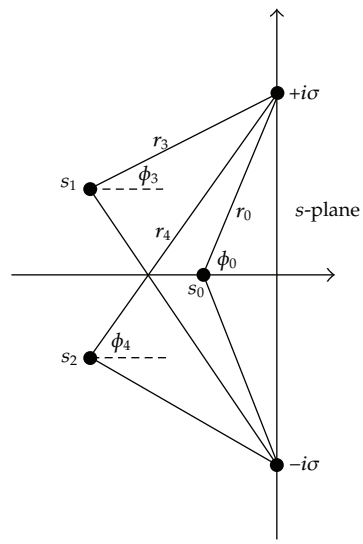

Figure 5: The relation between poles and s_i for the second problem for the case $\lambda > \lambda_r > 0$ with the condition $\omega_1 < \omega < \omega_2$.

where

$$
\begin{aligned}
r_3 &= \sqrt{\sigma^2 - \frac{\sigma}{\lambda}\sqrt{4\omega^2\nu\lambda - (1+\omega^2\nu\lambda_r)^2} + \frac{\omega^2\nu}{\lambda}}, \\
r_4 &= \sqrt{\sigma^2 + \frac{\sigma}{\lambda}\sqrt{4\omega^2\nu\lambda - (1+\omega^2\nu\lambda_r)^2} + \frac{\omega^2\nu}{\lambda}}, \\
\phi_3 &= \tan^{-1}\left(\frac{2\lambda\sigma - \sqrt{4\omega^2\nu\lambda - (1+\omega^2\nu\lambda_r)^2}}{1+\omega^2\nu\lambda_r}\right), \\
\phi_4 &= \tan^{-1}\left(\frac{2\lambda\sigma + \sqrt{4\omega^2\nu\lambda - (1+\omega^2\nu\lambda_r)^2}}{1+\omega^2\nu\lambda_r}\right),
\end{aligned}
\tag{3.23}
$$

where the relation is shown in Figure 5. Now the exact solution of u can be acquired by substituting (3.17), and (3.19)–(3.23) into (3.16) to calculate the inverse Fourier transform.

4. Concluding Remarks

The extended Stokes' problems for the Oldroyd-B fluid are revisited in this paper by applying the contour integration to calculate the inversion of velocity rather than using the series expansion in the earlier paper. Two cases classified according to the relation between rheological parameters are analyzed. Mathematical techniques used in this paper can be generalized and applied to investigate other fluids for the extended Stokes' problems.

Acknowledgment

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