Research Article

When Is the Complement of the Zero-Divisor Graph of a Commutative Ring Complemented?

S. Visweswaran

Department of Mathematics, Saurashtra University, Rajkot 360 005, India

Correspondence should be addressed to S. Visweswaran, s.visweswaran2006@yahoo.co.in

Received 12 March 2012; Accepted 3 April 2012

Academic Editors: D. Anderson, A. V. Kelarev, and C. Munuera

Copyright © 2012 S. Visweswaran. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let R be a commutative ring with identity which has at least two nonzero zero-divisors. Suppose that the complement of the zero-divisor graph of R has at least one edge. Under the above assumptions on R , it is shown in this paper that the complement of the zero-divisor graph of R is complemented if and only if R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ as rings. Moreover, if R is not isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ as rings, then, it is shown that in the complement of the zero-divisor graph of R , either no vertex admits a complement or there are exactly two vertices which admit a complement.

1. Introduction

The rings considered in this paper are commutative rings with identity satisfying the further condition that there exist two distinct zero-divisors whose product is nonzero. Let R be a commutative ring with identity. Recall from [1] that the zero-divisor graph of R denoted by $\Gamma(R)$ is an undirected graph whose vertex set is the set of all nonzero zero-divisors of R and two distinct nonzero zero-divisors x, y of R are joined by an edge in this graph if and only if $xy = 0$. Several researchers investigated the properties of zero-divisor graphs of commutative rings with identity. The following survey article [2] gives a very clear account of the problems solved in the area of zero-divisor graphs of commutative rings along with necessary history of the problems attempted in this area.

All graphs considered in this paper are undirected graphs. Let $G = (V, E)$ be a graph. Recall from [3, 4] that two distinct vertices a, b of G are said to be orthogonal, written $a \perp b$ if a and b are adjacent in G and there is no vertex c of G which is adjacent to both a and b in G ; that is, the edge $a - b$ of G is not a part of any triangle in G . Let a be a vertex of G . Recall from [3] that a vertex b of G is said to be a complement of a if $a \perp b$. Moreover, recall from [3] that

G is complemented if each vertex of G admits a complement in G . Furthermore, G is said to be uniquely complemented if G is complemented and whenever the vertices a, b, c of G are such that $a \perp b$ and $a \perp c$, then b, c are not adjacent in G and a vertex v of G is adjacent to b in G if and only if v is adjacent to c in G . In Section 3 of [3], the authors characterized commutative rings R such that $\Gamma(R)$ is complemented (resp., $\Gamma(R)$ is uniquely complemented).

Let $G = (V, E)$ be a simple graph. Recall from [5, Definition 1.1.13] that the complement of G denoted by G^c is defined as a graph with $V(G^c) = V(G) = V$ and two distinct elements $x, y \in V$ are joined by an edge in G^c if and only if there is no edge of G joining x and y .

Let R be a commutative ring with identity. We denote by $Z(R)^*$ the set of all nonzero zero-divisors of R . Suppose that $|Z(R)^*| \geq 2$. With this assumption on R , in [6, 7], we investigated the relationship between some graph-theoretic properties of $(\Gamma(R))^c$ and the ring-theoretic properties of R . Motivated by the interesting theorems proved in [3], in this paper, we discuss the question of when $(\Gamma(R))^c$ is complemented.

Before we proceed further, let us recall the following definitions and results from commutative ring theory. Let R be a commutative ring with identity. Let I be an ideal of R . Recall from [8] that a prime ideal P of R is said to be a maximal N -prime of I if P is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R : rs \in I \text{ for some } s \in R \setminus I\}$. Thus a prime ideal P of R is a maximal N -prime of (0) if P is maximal with respect to the property of being contained in $Z(R)$. Note that $S = R \setminus Z(R)$ is a multiplicatively closed subset of R . If $x \in Z(R)$, then $Rx \cap S = \emptyset$. Hence it follows from Zorn's lemma and [9, Theorem 1.1] that there exists a maximal N -prime P of (0) in R such that $x \in P$. Thus if $\{P_\alpha\}_{\alpha \in \Lambda}$ denotes the set of all maximal N -primes of (0) in R , then $Z(R) = \bigcup_{\alpha \in \Lambda} P_\alpha$.

Let I be an ideal of a ring R . Recall from [10] that a prime ideal P of R is said to be an associated prime of I in the sense of Bourbaki if $P = (I :_R x)$ for some $x \in R$. In this case we say that P is a B -prime of I .

Let R be a commutative ring with identity and suppose that $|Z(R)^*| \geq 2$. In Section 2 of this paper we assume that R has exactly one maximal N -prime of (0) . Let P be the unique maximal N -prime of (0) in R . If P is not a B -prime of (0) in R , then it is proved in Proposition 2.1 that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. Suppose that P is a B -prime of (0) in R and if $|P| \geq 5$, then it is shown in Proposition 2.2 that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. If $|P| < 5$ and if $(\Gamma(R))^c$ admits at least one edge (i.e., equivalently, if there exist distinct $x, y \in Z(R)^*$ with $xy \neq 0$), then it is observed in Remark 2.3 that $|P| = 4$, moreover, we note that, except for the unique isolated vertex of $(\Gamma(R))^c$, the other two vertices are complements of each other, and furthermore, with the help of results from [11], it is deduced that R is isomorphic to exactly one of the rings from the collection $\{Z/8Z, (Z/4Z)[x]/((2+4Z)x(Z/4Z)[x] + (x^2 - (2+4Z))(Z/4Z)[x]), (Z/2Z)[x]/x^3(Z/2Z)[x]\}$ where $(Z/4Z)[x]$ (resp., $(Z/2Z)[x]$) denotes the polynomial ring in one variable over $Z/4Z$ (resp., over $Z/2Z$). Throughout this paper unless otherwise specified we denote by $(Z/nZ)[x]$, the polynomial ring in one variable over Z/nZ for any $n > 1$.

In Section 3 we consider commutative rings R with identity such that R has exactly two maximal N -primes of (0) . Let $\{P_1, P_2\}$ denote the set of all maximal N -primes of (0) in R . The main results proved in Section 3 are Theorem 3.9 and Proposition 3.11. If $P_1 \cap P_2 \neq (0)$, then it is proved in Theorem 3.9 that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$ if and only if either R is isomorphic to $Z/4Z \times Z/2Z$ or R is isomorphic to $(Z/2Z)[x]/x^2(Z/2Z)[x] \times Z/2Z$ as rings if and only if $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$. If $P_1 \cap P_2 = (0)$ and if $(\Gamma(R))^c$ admits at least one edge, then it is shown in Proposition 3.11 that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$ if and only if R is isomorphic to $Z/3Z \times T$, where T is an integral domain, and it is noted that

$(\Gamma(R))^c$ is complemented if and only if R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ as rings, and in the case, when T is not isomorphic to $\mathbf{Z}/3\mathbf{Z}$, then $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

In Section 4 we consider commutative rings R with identity such that R has at least three maximal N -primes of (0) and it is shown in Proposition 4.1 that, for such rings, no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

2. R Has Exactly One Maximal N -Prime of (0)

Let R be a commutative ring with identity such that $|Z(R)^*| \geq 2$ and R has exactly one maximal N -prime of (0) . Let P be the unique maximal N -prime of (0) in R . Observe that either P is not a B -prime of (0) in R or P is a B -prime of (0) in R ; that is, equivalently in terms of graph theoretic terms, either $(\Gamma(R))^c$ is connected or $(\Gamma(R))^c$ is not connected [6, Theorem 1.1(a)]. In this section, we prove that $(\Gamma(R))^c$ is not complemented if P is not a B -prime of (0) in R . Indeed, if P is not a B -prime of (0) in R , then we prove in Proposition 2.1 that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. If P is a B -prime of (0) in R and if $|P| \geq 5$, then we prove in Proposition 2.2 that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. Moreover, if P is a B -prime of (0) in R and if $|P| < 5$, then with the help of [11, Theorem 3.2] we describe, in Remark 2.3, rings R such that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. We often make use of the following fact (which is an immediate consequence of the definition of a complement of a vertex in a graph) that, in a graph $G = (V, E)$, no vertex admits a complement in G if and only if each edge of G is an edge of a triangle in G . We begin with the following proposition.

Proposition 2.1. *Let R be a commutative ring with identity. Let $|Z(R)^*| \geq 2$, and suppose that R has exactly one maximal N -prime of (0) . Let P be the unique maximal N -prime of (0) in R . If P is not a B -prime of (0) in R , then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. In view of the hypothesis that P is the only maximal N -prime of (0) in R , it follows that $Z(R) = P$. Let $x - y$ be any edge of $(\Gamma(R))^c$. We prove that the edge $x - y$ is an edge of a triangle in $(\Gamma(R))^c$; that is, there exists $z \in P \setminus \{x, y\}$ such that $xz \neq 0$ and $yz \neq 0$. Proceeding as in the proof of [7, Lemma 3.2], it can be shown that there exists $z \in P \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. This shows that any edge of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$. Thus if x is any element of $Z(R)^*$, then x does not admit a complement in $(\Gamma(R))^c$. $\square$

With the assumption that P is a B -prime of (0) in R , the following proposition provides a sufficient condition under which no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

Proposition 2.2. *Let R be a commutative ring with identity. Suppose that R has exactly one maximal N -prime of (0) , and let it be P . If P is a B -prime of (0) in R and if $|P| \geq 5$, then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. As P is a B -prime of (0) in R , there exists $c \in P \setminus \{0\}$ such that $P = ((0):_{RC})$. Let $x - y$ be any edge of $(\Gamma(R))^c$. We now verify that there exists $z \in Z(R)^* = P \setminus \{0\}$ such that $z \notin \{x, y\}$, $zx \neq 0$, and $zy \neq 0$. Proceeding as in the proof of [7, Proposition 3.7], it can be shown that there exists $z \in Z(R)^*$ such that $z \notin \{x, y\}$, $zx \neq 0$, and $zy \neq 0$. This proves that any edge $x - y$ of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$. Hence we obtain that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. $\square$

The following remark characterizes commutative rings R with identity satisfying the following conditions: (i) R has exactly one maximal N -prime of (0) , (ii) the unique maximal N -prime of (0) in R is a B -prime of (0) in R , and (iii) $(\Gamma(R))^c$ has at least one edge and has at least one vertex which admits a complement in $(\Gamma(R))^c$.

Remark 2.3. First observe that R is one of the rings from the collection $\{\mathbf{Z}/8\mathbf{Z}, (\mathbf{Z}/4\mathbf{Z})[x]/((2+4\mathbf{Z})x(\mathbf{Z}/4\mathbf{Z})[x] + (x^2 - (2+4\mathbf{Z}))(\mathbf{Z}/4\mathbf{Z})[x]), (\mathbf{Z}/2\mathbf{Z})[x]/x^3(\mathbf{Z}/2\mathbf{Z})[x]\}$, then R has properties (i), (ii), and (iii) mentioned above. We show in this remark with the help of [11, Theorem 3.2] that if R is a ring with the above three properties, then R is isomorphic to exactly one of the rings given in the above collection. Let R, P be as mentioned in the beginning of this section. Suppose that P is a B -prime of (0) in R and moreover, $(\Gamma(R))^c$ contains at least one edge that is, there exist distinct $x, y \in P \setminus \{0\}$ such that $xy \neq 0$. Let $c \in P \setminus \{0\}$ be such that $P = ((0) :_R c)$. We want to characterize R such that $(\Gamma(R))^c$ has at least one vertex which admits a complement in $(\Gamma(R))^c$. It follows from Proposition 2.2 that $|P| < 5$. Note that $\{0, x, y, c\} \subseteq P$, and hence it follows that $|P| = 4$. Now proceeding as in [7, Remark 3.8], it follows using [11, Theorem 3.2] that R must be isomorphic to one of the rings given in the above collection. Moreover, note that $(\Gamma(R))^c$ has exactly three vertices with one isolated vertex and the other two vertices being complements of each other in $(\Gamma(R))^c$.

3. R Has Exactly Two Maximal N -Primes of (0)

Let R be a commutative ring with identity. Suppose that R has exactly two maximal N -primes of (0) . Let $\{P_1, P_2\}$ denote the set of all maximal N -primes of (0) in R .

Suppose that $P_1 \cap P_2 \neq (0)$. It is useful to remark here that $P_1 \cap P_2 \neq (0)$ if and only if $(\Gamma(R))^c$ is connected [6, Theorem 1.1.(b)]. We prove in Theorem 3.9 that $(\Gamma(R))^c$ has at least one vertex which admits a complement in $(\Gamma(R))^c$ if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ where $(\mathbf{Z}/2\mathbf{Z})[x]$ denotes the polynomial ring in one variable over $\mathbf{Z}/2\mathbf{Z}$.

Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that $P_1 \cap P_2 = (0)$. In Proposition 3.11, we determine up to isomorphism of rings, rings R such that $(\Gamma(R))^c$ has at least one vertex which admits a complement in $(\Gamma(R))^c$. It is indeed proved in Proposition 3.11 that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$ if and only if R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times T$, where T is an integral domain. Moreover, it is noted in Proposition 3.11 that $(\Gamma(R))^c$ is complemented if and only if R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ as rings. Furthermore, if T is not isomorphic to $\mathbf{Z}/3\mathbf{Z}$, then it is observed that $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

We first state and prove several lemmas that are needed for proving Theorem 3.9. We start with the following.

Lemma 3.1. *Let R, P_1, P_2 be as mentioned in the beginning of this section. If one between P_1 and P_2 is not a B -prime of (0) in R , then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. Without loss of generality we may assume that P_1 is not a B -prime of (0) in R . Observe that, to prove the lemma, it is enough to prove the following: if $x - y$ is any edge of $(\Gamma(R))^c$, then there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. We consider the following cases.

Case 1. Both x and y belong to P_1 .

If $P_2 \not\subseteq ((0):_R x) \cup ((0):_R y)$, then as $P_2 \not\subseteq P_1$, it follows from [9, Theorem 81] that $P_2 \not\subseteq P_1 \cup ((0):_R x) \cup ((0):_R y)$. Hence there exists $z \in P_2 \setminus P_1$ such that $zx \neq 0$ and $zy \neq 0$. Since $x, y \in P_1$, whereas $z \notin P_1$, it is clear that $z \notin \{x, y\}$.

Suppose that $P_2 \subseteq ((0):_R x) \cup ((0):_R y)$, then it follows that either $P_2 = ((0):_R x)$ or $P_2 = ((0):_R y)$. Without loss of generality we may assume that $P_2 = ((0):_R x)$. Since $xy \neq 0$, it follows that $y \notin P_2$. Thus $y \in P_1 \setminus P_2$. Since P_1 is not a B -prime of (0) in R , $P_1 \not\subseteq Rx$ and $P_1 \not\subseteq Ry$. Now it follows from [9, Theorem 81] that $P_1 \not\subseteq Rx \cup Ry \cup P_2$. Hence there exists $z \in P_1 \setminus \{v, y\}$ such that $z \notin P_2$. Since $P_2 = ((0):_R x)$, it follows that $zx \neq 0$. As $y, z \in P_1 \setminus P_2$, we obtain that $yz \in P_1 \setminus P_2$ and so $zy \neq 0$.

Case 2. Both x and y belong to P_2 .

Since P_1 is not a B -prime of (0) in R , it follows that $P_1 \not\subseteq ((0):_R x)$ and $P_1 \not\subseteq ((0):_R y)$. Hence we obtain from [9, Theorem 81] that $P_1 \not\subseteq P_2 \cup ((0):_R x) \cup ((0):_R y)$. So there exists $z \in P_1 \setminus P_2$ such that $zx \neq 0$ and $zy \neq 0$. Since $x, y \in P_2$, whereas $z \notin P_2$, it is clear that $z \notin \{x, y\}$.

Case 3. $x \in P_1 \setminus P_2, y \in P_2 \setminus P_1$.

Now $x \in P_1 \setminus P_2, y \in P_2 \setminus P_1$ are such that $xy \neq 0$. Proceeding as in the proof of [7, Lemma 3.4(ii)], we obtain that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$.

Thus it is shown that any edge of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$. Hence we obtain that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. $\square$

Suppose that both P_1 and P_2 are B -primes of (0) in R . The following lemma gives a sufficient condition under which no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

Lemma 3.2. *Let R, P_1, P_2 be as mentioned in the beginning of this section. If both P_1 and P_2 are B -primes of (0) in R and if $|P_1 \cap P_2| \geq 3$, then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. By assumption both P_1 and P_2 are B -primes of (0) in R . Hence there exist $u, v \in R$ such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. We know from [12, Lemma 3.6] that $uv = 0$. We now proceed to show that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. As is remarked in the introduction, it is enough to show that any edge of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$. Let $x - y$ be any edge of $(\Gamma(R))^c$. We want to show that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. We consider the following cases.

Case 1. Both x and y belong to P_1 .

It now follows as in the proof of Lemma 3.1 Case 1 that we may assume without loss of generality that $P_2 = ((0):_R x)$. Since $xy \neq 0$, we obtain that $y \notin P_2$. Thus $y \in P_1 \setminus P_2$. Note that for any $w \in P_1 \cap P_2$, $y + w \in P_1 \setminus P_2$ and so $(y + w)y \in P_1 \setminus P_2$. Hence $(y + w)y \neq 0$. Moreover, since $xw = 0$, we obtain that $x(y + w) = xy \neq 0$. By hypothesis, $|P_1 \cap P_2| \geq 3$. Hence there exist distinct $w_1, w_2 \in (P_1 \cap P_2) \setminus \{0\}$. It is clear that $y + w_1 \neq y$ and $y + w_2 \neq y$. As $w_1 \neq w_2$, it follows that either $y + w_1 \neq x$ or $y + w_2 \neq x$. We may assume without loss of generality that $y + w_1 \neq x$. Now $z = y + w_1$ is such that $z \in Z(R)^* \setminus \{x, y\}$, $zx \neq 0$, and $zy \neq 0$.

Case 2. Both x and y belong to P_2 .

The proof of the fact that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$ is similar to the proof of Case 1 of this lemma.

Case 3. $x \in P_1 \setminus P_2, y \in P_2 \setminus P_1$.

Now $xy \neq 0$. It follows as in the proof of [7, Lemma 3.4(ii)] that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$.

This proves that any edge of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$, and so we obtain that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. $\square$

If both P_1 and P_2 are B -primes of (0) in R , we provide in the following lemma some conditions under which no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

Lemma 3.3. *Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that both P_1 and P_2 are B -primes of (0) in R . If $|P_1 \setminus P_2| \geq 3$ and $|P_2 \setminus P_1| \geq 3$, then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. Let $x - y$ be any edge of $(\Gamma(R))^c$. We prove that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. We consider the following cases.

Case 1. Both x and y belong to P_1 .

Proceeding as in the proof of Lemma 3.1 Case 1, we may assume without loss of generality that $P_2 = ((0):_R x)$. Since $xy \neq 0$, it follows that $y \notin P_2$. Thus $y \in P_1 \setminus P_2$. If $x^2 \neq 0$, then $x \notin P_2$. Hence $x \in P_1 \setminus P_2$. By hypothesis $|P_1 \setminus P_2| \geq 3$. Hence there exists $z \in (P_1 \setminus P_2) \setminus \{x, y\}$. Now it is clear that $zx \neq 0$ and $zy \neq 0$. Suppose that $x^2 = 0$. Then $x \in P_2$ and so $x \in P_1 \cap P_2$. Note that $z = x + y$ is such that $z \in P_1 \setminus P_2$ and $z \notin \{x, y\}$. Moreover, $zx = (x + y)x = xy \neq 0$, $zy \in P_1 \setminus P_2$, and so $zy \neq 0$.

Case 2. Both x and y belong to P_2 .

The hypotheses regarding P_1 and P_2 are symmetric. Hence it follows as in the proof of Case 1 of this lemma that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$.

Case 3. $x \in P_1 \setminus P_2, y \in P_2 \setminus P_1$.

Now it follows as in the proof of [7, Lemma 3.4(ii)] that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$.

This shows that any edge of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$, and so we obtain that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. $\square$

If $P_1 \cap P_2 \neq (0)$, then we prove in Lemma 3.5 that $(\Gamma(R))^c$ is not complemented; that is, there exists at least one vertex of $(\Gamma(R))^c$ which does not admit a complement in $(\Gamma(R))^c$. We make use of the following lemma in the proof of Lemma 3.5.

Lemma 3.4. *Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that both P_1 and P_2 are B -primes of (0) in R . Let $u, v \in R$ be such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. If $P_1 \cap P_2 \neq (0)$, then either $u \in P_1 \cap P_2$ or $v \in P_1 \cap P_2$.*

Proof. Since P_1 and P_2 are the only maximal N -primes of (0) in R , it follows that $Z(R) = P_1 \cup P_2$. By assumption, $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. Now for any $x \in P_1 \setminus P_2$, from $xu = 0$, it follows that $u \in P_2$. Similarly for any $y \in P_2 \setminus P_1$, from $yv = 0$, we obtain that $v \in P_1$. By hypothesis, $P_1 \cap P_2 \neq (0)$. Let $z \in (P_1 \cap P_2) \setminus \{0\}$. Now $zu = zv = 0$ and so $(u + v)z = 0$. Hence $u + v \in Z(R) = P_1 \cup P_2$. Thus either $u + v \in P_1$ or $u + v \in P_2$. If $u + v \in P_1$, then as $v \in P_1$ and $u \in P_2$, we obtain that $u \in P_1 \cap P_2$. If $u + v \in P_2$, then it follows that $v \in P_1 \cap P_2$ since $u \in P_2$ and $v \in P_1$. This proves that either $u \in P_1 \cap P_2$ or $v \in P_1 \cap P_2$. $\square$

We next have the following lemma which shows that $(\Gamma(R))^c$ is not complemented if $P_1 \cap P_2 \neq (0)$.

Lemma 3.5. *Let R, P_1, P_2 be mentioned in the beginnig of this section. If $P_1 \cap P_2 \neq (0)$, then there exists at least one vertex of $(\Gamma(R))^c$ which does not admit a complement in $(\Gamma(R))^c$.*

Proof. If at least one between P_1 and P_2 is not a B -prime of (0) in R , then it is proved in Lemma 3.1 that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. Hence we may assume that both P_1 and P_2 are B -primes of (0) in R . Let $u, v \in R$ be such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. We know from Lemma 3.4 that either $u \in P_1 \cap P_2$ or $v \in P_1 \cap P_2$. Without loss of generality we may assume that $u \in P_1 \cap P_2$.

We assert that u does not admit a complement in $(\Gamma(R))^c$. It is enough to prove the following: if $u - w$ is any edge of $(\Gamma(R))^c$, then there exists $z \in Z(R)^* \setminus \{u, w\}$ such that $zu \neq 0$ and $zw \neq 0$. Since $uw \neq 0$, it follows that $w \in P_2 \setminus P_1$. Note that $z = w + u$ is such that $z \in P_2 \setminus P_1$, $z \notin \{u, w\}$, $zu \neq 0$, and $zw \neq 0$.

This proves that $(\Gamma(R))^c$ has at least one vertex which does not admit a complement in $(\Gamma(R))^c$. $\square$

With the assumption that $P_1 \cap P_2 \neq (0)$, we next attempt to characterize rings R such that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. Towards that goal, we begin with the following lemma.

Lemma 3.6. *Let R, P_1, P_2 be as mentioned in the beginning of this section. If $P_1 \cap P_2 \neq (0)$, then $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$ if and only if one of the following holds:*

- (i) $|P_2 \setminus P_1| = 2$ and $P_1 = ((0):_R b)$ for some $b \in P_2 \setminus P_1$.
- (ii) $|P_1 \setminus P_2| = 2$ and $P_2 = ((0):_R a)$ for some $a \in P_1 \setminus P_2$.

Proof. Suppose that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. Then it follows from Lemma 3.1 that there exist $u, v \in R$ such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. Moreover, it follows from Lemma 3.2 that $|P_1 \cap P_2| = 2$. Furthermore, we know from Lemma 3.3 that either $|P_1 \setminus P_2| < 3$ or $|P_2 \setminus P_1| < 3$.

Let $P_1 \cap P_2 = \{0, w\}$. Note that, for any $x \in P_1 \setminus P_2$, $x + w \in (P_1 \setminus P_2) \setminus \{x\}$. Hence $|P_1 \setminus P_2| \geq 2$. Similarly it follows that $|P_2 \setminus P_1| \geq 2$. Suppose that $|P_2 \setminus P_1| < 3$. Then we obtain that $|P_2 \setminus P_1| = 2$. It is shown in the proof of Lemma 3.4 that $u \in P_2$ and $v \in P_1$. If $u \notin P_1$, then we arrive at $|P_2 \setminus P_1| = 2$ and $P_1 = ((0):_R u)$ with $u \in P_2 \setminus P_1$. Hence (i) holds. Suppose that $u \in P_1 \cap P_2$. Then as $|P_1 \cap P_2| = 2$, it follows that $v \notin P_1 \cap P_2$. Hence $v \in P_1 \setminus P_2$. We assert that $|P_1 \setminus P_2| = 2$. Suppose that it does not hold. Then $|P_1 \setminus P_2| \geq 3$. Let $x \in Z(R)^*$ be such that x admits a complement y in $(\Gamma(R))^c$. It is shown in the proof of Lemma 3.5 that u does not admit a complement in $(\Gamma(R))^c$. Hence it follows that $x, y \notin \{u\}$. As $P_1 \cap P_2 = \{0, u\}$, it follows that $x, y \notin P_1 \cap P_2$. If one between x and y is in $P_1 \setminus P_2$ and the other is in $P_2 \setminus P_1$, then it follows as in the proof of [7, Lemma 3.4(ii)] that there exists $z \in Z(R)^*$ which is adjacent to both x and y in $(\Gamma(R))^c$. This is not possible since y is a complement of x in $(\Gamma(R))^c$. Thus either both x and y belong to $P_2 \setminus P_1$ or both x and y belong to $P_1 \setminus P_2$. If both x and y belong to $P_2 \setminus P_1$, then $u \in Z(R)^*$ is such that $xu \neq 0$ and $yu \neq 0$. Hence u is adjacent to both x and y in $(\Gamma(R))^c$. This is impossible. If both x and y belong to $P_1 \setminus P_2$, then, since we are assuming that $|P_1 \setminus P_2| \geq 3$, any $z \in (P_1 \setminus P_2) \setminus \{x, y\}$ satisfies $zx \neq 0$ and $zy \neq 0$. This cannot happen as y is a complement of x in $(\Gamma(R))^c$. Hence $|P_1 \setminus P_2| = 2$. This together with the fact that $v \in P_1 \setminus P_2$ and

$P_2 = ((0):_R v)$ implies that (ii) holds. Thus if $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$, then either (i) or (ii) holds.

Conversely assume that either (i) or (ii) holds. Suppose that (i) holds. Now $P_1 = ((0):_R b)$ for some $b \in P_2 \setminus P_1$, and $|P_2 \setminus P_1| = 2$. Let $P_2 \setminus P_1 = \{b, c\}$. Since $b, c \in P_2 \setminus P_1$, it follows that $bc \in P_2 \setminus P_1$ and hence $bc \neq 0$. Moreover, if $z \in Z(R)^* \setminus \{b, c\}$, then z must be in P_1 and so $zb = 0$. Hence there exists no vertex of $(\Gamma(R))^c$ which is adjacent to both b and c in $(\Gamma(R))^c$. Therefore, c is a complement of b in $(\Gamma(R))^c$. This shows that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. Similarly if (ii) holds and if $P_1 \setminus P_2 = \{a, d\}$, then it follows that a and d are complements of each other in $(\Gamma(R))^c$. This proves that either (i) or (ii) holds, then $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.

This completes the proof of Lemma 3.6. $\square$

Suppose that $P_1 \cap P_2 \neq (0)$. In Theorem 3.9 we characterize rings R such that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. We need the following lemma for proving Theorem 3.9.

Lemma 3.7. *Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that $P_1 \cap P_2 \neq (0)$, and if $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$, then R is isomorphic to $T_1 \times T_2$ as rings where T_1 is a ring with $|Z(T_1)^*| = 1$ and T_2 is an integral domain.*

Proof. We are assuming that there are vertices of $(\Gamma(R))^c$ which admit a complement in $(\Gamma(R))^c$. Hence we obtain from Lemma 3.1 that there exist $u, v \in R$ such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. Now it follows from Lemma 3.2 that $|P_1 \cap P_2| = 2$. Let $P_1 \cap P_2 = \{0, w\}$. We claim that $P_1 + P_2 = R$. Suppose that $P_1 + P_2 \neq R$. Then there exists a maximal ideal M of R such that $P_1 + P_2 \subseteq M$. Let $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Now $a + b \in P_1 + P_2 \subseteq M$, and since $Z(R) = P_1 \cup P_2$, it follows from the choice of the elements a and b that $a + b \notin Z(R)$. Let $z = a + b$. Note that $z \in M \setminus Z(R)$, and as $w \neq 0$, we obtain that $zw \neq 0$. Since $P_1 \cap P_2 = \{0, w\}$, it follows that $zw = w$. Thus $w(1 - z) = 0$, and so $1 - z \in Z(R)$. This is impossible since $Z(R) = P_1 \cup P_2 \subseteq M$, whereas $1 - z \notin M$. Hence we obtain that $P_1 + P_2 = R$. Observe that $w^2 \in P_1 \cap P_2 = \{0, w\}$, and as $1 - w \notin Z(R)$, it follows that $w^2 = 0$. Hence $(P_1 \cap P_2)^2 = (0)$. As $P_1 + P_2 = R$, it follows that $(0) = (P_1 \cap P_2)^2 = P_1^2 \cap P_2^2$. Now we obtain from the Chinese remainder theorem [13, Proposition 1.10(ii)] that R is isomorphic to $R/P_1^2 \times R/P_2^2$ as rings. Indeed, it follows from the Chinese remainder theorem that the mapping $f : R \rightarrow R/P_1^2 \times R/P_2^2$ given by $f(r) = (r + P_1^2, r + P_2^2)$ for any $r \in R$ is an isomorphism of rings. The isomorphism f maps $P_1 \cap P_2$ onto $P_1/P_1^2 \times P_2/P_2^2$. Hence $2 = |P_1 \cap P_2| = |f(P_1 \cap P_2)| = |P_1/P_1^2 \times P_2/P_2^2| = |P_1/P_1^2| |P_2/P_2^2|$. Thus either $|P_1/P_1^2| = 2, |P_2/P_2^2| = 1$ or $|P_1/P_1^2| = 1, |P_2/P_2^2| = 2$. We may assume without loss of generality that $|P_1/P_1^2| = 2$ and $|P_2/P_2^2| = 1$. Let $T_1 = R/P_1^2$ and $T_2 = R/P_2^2$. Thus R is isomorphic to $T_1 \times T_2$ as rings. Since $P_2 = P_2^2$, it follows that $T_2 = R/P_2$ is an integral domain. We next verify that $|Z(T_1)^*| = 1$. Note that $f(Z(R)) = Z(T_1 \times T_2) = (Z(T_1) \times T_2) \cup (T_1 \times Z(T_2))$. As $Z(R) = P_1 \cup P_2$ and $f(P_1 \cup P_2) = (P_1/P_1^2 \times T_2) \cup (T_1 \times P_2/P_2^2)$, it follows that $Z(T_1) = P_1/P_1^2$. Since $|P_1/P_1^2| = 2$, we obtain that $|Z(T_1)^*| = 1$.

Thus if $P_1 \cap P_2 \neq (0)$ and if $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$, then R is isomorphic to $T_1 \times T_2$ as rings with $|Z(T_1)^*| = 1$ and T_2 is an integral domain. $\square$

Let T be a commutative ring with identity. It is well known [1, Example 2.1(a)] that $|Z(T)^*| = 1$ (i.e., equivalently, $\Gamma(T)$ is a graph on a single vertex) if and only if T is either isomorphic to $T_{11} = \mathbf{Z}/4\mathbf{Z}$ or T is isomorphic to $T_{12} = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ as rings. Let $R = T_1 \times T_2$, where either $T_1 = T_{11}$ or $T_1 = T_{12}$ and T_2 is an integral domain. In the following lemma, we determine when $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.

Lemma 3.8. *Let T_{11}, T_{12} be as in the previous paragraph. Let T_2 be an integral domain. Let $R = T_1 \times T_2$, where either $T_1 = T_{11}$ or $T_1 = T_{12}$. Then the following statements are equivalent.*

- (i) $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.
- (ii) T_2 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ as rings.
- (iii) $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

Proof. Let $R = T_{11} \times T_2 = \mathbf{Z}/4\mathbf{Z} \times T_2$ where T_2 is an integral domain. Note that $Z(R)$ is the union of two prime ideals $P_1 = (2 + 4\mathbf{Z})T_{11} \times T_2$ and $P_2 = T_{11} \times (0)$. Moreover, P_1 and P_2 are the only maximal N -primes of the zero ideal in R . Furthermore, $(2 + 4\mathbf{Z}, 0) \in P_1 \cap P_2$, and hence $P_1 \cap P_2$ is not the zero ideal of R . Furthermore, $P_1 = ((0 + 4\mathbf{Z}, 0) :_R (2 + 4\mathbf{Z}, 0))$ and $P_2 = ((0 + 4\mathbf{Z}, 0) :_R (0 + 4\mathbf{Z}, 1))$. We now show that the statements (i) to (iii) are equivalent.

(i) $\Rightarrow$ (ii) Suppose that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. We claim that $P_1 \neq ((0 + 4\mathbf{Z}, 0) :_R b)$ for any $b \in P_2 \setminus P_1$. Suppose that $P_1 = ((0 + 4\mathbf{Z}, 0) :_R b)$ for some $b \in P_2 \setminus P_1$. Note that either $b = (1 + 4\mathbf{Z}, 0)$ or $b = (3 + 4\mathbf{Z}, 0)$. Observe that $a = (2 + 4\mathbf{Z}, 0) \in P_1$ is such that $a(1 + 4\mathbf{Z}, 0) = a = a(3 + 4\mathbf{Z}, 0) \neq (0 + 4\mathbf{Z}, 0)$. This shows that $P_1 \neq ((0 + 4\mathbf{Z}, 0) :_R b)$ for any $b \in P_2 \setminus P_1$. Hence condition (i) of Lemma 3.6 does not hold. Therefore condition (ii) of Lemma 3.6 must hold. Thus $|P_1 \setminus P_2| = 2$ and $P_2 = ((0 + 4\mathbf{Z}, 0) :_R c)$ for some $c \in P_1 \setminus P_2$. We now show that $|T_2| = 2$. Suppose that $|T_2| \geq 3$. Let $y, z \in T_2 \setminus \{0\}$ with $y \neq z$. Note that $\{(0 + 4\mathbf{Z}, y), (0 + 4\mathbf{Z}, z), (2 + 4\mathbf{Z}, y)\} \subseteq P_1 \setminus P_2$, and this implies that $|P_1 \setminus P_2| \geq 3$ which contradicts the fact that $|P_1 \setminus P_2| = 2$. This proves that $|T_2| = 2$, and so T_2 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ as rings.

(ii) $\Rightarrow$ (iii) Now $T_2 = \{0, 1\}$. Observe that $Z(R)^* = \{(0 + 4\mathbf{Z}, 1), (2 + 4\mathbf{Z}, 0), (2 + 4\mathbf{Z}, 1), (1 + 4\mathbf{Z}, 0), (3 + 4\mathbf{Z}, 0)\}$. Observe that the edge $(0 + 4\mathbf{Z}, 1) - (2 + 4\mathbf{Z}, 1)$ of $(\Gamma(R))^c$ is not an edge of any triangle in $(\Gamma(R))^c$. Indeed, $(0 + 4\mathbf{Z}, 1)$ is a pendant vertex of $(\Gamma(R))^c$. Thus the vertices $(0 + 4\mathbf{Z}, 1)$ and $(2 + 4\mathbf{Z}, 1)$ are complements of each other in $(\Gamma(R))^c$. It can be easily verified that each of the other edges of $(\Gamma(R))^c$ is an edge of a triangle in $(\Gamma(R))^c$. Hence $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

(iii) $\Rightarrow$ (i) This is clear.

If $R = T_{12} \times T_2$ where $T_{12} = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ and T_2 is an integral domain, then the proof of the fact that the statements (i), (ii), and (iii) are equivalent is exactly similar and hence is omitted. $\square$

Suppose that $P_1 \cap P_2 \neq (0)$. The following theorem characterizes rings R such that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.

Theorem 3.9. *Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that $P_1 \cap P_2 \neq (0)$. The following statements are equivalent.*

- (i) $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.
- (ii) Either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings.
- (iii) $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

Proof. (i) $\Rightarrow$ (ii) We know from Lemma 3.7 that R is isomorphic to $T_1 \times T_2$ as rings with $|Z(T_1)^*| = 1$ and T_2 is an integral domain. Since $|Z(T_1)^*| = 1$, it follows from [1, Example 2.1(a)] that either T_1 is isomorphic to $\mathbf{Z}/4\mathbf{Z}$ or T_1 is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ as rings. Let $T_{11} = \mathbf{Z}/4\mathbf{Z}$ and $T_{12} = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$. Since either R is isomorphic to $T_{11} \times T_2$ or R is isomorphic to $T_{12} \times T_2$, (i) implies that either $(\Gamma(T_{11} \times T_2))^c$ has vertices which admit a complement in $(\Gamma(T_{11} \times T_2))^c$ or $(\Gamma(T_{12} \times T_2))^c$ has vertices which admit a complement

in $(\Gamma(T_{12} \times T_2))^c$. Now it follows from (i) $\Rightarrow$ (ii) of Lemma 3.8 that T_2 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ as rings. Hence we obtain that either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings.

(ii) $\Rightarrow$ (iii) It follows from (ii) $\Rightarrow$ (iii) of Lemma 3.8 that $(\Gamma(R))^c$ has exactly two vertices which admit a complement in $(\Gamma(R))^c$.

(iii) $\Rightarrow$ (i) This is clear.

This completes the proof of Theorem 3.9. $\square$

Suppose that $P_1 \cap P_2 = (0)$; that is, equivalently, by [6, Theorem 1.1(b)], $(\Gamma(R))^c$ is not connected. Assume that $(\Gamma(R))^c$ admits at least one edge. Our next aim is to determine when no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$ and to determine rings R such that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.

Lemma 3.10. *Let R, P_1, P_2 be as mentioned in the beginning of this section. Suppose that $P_1 \cap P_2 = (0)$. Then the following hold.*

- (i) *If $|P_1 \setminus P_2| \geq 3$, then no element of $P_1 \setminus P_2$ admits a complement in $(\Gamma(R))^c$.*
- (ii) *If $|P_2 \setminus P_1| \geq 3$, then no element of $P_2 \setminus P_1$ admits a complement in $(\Gamma(R))^c$.*
- (iii) *If $|P_1 \setminus P_2| \geq 3$ and $|P_2 \setminus P_1| \geq 3$, then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. (i) Let $x \in P_1 \setminus P_2$. Suppose that x admits a complement in $(\Gamma(R))^c$. Let $y \in Z(R)^*$ be a complement of x in $(\Gamma(R))^c$. Then $x \neq y$ and $xy \neq 0$. Since $Z(R) = P_1 \cup P_2$, $P_1 \cap P_2 = (0)$, and $xy \neq (0)$, it follows that $y \in P_1 \setminus P_2$. By hypothesis, $|P_1 \setminus P_2| \geq 3$. Now for any $z \in (P_1 \setminus P_2) \setminus \{x, y\}$, $zx \neq 0$ and $zy \neq 0$. This is impossible since y is a complement of x in $(\Gamma(R))^c$. This proves that if $|P_1 \setminus P_2| \geq 3$, then no element of $P_1 \setminus P_2$ admits a complement in $(\Gamma(R))^c$.

(ii) The proof of (ii) is similar to the proof of (i).

(iii) Since $Z(R) = P_1 \cup P_2$ and $P_1 \cap P_2 = (0)$, it follows that $Z(R)^* = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$. Note that $Z(R)^*$ is the vertex set of $(\Gamma(R))^c$. Thus if $|P_1 \setminus P_2| \geq 3$ and $|P_2 \setminus P_1| \geq 3$, then it follows from (i) and (ii) that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

This completes the proof of Lemma 3.10. $\square$

The following proposition describes rings R such that $P_1 \cap P_2 = (0)$, $(\Gamma(R))^c$ has at least one edge, and moreover, $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.

Proposition 3.11. *Let R, P_1, P_2 be as mentioned in the beginning of this section. If $P_1 \cap P_2 = (0)$ and if $(\Gamma(R))^c$ has at least one edge, then the following statements are equivalent.*

- (i) *$(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$.*
- (ii) *R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times T$ as rings, where T is an integral domain.*
- (iii) *Exactly one of the following holds.*
 - (a) *R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ as rings, and in this case $(\Gamma(R))^c$ is a graph on three vertices and it admits exactly one isolated vertex and the two vertices are complements of each other in $(\Gamma(R))^c$.*
 - (b) *R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ as rings, and in this case $(\Gamma(R))^c$ is a graph on four vertices and is complemented.*
 - (c) *R is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times T$ as rings, where T is an integral domain with $|T| \geq 4$, and in this case $(\Gamma(R))^c$ is a graph on more than four vertices and it has exactly two vertices which admit a complement in $(\Gamma(R))^c$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. Then it follows from Lemma 3.10(iii) that either $|P_1 \setminus P_2| < 3$ or $|P_2 \setminus P_1| < 3$. Without loss of generality we may assume that $|P_2 \setminus P_1| < 3$. Then either $|P_2 \setminus P_1| = 1$ or $|P_2 \setminus P_1| = 2$.

Suppose that $|P_2 \setminus P_1| = 1$. Let $P_2 \setminus P_1 = \{b\}$. Since $P_1 \cap P_2 = (0)$, it follows that $P_2 = \{0, b\}$. Moreover, $Z(R)^* = (P_1 \setminus P_2) \cup \{b\}$. We assert that $|P_1 \setminus P_2| = 2$. Since we are assuming that $(\Gamma(R))^c$ has at least one edge, we obtain that $|P_1 \setminus P_2| \geq 2$. Suppose that $|P_1 \setminus P_2| \geq 3$, then it follows from Lemma 3.10(i) that no element of $P_1 \setminus P_2$ admits a complement in $(\Gamma(R))^c$. Observe that b is an isolated vertex in $(\Gamma(R))^c$ and hence it does not admit a complement in $(\Gamma(R))^c$. Hence we obtain that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. This is in contradiction to the hypothesis that $(\Gamma(R))^c$ has vertices which admit a complement in $(\Gamma(R))^c$. Therefore, we obtain that $|P_1 \setminus P_2| \leq 2$. This shows that $|P_1 \setminus P_2| = 2$. Thus $|Z(R)| = |P_1 \cup P_2| = 4$. Now $Z(R)$ is a finite set, and hence it follows from [14, Theorem 1] that R is finite. Since any prime ideal of a finite ring is a maximal ideal, it follows that P_1 and P_2 are maximal ideals of R . Thus $P_1 + P_2 = R$. Since $P_1 \cap P_2 = (0)$, it follows from the Chinese remainder theorem [13, Proposition 1.10(ii)] that the mapping $f : R \rightarrow R/P_1 \times R/P_2$ given by $f(r) = (r + P_1, r + P_2)$ for any $r \in R$ is an isomorphism of rings. Note that $f(P_1) = P_1/P_1 \times R/P_2$ and $f(P_2) = R/P_1 \times P_2/P_2$. Hence $3 = |P_1| = |f(P_1)| = |R/P_2|$ and $2 = |P_2| = |f(P_2)| = |R/P_1|$. So we obtain that $R/P_1 \cong \mathbb{Z}/2\mathbb{Z}$ and $R/P_2 \cong \mathbb{Z}/3\mathbb{Z}$ as rings. Thus $R \cong R/P_1 \times R/P_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as rings. Thus with $T = \mathbb{Z}/2\mathbb{Z}$, we obtain that $R \cong \mathbb{Z}/3\mathbb{Z} \times T$.

Suppose that $|P_2 \setminus P_1| = 2$. Let $P_2 \setminus P_1 = \{b_1, b_2\}$. Note that $P_2 = \{0, b_1, b_2\}$. We claim that $P_1 + P_2 = R$. Suppose that $P_1 + P_2 \neq R$. Then there exists a maximal ideal M of R such that $P_1 + P_2 \subseteq M$. Let $a \in P_1 \setminus P_2$. Now $a + b_1 \in P_1 + P_2 \subseteq M$, and as $Z(R) = P_1 \cup P_2$, it follows from the choice of the elements a and b_1 that $a + b_1 \notin Z(R)$. Let $z = a + b_1$. Note that $zb_1 \in P_2 \setminus \{0\} = \{b_1, b_2\}$. Observe that $zb_1 \neq b_1$. If $zb_1 = b_1$, then $(1 - z)b_1 = 0$, and this implies that $1 - z \in Z(R) = P_1 \cup P_2 \subseteq M$. Thus both z and $1 - z$ belong to M . Hence $1 = z + 1 - z \in M$, and this is impossible since $M \neq R$. This shows that $zb_1 = b_2$. Similarly it follows that $zb_2 = b_1$. Hence we obtain that $z^2 b_1 b_2 = b_1 b_2$. Hence $(1 - z^2)b_1 b_2 = 0$. As $b_1 b_2 \notin P_1$, it follows that $1 - z^2 \in P_1 \subseteq M$. This is impossible since $z^2 \in M$. This proves that $P_1 + P_2 = R$. Since $P_1 \cap P_2 = (0)$, we obtain from the Chinese remainder theorem that the mapping $f : R \rightarrow R/P_1 \times R/P_2$ given by $f(r) = (r + P_1, r + P_2)$ is an isomorphism of rings. As $|P_2| = 3$ and as $f(P_2) = R/P_1 \times P_2/P_2$, it follows that $|R/P_1| = 3$. Hence $R/P_1 \cong \mathbb{Z}/3\mathbb{Z}$ as rings. Let $T = R/P_2$. Then T is an integral domain and $R \cong R/P_1 \times R/P_2 \cong \mathbb{Z}/3\mathbb{Z} \times T$ as rings.

This proves (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Now by hypothesis R is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times T$ as rings where T is an integral domain. If $|T| = 2$, then $R \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as rings. Consider the ring $S_1 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Note that $Z(S_1)^* = \{(0+3\mathbb{Z}, 1+2\mathbb{Z}), (1+3\mathbb{Z}, 0+2\mathbb{Z}), (2+3\mathbb{Z}, 0+2\mathbb{Z})\}$. Note that $(\Gamma(S_1))^c$ is a graph on three vertices and it admits exactly one isolated vertex and the other two vertices are complements of each other in $(\Gamma(S_1))^c$. Since $R \cong S_1$ as rings, it follows that $(\Gamma(R))^c$ also has the same properties. Hence (a) holds.

Suppose that $|T| = 3$. Then $T \cong \mathbb{Z}/3\mathbb{Z}$ as rings. Hence R is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ as rings. Let $S_2 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Note that $Z(S_2)^* = \{(0+3\mathbb{Z}, 1+3\mathbb{Z}), (0+3\mathbb{Z}, 2+3\mathbb{Z}), (1+3\mathbb{Z}, 0+3\mathbb{Z}), (2+3\mathbb{Z}, 0+3\mathbb{Z})\}$. Note that $(\Gamma(S_2))^c$ is a graph on four vertices, $(0+3\mathbb{Z}, 1+3\mathbb{Z})$, $(0+3\mathbb{Z}, 2+3\mathbb{Z})$ are complements of each other in $(\Gamma(S_2))^c$, and $(1+3\mathbb{Z}, 0+3\mathbb{Z})$, $(2+3\mathbb{Z}, 0+3\mathbb{Z})$ are complements of each other in $(\Gamma(S_2))^c$. Thus $(\Gamma(S_2))^c$ is complemented. Since $R \cong S_2$ as rings, it follows that $(\Gamma(R))^c$ is a graph on four vertices and is complemented. Hence (b) holds.

Suppose that $|T| \geq 4$. Consider the ring $S_3 = \mathbb{Z}/3\mathbb{Z} \times T$. Note that $Z(S_3)^* = \{(0+3\mathbb{Z}, t), (1+3\mathbb{Z}, 0), (2+3\mathbb{Z}, 0) | t \in T \setminus \{0\}\}$. Since $|T| \geq 4$, it is clear that $(\Gamma(S_3))^c$ is a graph on

more than four vertices. The vertex $(2 + 3\mathbf{Z}, 0)$ is a complement of $(1 + 3\mathbf{Z}, 0)$ in $(\Gamma(S_3))^c$. Since $|T \setminus \{0\}| \geq 3$, it is easy to verify that no vertex of the form $(0 + 3\mathbf{Z}, t)$, where $t \in T \setminus \{0\}$, admits a complement in $(\Gamma(S_3))^c$. Since $R \cong S_3$ as rings, it follows that $(\Gamma(R))^c$ is a graph on more than four vertices, and moreover, there are exactly two vertices which admit a complement in $(\Gamma(R))^c$. Hence (c) holds.

(iii) $\Rightarrow$ (i) This is clear.

This completes the proof of Proposition 3.11. $\square$

4. R Has More Than Two Maximal N -Primes of (0)

Let R be a commutative ring with identity. Suppose that R has more than two maximal N -primes of (0) . In the following proposition, we prove that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.

Proposition 4.1. *Let R be a commutative ring with identity. If R has more than two maximal N -primes of (0) , then no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$.*

Proof. Note that to prove that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$, it is enough to prove the following: if $x - y$ is any edge of $(\Gamma(R))^c$, then there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. Let $\{P_\alpha\}_{\alpha \in \Lambda}$ denote the set of all maximal N -primes of (0) in R . By hypothesis R has more than two maximal N -primes of (0) . Hence $|\Lambda| \geq 3$. Note that $Z(R) = \bigcup_{\alpha \in \Lambda} P_\alpha$. We consider the following cases.

Case 1. There exists $\alpha \in \Lambda$ such that both x and y belong to P_α . Let $\beta, \gamma \in \Lambda \setminus \{\alpha\}$ be such that $\beta \neq \gamma$. Since $P_\beta \neq P_\gamma$ and $xy \neq 0$, it follows that either $P_\beta \not\subseteq ((0) :_R xy)$ or $P_\gamma \not\subseteq ((0) :_R xy)$. We may assume without loss of generality that $P_\beta \not\subseteq ((0) :_R xy)$. Note that $P_\beta \not\subseteq P_\alpha \cup ((0) :_R xy)$. Hence there exists $z \in P_\beta$ such that $z \notin P_\alpha$ and $zxy \neq 0$. Since both x and y belong to P_α , whereas $z \notin P_\alpha$, it is clear that $z \notin \{x, y\}$. As $z \in P_\beta \subseteq Z(R)$, we obtain that $z \in Z(R)$. From $zxy \neq 0$, it follows that $zx \neq 0$ and $zy \neq 0$.

Case 2. x and y do not belong to the same maximal N -prime of (0) in R .

Let α, β be distinct elements of Λ such that $x \in P_\alpha \setminus P_\beta$ and $y \in P_\beta \setminus P_\alpha$. Now $xy \neq 0$. Proceeding as in the proof of [7, Lemma 3.4(ii)], it can be shown that there exists $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. We give another argument to exhibit an element $z \in Z(R)^* \setminus \{x, y\}$ such that $zx \neq 0$ and $zy \neq 0$. Since $x \in P_\alpha \setminus P_\beta$, it follows that $((0) :_R x) \subseteq P_\beta$. Similarly since $y \in P_\beta \setminus P_\alpha$, it follows that $((0) :_R y) \subseteq P_\alpha$. By hypothesis $|\Lambda| \geq 3$. Let $\gamma \in \Lambda \setminus \{\alpha, \beta\}$. Note that $P_\gamma \not\subseteq P_\alpha \cup P_\beta$. Hence there exists $z \in P_\gamma$ such that $z \notin P_\alpha \cup P_\beta$. It is now clear that $z \in Z(R) \setminus \{x, y\}$. Since $((0) :_R x) \subseteq P_\beta$ and $z \notin P_\beta$, it follows that $zx \neq 0$. Similarly since $((0) :_R y) \subseteq P_\alpha$ and $z \notin P_\alpha$, we obtain that $zy \neq 0$.

This shows that no vertex of $(\Gamma(R))^c$ admits a complement in $(\Gamma(R))^c$. $\square$

Acknowledgments

The author is very much thankful to the Academic Editors Professor D. F. Anderson, Professor A. V. Kelarev, and Professor C. Munuera for their valuable suggestions and for their support. Moreover, he is very much thankful to Professor A. V. Kelarev for bringing to his attention two useful and valuable books pertaining to the investigation of graphs associated to algebraic structures [15, 16].

References

- [1] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, no. 2, pp. 434–447, 1999.
- [2] D. F. Anderson, M. C. Axtell, and J. A. Stickles, "Zero-divisor graphs in commutative rings," in *Commutative Algebra: Noetherian and Non-Noetherian Perspectives*, pp. 23–45, Springer, New York, NY, USA, 2011.
- [3] D. F. Anderson, R. Levy, and J. Shapiro, "Zero-divisor graphs, von Neumann regular rings, and Boolean algebras," *Journal of Pure and Applied Algebra*, vol. 180, no. 3, pp. 221–241, 2003.
- [4] R. Levy and J. Shapiro, "The zero-divisor graph of von Neumann regular rings," *Communications in Algebra*, vol. 30, no. 2, pp. 745–750, 2002.
- [5] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, New York, NY, USA, 2000.
- [6] S. Visweswaran, "Some results on the complement of the zero-divisor graph of a commutative ring," *Journal of Algebra and its Applications*, vol. 10, no. 3, pp. 573–595, 2011.
- [7] S. Visweswaran, "Some properties of the complement of zero-divisor graph of a commutative ring," *ISRN Algebra*, vol. 2011, Article ID 591041, 24 pages, 2011.
- [8] W. Heinzer and J. Ohm, "On the Noetherian-like rings of E. G. Evans," *Proceedings of the American Mathematical Society*, vol. 34, no. 1, pp. 73–74, 1972.
- [9] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, Ill, USA, 1974.
- [10] W. Heinzer and J. Ohm, "Locally noetherian commutative rings," *Transactions of the American Mathematical Society*, vol. 158, pp. 273–284, 1971.
- [11] D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, "The zero-divisor graph of a commutative ring. II," in *Ideal Theoretic Methods in Commutative Algebra*, vol. 220 of *Lecture Notes in Pure and Applied Mathematics*, pp. 61–72, Marcel Dekker, New York, NY, USA, 2001.
- [12] I. Beck, "Coloring of commutative rings," *Journal of Algebra*, vol. 116, no. 1, pp. 208–226, 1988.
- [13] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
- [14] N. Ganesan, "Properties of rings with a finite number of zero divisors," *Mathematische Annalen*, vol. 157, pp. 215–218, 1964.
- [15] A. V. Kelarev, *Ring Constructions and Applications*, vol. 9 of *Series in Algebra*, World Scientific, River Edge, NJ, USA, 2002.
- [16] A. Kelarev, *Graph Algebras and Automata*, vol. 257, Marcel Dekker, New York, NY, USA, 2003.