Research Article

On Submersion of CR-Submanifolds of l.c.q.K. Manifold

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Received 27 September 2012; Accepted 15 October 2012

Academic Editors: A. Ferrandez and T. Friedrich

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We study submersion of CR-submanifolds of an l.c.q.K. manifold. We have shown that if an almost Hermitian manifold B admits a Riemannian submersion $\pi : M \rightarrow B$ of a CR-submanifold M of a locally conformal quaternion Kaehler manifold $\overline{M}$, then B is a locally conformal quaternion Kaehler manifold.

1. Introduction

The concept of locally conformal Kaehler manifolds was introduced by Vaisman in [1]. Since then many papers appeared on these manifolds and their submanifolds (see [2] for details). However, the geometry of locally conformal quaternion Kaehler manifolds has been studied in [2–4] and their QR-submanifolds have been studied in [5].

A locally conformal quaternion Kaehler manifold (shortly, l.c.q.K. manifold) is a quaternion Hermitian manifold whose metric is conformal to a quaternion Kaehler metric in some neighborhood of each point. The main difference between locally conformal Kaehler manifolds and l.c.q.K. manifolds is that the Lee form of a compact l.c.q.K. manifold can be chosen as parallel form without any restrictions [2].

The study of the Riemannian submersion $\pi : M \rightarrow B$ of a Riemannian manifold M onto a Riemannian manifold B was initiated by O'Neill [6]. A submersion naturally gives rise to two distributions on M called the horizontal and vertical distributions, respectively, of which the vertical distribution is always integrable giving rise to the fibers of the submersion which are closed submanifold of M . The notion of Cauchy-Riemann (CR) submanifold was introduced by Bejancu [7] as a natural generalization of complex submanifolds and totally real submanifolds. A CR-submanifolds M of a l.c.q.K. manifold $\overline{M}$ requires a differentiable

holomorphic distribution D , that is, $J_x D_x = D_x$ for all $x \in M$, whose orthogonal complement $D^\perp$ is totally real distribution on M , that is, $J_x D_x^\perp \subset TM^\perp$ for all $x \in M$. A CR-submanifold is called holomorphic submanifold if $\dim D_x^\perp = 0$, totally real if $\dim D_x = 0$ and proper if it is neither holomorphic nor totally real.

A CR-submanifold of a l.c.q.K. manifold $\overline{M}$ is called a CR-product if it is Riemannian product of a holomorphic submanifold $N^\top$ and a totally real submanifold $N^\perp$ of $\overline{M}$. Kobayashi [8] has proved that if an almost Hermitian manifold B admits a Riemannian submersion $\pi : M \rightarrow B$ of a CR-submanifold M of a Kaehler Manifold $\overline{M}$, then B is a Kaehler manifold. However, Deshmukh et al. [9] studied similar type of results for CR-submanifolds of manifolds in different classes of almost Hermitian manifolds, namely, Hermitian manifolds, quasi-Kaehler manifolds, and nearly Kaehler manifolds.

In the present paper, we investigate submersion of CR-submanifold of a l.c.q.K. manifold $\overline{M}$ and prove that if an almost Hermitian manifold B admits a Riemannian submersion $\pi : M \rightarrow B$ of a CR-submanifold M of a l.c.q.K. manifold $\overline{M}$, then B is an l.c.q.K. manifold.

2. Preliminaries

Let $(\overline{M}, g, H)$ be a quaternion Hermitian manifold, where H is a subbundle of end $(T\overline{M})$ of rank 3 which is spanned by almost complex structures J_1, J_2 , and J_3 . The quaternion Hermitian metric g is said to be a quaternion Kaehler metric if its Levi-Civita connection $\overline{\nabla}$ satisfies $\overline{\nabla}H \subset H$.

A quaternion Hermitian manifold with metric g is called a locally conformal quaternion Kaehler (l.c.q.K.) manifold if over neighborhoods $\{U_i\}$ covering $\overline{M}$, $g|_{U_i} = e^{f_i} g_i$ where g_i is a quaternion Kaehler metric on U_i . In this case, the Lee form ω is locally defined by $\omega|_{U_i} = df_i$ and satisfies [3]

$$d\theta = \omega \wedge \theta, \quad dw = 0. \quad (2.1)$$

Let $\overline{M}$ be l.c.q.K. manifold and $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$. Let B be the Lee vector field given by

$$g(X, B) = \omega(X). \quad (2.2)$$

Then for l.c.q.K. manifold, we have [3]

$$\begin{aligned} (\tilde{\nabla}_X J_a)Y &= \frac{1}{2} \{ \theta(Y)X - \omega(Y)J_a X - g(X, Y)A - \Omega(X, Y)B \} \\ &\quad + Q_{ab}(X)J_b Y + Q_{ac}(X)J_c Y \end{aligned} \quad (2.3)$$

for any $X, Y \in T\overline{M}$, where Q_{ab} is skew-symmetric matrix of local forms $\theta = \omega \circ J_a$ and $A = -J_a B$.

We also have

$$\theta(X) = g(J_a X, B), \quad \Omega(X, Y) = g(X, J_a Y). \quad (2.4)$$

Let M be a Riemannian manifold isometrically immersed in $\overline{M}$. Let $T(M)$ be the Lie algebra of vector fields in M and $TM^\perp$, the set of all vector fields normal to M . Denote by ∇ the Levi connection of M . Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.5)$$

$$\overline{\nabla}_X N = -\overline{A}_N X + \nabla_X^\perp N \quad (2.6)$$

for any $X, Y \in T(M)$, and $N \in TM^\perp$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, h is the second fundamental form, and $\overline{A}_N$ is the Weingarten endomorphism associated with N . The second fundamental form and shape operator are related by

$$g(\overline{A}_N X, Y) = g(h(X, Y), N). \quad (2.7)$$

The curvature tensor R of the submanifold M is related to the curvature tensor $\overline{R}$ of $\overline{M}$ by the following Gauss formula:

$$\begin{aligned} \overline{R}(X, Y; Z, W) &= R(X, Y; Z, W) + g(h(X, Z), h(Y, W)) \\ &\quad - g(h(X, W), h(Y, Z)), \end{aligned} \quad (2.8)$$

for any $X, Y, Z, W \in T(M)$.

For submersion of a l.c.q.K. manifold onto an almost Hermitian manifold, we have the following.

Definition 2.1. Let M be a CR-submanifold of a locally conformal quaternion Kaehler manifold $\overline{M}$. By a submersion $\pi : M \rightarrow B$ of M onto an almost Hermitian manifold B , we mean a Riemannian submersion $\pi : M \rightarrow B$ together with the following conditions:

- (i) $D^\perp$ is the kernel of π_* , that is, $\pi_* D^\perp = \{0\}$,
- (ii) $\pi_* : D_p \rightarrow D_{\pi(p)}^*$ is a complex isometry of the subspace D_p onto $D_{\pi(p)}^*$ for every $p \in M$, where $D_{\pi(p)}^*$ denotes the tangent space of B at $\pi(p)$,
- (iii) J interchanges $D^\perp$ and ν , that is, $J D^\perp = TM^\perp$.

For a vector field X on M , we set [8]

$$X = HX + VX, \quad (2.9)$$

where H and V denoted the horizontal and vertical part of X .

We recall that a vector field X on M for submersion $\pi : M \rightarrow B$ is said to be a basic vector field if $X \in D$ and X is π related to a vector field on B , that is, there is a vector field X_* on B such that

$$(\pi_* X)_p = (X_*)_{\pi(p)} \quad \text{for each } p \in M. \quad (2.10)$$

If J and J' are the almost complex structures on $\overline{M}$ and B , respectively, then from Definition 2.1(ii) we have $\pi_* \circ J = J' \circ \pi_*$ on D .

We have the following lemma for basic vector fields [6].

Lemma 2.2. *Let X and Y be basic vector fields on M . Then*

- (i) $g(X, Y) = g_*(X_*, Y_*) \circ \pi$, g is the metric on M , and g_* is the Riemannian metric on B ;
- (ii) the horizontal part $H[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, that is,

$$\pi_* H[X, Y] = [X_*, Y_*] \circ \pi; \quad (2.11)$$

- (iii) $H(\nabla_X Y)$ is a basic vector field corresponding to $\nabla_{X_*}^* Y_*$, where ∇^* is a Riemannian connection on B ;

- (iv) $[X, W] \in D^\perp$ for $W \in D^\perp$.

For a covariant differentiation operator ∇^* , we define a corresponding operator $\tilde{\nabla}^*$ for basic vector fields of M by

$$\tilde{\nabla}_X^* Y = H(\nabla_X Y), \quad X, Y \in D, \quad (2.12)$$

then $\tilde{\nabla}_X^* Y$ is a basic vector field, and from the above lemma we have

$$\pi_* (\tilde{\nabla}_X^* Y) = \nabla_{X_*}^* Y_* \circ \pi. \quad (2.13)$$

Now, we define a tensor field C on M by setting

$$\nabla_X Y = H(\nabla_X Y) + C(X, Y), \quad X, Y \in D, \quad (2.14)$$

that is, $C(X, Y)$ is the vertical component of $\nabla_X Y$.

In particular, if X and Y are basic vector fields, then we have

$$\nabla_X Y = \tilde{\nabla}_X^* Y + C(X, Y). \quad (2.15)$$

The tensor field C is skew-symmetric and it satisfies

$$C(X, Y) = \frac{1}{2} V[X, Y], \quad X, Y \in D. \quad (2.16)$$

For $X \in D$ and $V \in D^\perp$ define an operator A on M by setting $\nabla_X V = \nu(\nabla_X V) + A_X V$, that is, $A_X V$ is the horizontal component of $\nabla_X V$. Using (iv) of Lemma 2.2 we have

$$H(\nabla_X V) = H(\nabla_V X) = A_X V. \quad (2.17)$$

The operator C and A are related by

$$g(A_X V, Y) = -g(V, C(X, Y)), \quad X, Y \in D, \quad V \in D^\perp. \quad (2.18)$$

For a CR-submanifold M in a locally conformal quaternion Kaehler manifold $\overline{M}$, we denote by ν the orthogonal complement of $JD^\perp$ in $TM^\perp$. Hence, we have the following orthogonal decomposition of the normal bundle:

$$TM^\perp = JD^\perp \oplus \nu, \quad JD^\perp \perp \nu. \quad (2.19)$$

Set

$$\begin{aligned} PX &= \tan(JX), & FX &= \text{nor}(JX), & \text{for } X \in TM, \\ tN &= \tan(JN), & fN &= \text{nor}(JN), & \text{for } N \in TM^\perp. \end{aligned} \quad (2.20)$$

Here, $\tan_x$ and nor_x are the natural projections associated with the orthogonal direct sum decomposition

$$T_x \overline{M} = T_x M \oplus TM_x^\perp \quad \text{for any } x \in M. \quad (2.21)$$

Then the following identities hold:

$$P^2 = -I - tF, \quad FP + tF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft, \quad (2.22)$$

where I is the identity transformation.

We have following results.

Lemma 2.3. *Let M be a CR-submanifold in a l.c.q.K. manifold $\overline{M}$. Then*

(i) *holomorphic distribution D is integrable iff*

$$h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y) \text{ nor } (B) = 0, \quad \forall X, Y \in D \quad (2.23)$$

or equivalently,

$$\tilde{g}(h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y)B, J_a Z) = 0, \quad \forall X, Y \in D, \quad Z \in D^\perp; \quad (2.24)$$

(ii) *anti-invariant distribution $D^\perp$ of M is integrable iff*

$$A_{J_a W} T = A_{J_a T} W, \quad \forall W, T \in D^\perp. \quad (2.25)$$

Proof. (i) Using (2.3), we have

$$\begin{aligned} \nabla_X J_a Y &= P_a \nabla_X Y + t_a h(X, Y) + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - g(X, Y) \tan(A) - \Omega(X, Y) \tan(B) \} \\ &\quad + Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y, \\ h(X, J_a Y) &= F_a \nabla_X Y + f_a h(X, Y) - \frac{1}{2} \{ g(X, Y) \text{nor}(A) + \Omega(X, Y) \text{nor}(B) \}. \end{aligned} \quad (2.26)$$

From the second of these equations, we have

$$h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y) \text{nor}(B) = F_a[X, Y], \quad \forall X, Y \in D. \quad (2.27)$$

If we need D to be integrable, we have

$$h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y) \text{nor}(B) = 0 \quad (2.28)$$

or

$$g(h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y)(B), J_a Z) = 0, \quad \forall Z \in D^\perp. \quad (2.29)$$

ν -part of $h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y)B$ vanishes for all $X, Y \in D$.

(ii) We have

$$\begin{aligned} \bar{\nabla}_X J_a Y &= J_a \bar{\nabla}_X Y + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \} \\ &\quad + Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y. \end{aligned} \quad (2.30)$$

Then for any $T, W \in D^\perp$, and $X \in D$, we have

$$\begin{aligned} g(\bar{\nabla}_T J_a W, X) &= g(J_a \bar{\nabla}_T W, X) + \frac{1}{2} \theta(W) g(T, X) - \frac{1}{2} \omega(W) g(J_a T, X) \\ &\quad - \frac{1}{2} \Omega(T, W) g(B, X) + \frac{1}{2} g(T, W) g(J_a B, X) \\ &\quad + Q_{ab}(T) g(J_b W, X) + Q_{ac}(T) g(J_c W, X) \end{aligned} \quad (2.31)$$

$$\implies \langle -A_{J_a W} T, X \rangle = - \langle \bar{\nabla}_T, J_a X \rangle - \frac{1}{2} g(T, W) g(B, J_a X). \quad (2.32)$$

So, we have

$$\begin{aligned}\langle A_{J_a W} T, X \rangle &= \langle \bar{\nabla}_T W, J_a X \rangle + \frac{1}{2} g(T, W) g(B, J_a X), \\ \langle A_{J_a T} W, X \rangle &= \langle \bar{\nabla}_W T, J_a X \rangle + \frac{1}{2} g(W, T) g(B, J_a X).\end{aligned}\tag{2.33}$$

From these two equations, we have

$$\begin{aligned}\langle A_{J_a W} T - A_{J_a T} W, X \rangle &= \langle \bar{\nabla}_T W - \bar{\nabla}_W T, J_a X \rangle \\ \Rightarrow \langle A_{J_a W} T - A_{J_a T} W, X \rangle &= \langle [W, T], J_a X \rangle.\end{aligned}\tag{2.34}$$

So, we conclude that if $A_{J_a W} T = A_{J_a T} W$ then $D^\perp$ is integrable. Converse is obvious. $\square$

Lemma 2.4. *Let M be a CR-submanifold of l.c.q.K. manifold. Then*

$$\nabla_X J_a Y = \nabla_Y J_a X\tag{2.35}$$

iff Lee vector field B is orthogonal to anti-invariant distribution $D^\perp$.

Proof. Since $\bar{\nabla}$ is metric connection, for $X, Y \in D$, and $Z \in D^\perp$, using (2.3), we have

$$\begin{aligned}\langle \bar{\nabla}_X J_a Y, Z \rangle &= \langle J_a \bar{\nabla}_X Y, Z \rangle + \frac{1}{2} \theta(Y) \langle X, Z \rangle - \frac{1}{2} \Omega(X, Y) g(B, Z) \\ &\quad - \frac{1}{2} \omega(Y) g(J_a X, Z) + \frac{1}{2} g(X, Y) g(J_a B, Z) \\ &\quad + Q_{ab}(X) g(J_b Y, Z) + Q_{ac}(X) g(J_c Y, Z) \\ &= - \langle \bar{\nabla}_X Y, J_a Z \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_a Z) \\ &= \langle Y, \bar{\nabla}_X J_a Z \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_a Z)\end{aligned}\tag{2.36}$$

or

$$\begin{aligned}\langle \nabla_X J_a Y + h(X, J_a Y) Z \rangle &= \left\langle Y, -A_{J_a Z} X + \bar{\nabla}_X^\perp J_a Z \right\rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) \\ &\quad - \frac{1}{2} g(X, Y) g(B, J_a Z).\end{aligned}\tag{2.37}$$

This gives

$$\begin{aligned}\langle \nabla_X J_a Y, Z \rangle &= -\langle Y, A_{J_a Z} X \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_a Z), \\ \langle \nabla_Y J_a X, Z \rangle &= -\langle X, A_{J_a Z} Y \rangle - \frac{1}{2} \Omega(Y, X) \omega(Z) - \frac{1}{2} g(Y, X) g(B, J_a Z).\end{aligned}\quad (2.38)$$

The above two equations give

$$\begin{aligned}\langle \nabla_X J_a Y - \nabla_Y J_a X, Z \rangle &= -\langle A_{J_a Z} X, Y \rangle + \langle X, A_{J_a Z} Y \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) + \frac{1}{2} \Omega(Y, X) \omega(Z) \\ &= \Omega(Y, X) \omega(Z)\end{aligned}\quad (2.39)$$

or

$$\langle \nabla_X J_a Y - \nabla_Y J_a X, Z \rangle = \Omega(Y, X) g(B, Z). \quad (2.40)$$

This gives $\nabla_X J_a Y = \nabla_Y J_a X$ iff $\omega(Z) = 0$. $\square$

3. Submersions of CR-Submanifolds

On a Riemannian manifold M , a distribution S is said to be parallel if $\nabla_X Y \in S$, $X, Y \in S$, where ∇ is a Riemannian connection on M . It is proved earlier that horizontal distribution D is integrable. If, in addition, $D^\perp$ is parallel, then we prove the following.

Proposition 3.1. *Let $\pi : M \rightarrow B$ be a submersion of a CR-submanifold M of a locally conformal quaternion Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B . If (horizontal distribution) D is integrable and (vertical distribution) $D^\perp$ is parallel, then M is a CR-product (Riemannian product $M_1 \times M_2$, where M_1 is an invariant submanifold and M_2 is a totally real submanifold of $\overline{M}$).*

Proof. Since the horizontal distribution D is integrable for $X, Y \in D$, we have $[X, Y] \in D$. Therefore, $V[X, Y] = 0$. Then from (2.16), we have

$$C(X, Y) = 0, \quad \forall X, Y \in D. \quad (3.1)$$

Thus, from the definition of C , we have

$$\nabla_X Y = \tilde{\nabla}_X^* Y \in D, \quad \text{that is, } D \text{ is parallel.} \quad (3.2)$$

Since D and $D^\perp$ are both parallel, using de Rham's theorem, it follows that M is the product $M_1 \times M_2$, where M_1 is invariant submanifold of $\overline{M}$ and M_2 is totally real submanifold of $\overline{M}$. Hence, M is a CR-product. $\square$

In [10], Simons defined a connection and an invariant inner product on $H(T, V) = \text{Hom}(T(M), V(M))$, where $V(M)$ is vector bundle over M and $T(M)$ be tangent bundle of M . In fact, if $r, s \in H(T, V)_m$, we set

$$\langle r, s \rangle = \sum_{i=1}^p \langle r(e_i), s(e_i) \rangle, \quad \text{where } \{e_i\} \text{ is a frame in } T(M)_m. \quad (3.3)$$

Define $Q_{ab}(X) = \langle \bar{D}J_a, J_b \rangle$, which implies $Q_{ab}(X)J_a Y = \langle \bar{D}J_a, J_b \rangle J_a Y$.

Let D be $4n$ dimensional distribution whose basis is given by $\{e_1, \dots, e_n, e_{a_1}, \dots, e_{a_n}, e_{b_1}, \dots, e_{b_n}, e_{c_1}, \dots, e_{c_n}\}$ where $e_{a_i} = J_a(e_i)$, $e_{b_i} = J_b(e_i)$, $e_{c_i} = J_c(e_i)$ and $J_a \circ J_b = J_c$, $J_b \circ J_c = J_a$, $J_c \circ J_a = J_b$.

Now, component of $Q_{ab}(X)$ is defined as follows:

$$\begin{aligned} Q_{ab}(X) &= \langle \bar{D}_X J_a, J_b \rangle = \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^n \langle \bar{D}_X J_a(e_{a_i}), J_b(e_{a_i}) \rangle \\ &\quad + \sum_{i=1}^n \langle \bar{D}_X J_a(e_{b_i}), J_b(e_{b_i}) \rangle + \sum_{i=1}^n \langle \bar{D}_X J_a(e_{c_i}), J_b(e_{c_i}) \rangle \\ &\quad + \sum_{j=1}^q \langle \bar{D}_X J_a(e_j), J_b(e_j) \rangle. \end{aligned} \quad (3.4)$$

So, we have

$$\begin{aligned} Q_{ab}(X) &= \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^n \langle (\bar{D}_X J_a)(J_a e_i), J_b(J_a e_i) \rangle \\ &\quad + \sum_{i=1}^n \langle \bar{D}_X J_a(J_b e_i), J_b(J_b e_i) \rangle + \sum_{i=1}^n \langle \bar{D}_X J_a(J_c e_i), J_b(J_c e_i) \rangle \\ &\quad + \sum_{j=1}^q \langle \bar{D}_X J_a(e_j), J_b(e_j) \rangle \\ &= \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^n \langle \bar{D}_X e_i, J_c(e_i) \rangle + \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle \\ &\quad - \sum_{i=1}^n \langle \bar{D}_X J_c(e_i), e_i \rangle - \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle - \sum_{i=1}^n \langle \bar{D}_X J_b(e_i), J_a(e_i) \rangle \\ &\quad + \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^q \langle \bar{D}_X J_a(e_j), J_b(e_j) \rangle \end{aligned} \quad (3.5)$$

or

$$\begin{aligned}
Q_{ab}(X)J_bY &= \sum_{i=1}^n \langle \bar{D}_X J_a(e_i), J_b(e_i) \rangle J_bY + \sum_{i=1}^n \langle \bar{D}_X e_i, J_c(e_i) \rangle J_bY \\
&\quad + \sum_{i=1}^n \langle \bar{D}_X J_a e_i, J_b e_i \rangle J_bY - \sum_{i=1}^n \langle \bar{D}_X J_c(e_i), e_i \rangle J_bY \\
&\quad - \sum_{i=1}^n \langle \bar{D}_X J_b e_i, J_a e_i \rangle J_bY + \sum_{i=1}^q \langle \bar{D}_X J_a(e_j), J_b(e_j) \rangle J_bY.
\end{aligned} \tag{3.6}$$

Applying π^* and using Lemma 2.2, we get

$$\begin{aligned}
\pi^* Q_{ab}(X)J_bY &= \sum_{i=1}^n \langle \bar{D}_{X^*} J'_a e_i^*, J'_b e_i^* \rangle J'_b Y_* + \sum_{i=1}^n \langle \bar{D}_{X^*} e_i^*, J'_c e_i^* \rangle J'_b Y_* \\
&\quad + \sum_{i=1}^n \langle \bar{D}_{X^*} J'_a e_i^*, J'_b e_i^* \rangle J'_b Y_* - \sum_{i=1}^n \langle \bar{D}_{X^*} J'_c e_i^*, e_i^* \rangle J'_b Y_* \\
&\quad - \sum_{i=1}^n \langle \bar{D}_{X^*} J'_b e_i^*, J'_a e_i^* \rangle J'_b Y_* + \sum_{i=1}^q \langle \bar{D}_{X^*} J'_a e_j^*, J'_b e_j^* \rangle J'_b Y_* = \langle \bar{D}_{X^*} J'_a, J'_b \rangle J'_b Y_*
\end{aligned} \tag{3.7}$$

or

$$\pi^* Q_{ab}(X)J_bY = Q_{ab}^*(X_*)J_b^*Y_*. \tag{3.8}$$

Now, we prove the main result of this paper.

Theorem 3.2. *Let $\bar{M}$ be an l.c.q.K. manifold and M be a CR-submanifold of $\bar{M}$. Let B be an almost Hermitian manifold and $\pi : M \rightarrow B$ be a submersion. Then B is an l.c.q.K. manifold.*

Proof. Let $X, Y \in D$ be basic vector fields. Then from (2.5) and (2.15), we have

$$\bar{\nabla}_X Y = \tilde{\nabla}_X^* Y + C(X, Y) + h(X, Y). \tag{3.9}$$

Replacing Y by $J_a Y$ in (3.9), we have

$$\bar{\nabla}_X J_a Y = \tilde{\nabla}_X^* J_a Y + C(X, J_a Y) + h(X, J_a Y). \tag{3.10}$$

Using (2.3), we get

$$\begin{aligned}
 & \tilde{\nabla}_X^* J_a Y + C(X, J_a Y) + h(X, J_a Y) \\
 &= J_a \tilde{\nabla}_X^* Y + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \} \\
 & \quad + Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y
 \end{aligned} \tag{3.11}$$

or

$$\begin{aligned}
 & \tilde{\nabla}_X^* J_a Y + C(X, J_a Y) + h(X, J_a Y) = J_a \tilde{\nabla}_X^* Y + J_a C(X, Y) + J_a h(X, Y) \\
 & \quad + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \} \\
 & \quad + Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y.
 \end{aligned} \tag{3.12}$$

Thus, we have

$$\begin{aligned}
 & \left(\tilde{\nabla}_X^* J_a \right) Y + C(X, J_a Y) + h(X, J_a Y) - J_a C(X, Y) - J_a h(X, Y) \\
 & \quad - \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \} \\
 & \quad - Q_{ab}(X) J_b Y - Q_{ac}(X) J_c Y = 0.
 \end{aligned} \tag{3.13}$$

Comparing horizontal, vertical, and normal components in the above equation to get

$$\begin{aligned}
 & \left(\tilde{\nabla}_X^* J_a \right) Y - \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \} \\
 & \quad - Q_{ab}(X) J_b Y - Q_{ac}(X) J_c Y = 0,
 \end{aligned} \tag{3.14}$$

$$C(X, J_a Y) = J_a h(X, Y), \tag{3.15}$$

$$h(X, J_a Y) = J_a C(X, Y) \tag{3.16}$$

from (3.14), we have

$$\begin{aligned}
 & \tilde{\nabla}_X^* J_a Y - J_a \tilde{\nabla}_X^* Y - \frac{1}{2} \{ g(J_a Y, B) X - g(B, Y) J_a X - g(X, J_a Y) B + g(X, Y) J_a B \} \\
 & \quad - Q_{ab}(X) J_b Y - Q_{ac}(X) J_c Y = 0.
 \end{aligned} \tag{3.17}$$

Then for any $X_*, Y_* \in \chi(B)$, and J' being almost complex structure on B , we have after operating π^* on the above equation

$$\begin{aligned} \nabla_{X_*}^* J'_a Y_* - J'_a \nabla_{X_*}^* Y_* - \frac{1}{2} \{ g_*(J'_a Y_*, B_*) X_* - g_*(B_*, Y_*) J'_a X_* - g_*(X_*, J'_a Y_*) B_* + g_*(X_*, Y_*) J'_a B_* \} \\ - Q_{ab}^*(X_*) J_a^*(Y_*) - Q_{ac}^*(X_*) J_c^*(Y_*) = 0. \end{aligned} \quad (3.18)$$

This gives

$$\begin{aligned} (\nabla_{X_*}^* J'_a) Y_* - \frac{1}{2} \{ \theta'(Y_*) X_* - \omega'(Y_*) J'_a X_* - \Omega'(X_*, Y_*) B_* + g_*(X_*, Y_*) J'_a B_* \} \\ - Q'_{ab}(X_*) J'_a Y_* - Q'_{ac}(X_*) J'_c Y_* = 0. \end{aligned} \quad (3.19)$$

This shows that B is l.c.q.K. manifold. $\square$

Now, using (2.17) and (2.18), we obtain a relation between curvature tensor R on M and curvature tensor R^* of B as follows:

$$\begin{aligned} R(X, Y, Z, W) = R^*(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)), \end{aligned} \quad (3.20)$$

where $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$, and $\pi_* W = W_* \in B$.

Now, using the above equation together with (2.8) and using the fact that C is skew-symmetric, we obtain

$$\begin{aligned} \overline{H}(X) &= \overline{R}(X, J_a X, J_a X, X) \\ &= H^*(X_*) + \|h(X, J_a X)\|^2 - g(h(J_a X, J_a X), h(X, X)) \\ &\quad - 3\|C(X, J_a X)\|^2, \end{aligned} \quad (3.21)$$

where $\overline{H}(X)$ and $H^*(X_*)$ are the holomorphic sectional curvature tensors of $\overline{M}$ and B , respectively.

If we assume that D is integrable then using Lemma 2.3(i), we have

$$h(J_a X, J_a X) = -h(X, X). \quad (3.22)$$

Also from (3.15), we have $C(X, J_a X) = 0$. Then, (3.21) reduces to

$$\overline{H}(X) = H^*(X_*) + \|h(X, J_a X)\|^2 + \|h(X, X)\|^2, \quad \forall X \in D. \quad (3.23)$$

This gives $\overline{H}(X) \geq H^*(X_*)$.

Thus, we have the following result.

Theorem 3.3. Let M be a CR-submanifold of a l.c.q.K. manifold $\overline{M}$ with integrable D . Let B be an almost Hermitian manifold and $\pi : \overline{M} \rightarrow B$ be a submersion. Then holomorphic sectional curvatures $\overline{H}$ and H^* of $\overline{M}$ and B , respectively, satisfy

$$\overline{H}(X) \geq H^*(X_*), \quad \text{for all unit vectors } X \in D. \quad (3.24)$$

Note. The above result was obtained in [9] by taking $\overline{M}$ to be quasi-Kaehler manifold. Later, similar type of relation was derived in [11], considering $\overline{M}$ to be l.c.K manifold.

Acknowledgments

The first author is thankful to the Department of Science and Technology, Government of India, for its financial assistance provided through INSPIRE fellowship no. DST/INSPIRE Fellowship/2009/[XXV] to carry out this research work.

References

- [1] I. Vaisman, "On locally conformal almost kähler manifolds," *Israel Journal of Mathematics*, vol. 24, no. 3-4, pp. 338–351, 1976.
- [2] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, vol. 155 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 1998.
- [3] L. Ornea and P. Piccinni, "Locally conformal kähler structures in quaternionic geometry," *Transactions of the American Mathematical Society*, vol. 349, no. 2, pp. 641–655, 1997.
- [4] L. Ornea, "Weyl structure on quaternionic manifolds, a state of the art," <http://arxiv.org/abs/math/0105041>.
- [5] B. Sahin and R. Günes, "QR-submanifolds of a locally conformal quaternion kaehler manifold," *Publicationes Mathematicae Debrecen*, vol. 63, no. 1-2, pp. 157–174, 2003.
- [6] B. O'Neill, "The fundamental equations of a submersion," *The Michigan Mathematical Journal*, vol. 13, pp. 459–469, 1966.
- [7] A. Bejancu, "CR submanifolds of a kaehler manifold. I," *Proceedings of the American Mathematical Society*, vol. 69, no. 1, pp. 135–142, 1978.
- [8] S. Kobayashi, "Submersions of CR submanifolds," *The Tohoku Mathematical Journal*, vol. 39, no. 1, pp. 95–100, 1987.
- [9] S. Deshmukh, T. Ghazal, and H. Hashem, "Submersions of CR-submanifolds on an almost hermitian manifold," *Yokohama Mathematical Journal*, vol. 40, no. 1, pp. 45–57, 1992.
- [10] J. Simons, "Minimal varieties in riemannian manifolds," *Annals of Mathematics*, vol. 88, no. 1, pp. 62–105, 1968.
- [11] R. Al-Ghefari, M. H. Shahid, and F. R. Al-Solamy, "Submersion of CR-submanifolds of locally conformal kaehler manifold," *Contributions to Algebra and Geometry*, vol. 47, no. 1, pp. 147–159, 2006.