Research Article

On Generalized Sasakian-Space-Forms

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Received 31 October 2012; Accepted 16 December 2012

Academic Editors: A. Morozov, A. Popov, and E. H. Saidi

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The purpose of the present paper is to characterize pseudoprojectively flat and pseudoprojective semisymmetric generalized Sasakian-space-forms.

1. Introduction

Alegre et al. [1] introduced and studied the generalized Sasakian-space-forms. The authors Alegre and Carriazo [2], Somashekara and Nagaraja [3, 4], and De and Sarkar [5, 6] studied the generalized Sasakian-space-forms. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian-space-form if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of M is given by

$$R(X, Y)Z = f_1 R_1(X, Y)Z + f_2 R_2(X, Y)Z + f_3 R_3(X, Y)Z, \quad (1.1)$$

for any vector fields X, Y, Z on M , where

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi. \end{aligned} \quad (1.2)$$

In this paper, we study the curvature properties like flatness, symmetry, and semisymmetry properties in a generalized Sasakian-space-form by considering a pseudoprojective curvature tensor.

The paper is organized as follows. Section 2 of this paper contains some preliminary results on the generalized Sasakian-space-forms. In Section 3, we study the pseudoprojectively flat generalized Sasakian-space-form and obtain necessary and sufficient conditions for a generalized Sasakian-space-form to be pseudoprojectively flat. In the next section, we deal with pseudoprojectively semisymmetric generalized Sasakian-space-forms, and it is proved that a generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is pseudoprojectively flat and $f_1 = f_3$. The last section is devoted to the study of τ -flat and τ - ϕ -semi symmetric generalized Sasakian-space-forms. In this section, we prove that the associated functions f_1, f_2, f_3 are linearly dependent.

In a $(2n + 1)$ -dimensional almost contact metric manifold, the pseudoprojective curvature tensor $\tilde{P}$ [7] is defined by

$$\begin{aligned} \tilde{P}(X, Y)Z = & aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ & - \frac{r}{2n+1} \left(\frac{a}{2n} + b \right) [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (1.3)$$

where a and b are constants and R, S , and r are the Riemannian curvature tensor of type $(0, 2)$, the Ricci tensor, and the scalar curvature of the manifold, respectively. If $a = 1$, $b = -(1/2n)$, then (1.3) takes the form

$$\tilde{P}(X, Y)Z = P(X, Y)Z, \quad (1.4)$$

where P is the projective curvature tensor. A manifold (M, ϕ, ξ, η, g) shall be called pseudoprojectively flat if the pseudoprojective curvature tensor $\tilde{P} = 0$. It is known that the pseudoprojectively flat manifold is either projectively flat (if $a \neq 0$) or Einstein (if $a = 0$ and $b \neq 0$).

2. Preliminaries

A $(2n+1)$ -dimensional C^∞ -differentiable manifold M is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad (2.1)$$

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad (2.4)$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \quad (2.5)$$

where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form, and g is a Riemannian metric on M . A manifold equipped with an almost contact metric structure is called

an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X, \phi Y) = d\eta(X, Y), \quad (2.6)$$

for all vector fields X and Y .

In a generalized Sasakian-space-form, the following hold:

$$\begin{aligned} R(X, Y)Z &= f_1 [g(Y, Z)X - g(X, Z)Y] + f_2 [g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ &\quad + f_3 [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi], \end{aligned} \quad (2.7)$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \quad (2.8)$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \quad (2.9)$$

$$r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3. \quad (2.10)$$

3. Pseudoprojectively Flat Generalized Sasakian-Space-Forms

If the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ under consideration is pseudoprojectively flat, then from (1.3) we have

$$\begin{aligned} 'R(X, Y, Z, W) &= \frac{b}{a} [S(X, Z)g(Y, W) - S(Y, Z)g(X, W)] \\ &\quad + \frac{r}{(2n + 1)a} \left(\frac{a}{2n} + b \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \end{aligned} \quad (3.1)$$

where a and b are constants and $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Now taking $Z = \xi$ in (3.1) and using (2.1), (2.2), (2.7), and (2.9), we get

$$\begin{aligned} (f_1 - f_3) [\eta(Y)g(X, W) - \eta(X)g(Y, W)] &= \frac{2nb}{a} (f_1 - f_3) (\eta(Y)g(X, W) - \eta(X)g(Y, W)) \\ &\quad + \frac{r}{(2n + 1)a} \left(\frac{a}{2n} + b \right) \\ &\quad \times (\eta(Y)g(X, W) - \eta(X)g(Y, W)). \end{aligned} \quad (3.2)$$

Again putting $X = \xi$ in (3.2), we get

$$\left[\left(\frac{a + 2nb}{a} \right) \left(\frac{2n(2n + 1)(f_1 - f_3) - r}{2n(2n + 1)} \right) \right] [\eta(Y)\eta(W) - g(Y, W)] = 0. \quad (3.3)$$

The aforementioned equation implies

$$\left(\frac{a+2nb}{a}\right)\left[\frac{2n(2n+1)(f_1-f_3)-r}{2n(2n+1)}\right]=0. \quad (3.4)$$

That is, either

$$(a+2nb)=0 \quad (3.5)$$

or

$$r=2n(2n+1)(f_1-f_3). \quad (3.6)$$

If $a+2nb=0$, $a \neq 0$ and $b \neq 0$, then, from (1.3), it follows that $\tilde{P}(X, Y)Z = aP(X, Y)Z$. Thus in this case pseudoprojective flatness and projective flatness are equivalent.

If $a+2nb \neq 0$, $a \neq 0$ and $b \neq 0$, then comparing (2.10) and (3.6), we get

$$3f_2 + (2n-1)f_3 = 0. \quad (3.7)$$

Using (3.7) in (2.9), we get

$$S(X, Y) = 2n(f_1 - f_3)g(X, Y). \quad (3.8)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X = Y = e_i$ and summing over $1 \leq i \leq 2n+1$, we obtain

$$r = 2n(2n+1)(f_1 - f_3). \quad (3.9)$$

This shows that $M(f_1, f_2, f_3)$ is Einstein with a scalar curvature $r = 2n(2n+1)(f_1 - f_3)$. Thus we state the following.

Theorem 3.1. *A pseudoprojectively flat generalized Sasakian-space-form is either projectively flat or an Einstein manifold with a scalar curvature $r = [2n(2n+1)(f_1 - f_3)]$.*

Suppose that (3.7) holds. Then in view of (2.7) and (2.9), we can write (1.3) as

$$\begin{aligned} \tilde{P}(X, Y, Z, W) = & af_1(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ & + af_2[g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)] \\ & + af_3[\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \eta(Y)g(X, Z)g(\xi, W) \\ & \quad - \eta(X)g(Y, Z)g(\xi, W)] + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] \\ & - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \end{aligned} \quad (3.10)$$

where

$$' \tilde{P}(X, Y, Z, W) = g(\tilde{P}(X, Y)Z, W). \quad (3.11)$$

Replacing X by ϕX and Y by ϕY , we get

$$\begin{aligned} \tilde{P}(\phi X, \phi Y, Z, W) &= af_3(g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W)) \\ &\quad + af_2(g(\phi X, \phi Z)g(\phi^2 Y, W) - g(\phi Y, \phi Z)g(\phi^2 X, W)) \\ &\quad + 2g(\phi X, \phi^2 Y)g(\phi Z, W). \end{aligned} \quad (3.12)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold.

Taking $Y = W = e_i$ and summation over i , $1 \leq i \leq 2n + 1$, we get

$$\sum_{i=1}^{2n+1} \tilde{P}(\phi X, \phi e_i, Z, e_i) = af_3(g(\phi X, \phi Z)) + af_2(-g(\phi X, \phi Z)g(\phi e_i, \phi e_i) - g(\phi^2 X, \phi^2 Z)). \quad (3.13)$$

Again putting $X = Z = e_i$ and taking summation over i , $1 \leq i \leq 2n + 1$, we get $f_2 = 0$ with $a \neq 0$. In view of (3.7), we get $f_3 = 0$.

Now (2.7) reduces to the form

$$R(X, Y)Z = f_1[g(Y, Z)X - g(X, Z)Y], \quad (3.14)$$

from which we have $S(X, Y) = 2nf_1g(X, Y)$, and consequently

$$r = 2n(2n + 1)f_1. \quad (3.15)$$

By using (3.14) and (3.15) in (1.3), we get $\tilde{P}(X, Y)Z = 0$. This leads to the following.

Theorem 3.2. *A $(2n + 1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is pseudoprojectively flat if and only if $a + 2nb \neq 0$, $a \neq 0$, $b \neq 0$ and $3f_2 + (2n - 1)f_3 = 0$.*

Alegre and Carriazo [2] proved that any contact metric generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with a dimension greater than or equal to five is a Sasakian manifold and f_1 , f_2 , and f_3 must be constants.

Thus from (3.14), we have the following theorem.

Theorem 3.3. *A $(2n + 1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with a dimension greater than or equal to 5 is of constant curvature f_1 if and only if $a + 2nb \neq 0$, $a \neq 0$, $b \neq 0$, and $3f_2 + (2n - 1)f_3 = 0$.*

4. Pseudoprojective Semisymmetric Generalized Sasakian-Space-Form

Definition 4.1. If a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ satisfies

$$R(X, Y) \cdot \tilde{P} = 0, \quad (4.1)$$

then the manifold is said to be pseudoprojectively semisymmetric manifold.

By using (1.3), (2.1), (2.2), (2.7), and (2.9), we have

$$\begin{aligned} \eta(\tilde{P}(X, Y)Z) &= \left[a(f_1 - f_3) - \left(\frac{r}{2n+1} \right) \left(\frac{a}{2n} + b \right) \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &\quad + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \end{aligned} \quad (4.2)$$

Taking $Z = \xi$ in (4.2), we get

$$\eta(\tilde{P}(X, Y)\xi) = 0. \quad (4.3)$$

Again putting $X = \xi$ in (4.2), we get

$$\begin{aligned} (\eta(\tilde{P}(\xi, Y)Z)) &= \left[a(f_1 - f_3) - \left(\frac{r}{2n+1} \right) \left(\frac{a}{2n} + b \right) \right] [g(Y, Z) - \eta(Y)\eta(Z)] \\ &\quad + b[S(Y, Z) - 2n(f_1 - f_3)\eta(Y)\eta(Z)]. \end{aligned} \quad (4.4)$$

From (4.1), we have

$$\begin{aligned} (R(X, Y)\tilde{P}(U, V)W) - \tilde{P}(R(X, Y)U, V)W \\ - \tilde{P}(U, R(X, Y)V)W - \tilde{P}(U, V)R(X, Y)W = 0. \end{aligned} \quad (4.5)$$

Taking $X = \xi$ and contracting the above with respect to ξ , we get

$$\begin{aligned} (f_1 - f_3) \{ &\bar{P}(U, V, W, Y) - \eta(Y)\eta(\bar{P}(U, V)W) + \eta(U)\eta(\bar{P}(Y, V)W) \\ &- g(Y, U)\eta(\bar{P}(\xi, V)W) + \eta(V)\eta(\bar{P}(U, Y)W) - g(Y, V)\eta(\bar{P}(U, \xi)W) \\ &+ \eta(W)\eta(\bar{P}(U, V)Y) - g(Y, W)\eta(\bar{P}(U, V)\xi) \} = 0. \end{aligned} \quad (4.6)$$

Putting $U = Y$ in (4.6) and with the help of (4.2) and (4.3), we get either

$$f_1 = f_3 \quad (4.7)$$

or

$$\begin{aligned} & \bar{P}(Y, V, W, Y) - g(Y, Y)\eta(\bar{P}(\xi, V)W) \\ & - g(Y, V)\eta(\bar{P}(Y, \xi)W) + \eta(W)\eta(\bar{P}(Y, V)Y) = 0. \end{aligned} \quad (4.8)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold of the manifold. Putting $Y = e_i$ and taking summation over i , $1 \leq i \leq 2n + 1$, and using (4.2) and (4.4), we obtain

$$S(V, W) = Ag(V, W) + B\eta(V)\eta(W), \quad (4.9)$$

where

$$\begin{aligned} A &= 2nf_1 + 3f_2 - f_3, \\ B &= (2n + 1)[-3f_2 - (2n - 1)f_3]. \end{aligned} \quad (4.10)$$

Now contracting (4.9), we obtain

$$r = (2n + 1)A + B. \quad (4.11)$$

Using (4.10) in (4.11), we get

$$r = 2n(2n + 1)(f_1 - f_3). \quad (4.12)$$

In view of (2.10), (4.12) yields

$$3f_2 + (2n - 1)f_3 = 0. \quad (4.13)$$

From (2.9) and (4.13), we have

$$S(V, W) = 2n(f_1 - f_3)g(V, W). \quad (4.14)$$

Now using (4.12) and (4.14) in (4.2), we get

$$\eta(\tilde{P}(U, V)W) = 0. \quad (4.15)$$

Plugging (4.15) in (4.6), we obtain

$$\tilde{P}(U, V, W, Y) = 0. \quad (4.16)$$

Therefore by taking into account (4.7) and (4.16), we have either $f_1 = f_3$ or $M(f_1, f_2, f_3)$ is pseudoprojectively flat.

Conversely, suppose that $f_1 = f_3$. Then, from (2.1), (2.2) and (2.7), we have $R(\xi, X)Y = 0$. Hence $R(\xi, U) \cdot \tilde{P} = 0$. If the space-form is pseudoprojectively flat then clearly it is pseudoprojectively semisymmetric. Hence we can state the following.

Theorem 4.2. *A $2n+1$ -dimensional generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is either pseudoprojectively flat or $f_1 = f_3$.*

By combining Theorems 3.2 and 4.2, we have the following.

Corollary 4.3. *A $(2n+1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is pseudoprojectively flat if and only if $f_1 = f_3$ or $a + 2nb \neq 0$ and $3f_2 + (2n-1)f_3 = 0$.*

5. τ -Curvature Tensor in a Generalized Sasakian-Space-Form

In a $(2n+1)$ -dimensional Riemannian manifold M , the τ -curvature tensor is given by [8]

$$\begin{aligned} \tau(X, Y)Z = & a_0R(X, Y)Z + a_1S(Y, Z)X + a_2S(X, Z)Y + a_3S(X, Y)Z \\ & + a_4g(Y, Z)QX + a_5g(X, Z)QY + a_6g(X, Y)QZ \\ & + a_7r(g(Y, Z)X - g(X, Z)Y), \end{aligned} \quad (5.1)$$

where $a_0, \dots, a_7$ are some smooth functions on M . For different values of $a_0, \dots, a_7$, the τ -curvature tensor reduces to the curvature tensor, quasiconformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudoprojective curvature tensor, projective curvature tensor, M -projective curvature tensor, W_i -curvature tensors ($i = 0, \dots, 9$), and W_j^* -curvature tensors ($j = 0, 1$).

Suppose that $M(f_1, f_2, f_3)$ is τ -flat. Then from (5.1), we have

$$\begin{aligned} -a_0R(X, Y)Z = & a_1S(Y, Z)X + a_2S(X, Z)Y + a_3S(X, Y)Z \\ & + a_4g(Y, Z)QX + a_5g(X, Z)QY + a_6g(X, Y)QZ \\ & + a_7r(g(Y, Z)X - g(X, Z)Y). \end{aligned} \quad (5.2)$$

In view of (2.7), (2.8), and (2.9) in (5.2), we have

$$\begin{aligned} & -a_0\{f_1[g(Y, Z)X - g(X, Z)Y] + f_2[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi]\} \\ & = a_1[(2nf_1 + 3f_2 - f_3)g(Y, Z) - (3f_2 + (2n-1)f_3)\eta(Y)\eta(Z)]X \\ & + a_2[(2nf_1 + 3f_2 - f_3)g(X, Z) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Z)]Y \\ & + a_3[(2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)]Z \end{aligned}$$

$$\begin{aligned}
& + a_4 g(Y, Z) [(2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi] \\
& + a_5 g(X, Z) [(2nf_1 + 3f_2 - f_3)Y - (3f_2 + (2n - 1)f_3)\eta(Y)\xi] \\
& + a_6 g(X, Y) [(2nf_1 + 3f_2 - f_3)Z - (3f_2 + (2n - 1)f_3)\eta(Z)\xi] \\
& + a_7 r [g(Y, Z)X - g(X, Z)Y].
\end{aligned} \tag{5.3}$$

Putting $X = \phi Y$ in (5.3), we get

$$\begin{aligned}
& - a_0 \left\{ f_1 [g(Y, Z)\phi Y - g(\phi Y, Z)Y] + f_2 [g(\phi Y, \phi Z)\phi Y - g(Y, \phi Z)\phi^2 Y + 2g(\phi Y, \phi Y)\phi Z] \right. \\
& \quad \left. + f_3 [-\eta(Y)\eta(Z)\phi Y + g(\phi Y, Z)\eta(Y)\xi] \right\} \\
& = a_1 [(2nf_1 + 3f_2 - f_3)g(Y, Z) - (3f_2 + (2n - 1)f_3)\eta(Y)\eta(Z)]\phi Y \\
& \quad + a_2 (2nf_1 + 3f_2 - f_3)g(\phi Y, Z)Y + a_4 (2nf_1 + 3f_2 - f_3)g(Y, Z)\phi Y \\
& \quad + a_5 g(\phi Y, Z) [(2nf_1 + 3f_2 - f_3)Y - (3f_2 + (2n - 1)f_3)\eta(Y)\xi] \\
& \quad + a_7 r [g(Y, Z)\phi Y - g(\phi Y, Z)Y].
\end{aligned} \tag{5.4}$$

If we choose a unit vector U orthogonal to ξ and taking $Y = U$, then making use of (2.1) and (2.3) in (5.4), we obtain

$$\begin{aligned}
& [-a_0 f_1 + (a_2 + a_5)(2nf_1 + 3f_2 - f_3) - a_7 r + f_2]g(\phi U, Z)U \\
& \quad + [a_0(f_1 + f_2) + (a_1 + a_4)(2nf_1 + 3f_2 - f_3) + a_7 r]g(U, Z)\phi U \\
& \quad + 2a_0 f_2 g(U, U)\phi Z = 0.
\end{aligned} \tag{5.5}$$

Putting $Z = U$ in (5.5), we have

$$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0, \tag{5.6}$$

where

$$\begin{aligned}
\lambda_1 &= a_0 + 2n(a_1 + a_4) + 2n(2n + 1)a_7, \\
\lambda_2 &= 3(a_0 + a_1 + a_4 + 2na_7), \\
\lambda_3 &= -(a_1 + a_4 + 4na_7).
\end{aligned} \tag{5.7}$$

Thus we have the following.

Theorem 5.1. *If a $(2n + 1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is τ -flat, then (5.6) holds.*

From the above theorem, we discuss the following cases.

Case (i). (1) If $M(f_1, f_2, f_3)$ is quasiconformally flat, then $a_1 = -a_2 = a_4 = -a_5$, $a_3 = a_6 = 0$, $a_7 = (-1/(2n+1))(a_0/2n+2a_1)$. Putting these in (5.7), we obtain $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(2) If $M(f_1, f_2, f_3)$ is conharmonically flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 0$. Putting these in (5.7), we get $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for W_0^* -flat, W_1 -flat, W_3 -flat, W_9 -flat spaces, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Case (ii). If $M(f_1, f_2, f_3)$ is conformally flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 1/2n(2n-1)$.

Putting these in (5.7), we obtain $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. Hence $f_2 = 0$.

Case (iii). (a) If $M(f_1, f_2, f_3)$ is pseudoprojectively flat, then $a_1 = -a_2$, $a_3 = a_4 = a_5 = a_6 = 0$, $a_7 = -(1/(2n+1))(a_0/2n+a_1)$.

By putting these values in (5.7), we have $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(b) If $M(f_1, f_2, f_3)$ is projectively flat, then $a_0 = 1$, $a_1 = -a_2 = -(1/2n)$, $a_3 = a_4 = a_5 = a_6 = a_7 = 0$.

Making use of the above functional values in (5.7), we get $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for concircularly flat, M -projectively flat, W_0 -flat, W_1^* -flat, W_2 -flat, W_6 -flat, and W_8 -flat spaces, (5.7) gives $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Case (iv). (a) If $M(f_1, f_2, f_3)$ is W_4 -flat, then $a_0 = 1$, $a_5 = -a_6 = 1/2n$, $a_1 = a_2 = a_3 = a_4 = a_7 = 0$.

Putting these in (5.7), we obtain that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

(b) If $M(f_1, f_2, f_3)$ is W_5 -flat, then $a_0 = 1$, $a_2 = -a_5 = -(1/2n)$, $a_1 = a_3 = a_4 = a_6 = a_7 = 0$. Putting these in (5.7), we have $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Similarly, for a W_7 -flat space, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Summarizing the above cases, we have the following corollaries.

Corollary 5.2. *If a $(2n+1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either quasiconformally flat, conharmonically flat, W_0^* -flat, W_1 -flat, W_3 -flat, or W_9 -flat, then f_1 , f_2 , and f_3 are linearly dependent.*

Corollary 5.3. *If a $(2n+1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is conformally flat, then $f_2 = 0$.*

The above corollary was already proved by Kim [9] and Sarkar and De [10].

Corollary 5.4. *If a $(2n+1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either pseudoprojectively flat, projectively flat, concircularly flat, M -projectively flat, W_0 -flat, W_1^* -flat, W_2 -flat, W_6 -flat, or W_8 -flat, then f_2 and f_3 are linearly dependent.*

Corollary 5.5. *If a $(2n+1)$ -dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either W_4 -flat, W_5 -flat, or W_7 -flat, then f_1 and f_2 are linearly dependent.*

5.1. $\tau - \phi$ -Semisymmetric Generalized Sasakian-Space-Form

Definition 5.6. $M(f_1, f_2, f_3)$ is $\tau - \phi$ -semisymmetric if

$$\tau(X, Y) \cdot \phi = 0 \quad (5.8)$$

holds in $M(f_1, f_2, f_3)$.

We know that

$$(\tau(X, Y) \cdot \phi)Z = \tau(X, Y)\phi Z - \phi(\tau(X, Y)Z). \quad (5.9)$$

From (5.8) and (5.9), we have

$$\tau(X, Y)\phi Z - \phi(\tau(X, Y)Z) = 0. \quad (5.10)$$

By using (5.1) in (5.10), we have

$$\begin{aligned} & a_0 R(X, Y)\phi Z + a_1 S(Y, \phi Z)X + a_2 S(X, \phi Z)Y + a_3 S(X, Y)\phi Z \\ & + a_4 g(Y, \phi Z)QX + a_5 g(X, \phi Z)QY + a_6 g(X, Y)Q(\phi Z) \\ & + a_7 r[g(Y, \phi Z)X - g(X, \phi Z)Y] - a_7 r[g(Y, Z)\phi X - g(X, Z)\phi Y] \\ & - \{a_0 \phi(R(X, Y)\phi Z) + a_1 S(Y, Z)\phi X + a_2 S(X, Z)\phi Y + a_3 S(X, Y)\phi Z \\ & + a_4 g(Y, Z)\phi(QX) + a_5 g(X, Z)\phi(QY) + a_6 g(X, Y)\phi(QZ)\} = 0. \end{aligned} \quad (5.11)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Contracting (5.11) with respect to W and putting $Y = W = e_i$, also taking summation over i , $1 \leq i \leq 2n + 1$, and making use of (2.1), (2.4), (2.7), (2.9), and (2.8), we have

$$[2a_1 + (2n + 1)a_2]S(X, \phi Z) = Ag(X, \phi Z), \quad (5.12)$$

where

$$\begin{aligned} A = & [-(2n - 1)a_0 + 4na_4 + 2n(2n + 1)a_5 - 2n(4n^2 - 1)a_7]f_1 \\ & + [2(n - 1)a_0 + 6a_4 + 6na_5 - 6n(2n - 1)a_7]f_2 \\ & + [-2a_4 - 4na_5 + 4n(2n - 1)a_7]f_3. \end{aligned} \quad (5.13)$$

Changing Z to ϕZ in (5.12) and also in view of (2.1) and (2.2), (2.9) yields

$$\begin{aligned} S(X, Z) = & \left[\frac{A}{(2a_1 + (2n + 1)a_2)} \right] g(X, Z) \\ & + \left[\frac{2n(2a_1 + (2n + 1)a_2)(f_1 - f_3) - A}{(2a_1 + (2n + 1)a_2)} \right] \eta(X)\eta(Z). \end{aligned} \quad (5.14)$$

Thus we can state the following.

Theorem 5.7. *A $\tau - \phi$ -semisymmetric generalized Sasakian-space-form is η -Einstein provided $(2a_1 + (2n + 1)a_2) \neq 0$.*

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