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Research Article

Quantum Histories and Quantum Complementarity

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We discuss the origin of quantum complementarity using Feynman's sum-over-histories approach to Quantum Mechanics (QM). We analyze two experimental setups introduced by W. G. Unruh and S. Afshar and show how one should consistently calculate the existence of which-way information using interfering quantum histories. Then we discuss the possible problems associated with the notion of which way information and provide an explanation why Englert-Greenberger duality relation cannot be violated experimentally if the standard QM postulates are accepted.

1. Introduction

In the early days of quantum mechanics (QM) it was thought that quantum complementarity results from Heisenberg uncertainty principle; however later works have shown that the quantum complementarity could be more fundamental than previously thought and might be enforced via entanglements between the evolving quantum particle and measuring devices [1, 2]. For interferometric (or double slit) setups the quantum complementarity has been formulated in a mathematical form known as *Englert-Greenberger duality relation*:

$$V^2 + D^2 \leqslant 1, \tag{1.1}$$

where V stands for *visibility* of interference fringes and D stands for *distinguishability* of photon paths. If we denote by ψ_1 and ψ_2 the two photon beams passing through each interferometer arm (or each slit) and if we assume that ψ_1 and ψ_2 are *pure states* (cf. [3, 4]) then for V and D we will have

$$V = \frac{2|\psi_1||\psi_2|}{|\psi_1|^2 + |\psi_2|^2}, \tag{1.2}$$

$$D = \frac{\left||\psi_1|^2 - |\psi_2|^2\right|}{|\psi_1|^2 + |\psi_2|^2}. \tag{1.3}$$

Simple arithmetic substitution of (1.2) and (1.3) in (1.1) shows that indeed there is an equality: $V^2+D^2 = 1$. Inequality will be achieved if ψ_1 and ψ_2 are *mixed states*; however in such a case the expressions for V and D should take into account the degree of coherence within each beam and thus differ from (1.2) and (1.3) (cf. [3, 4]). In the subsequent discussion we will consider setups in which ψ_1 and ψ_2 are *pure states* therefore for our purpose we could safely use (1.2) and (1.3) without being mathematically inconsistent.

In 2004, Afshar and Cramer announced that Afshar had found an experimental way of showing that the principle of quantum complementarity is wrong [5, 6]. Particularly their claim was that the Englert-Greenberger duality relation is violated in a double-slit lens setup where one may both (1) verify the existent quantum interference pattern by placing wire grid at the dark fringes at the focal plane of the lens and subsequently (2) obtain the which-way information by detecting the photon at the image plane of the lens and finding out through which slit the photon passed. Furthermore, Cramer considered the results of Afshar experiment as an evidence against Copenhagen Interpretation of QM and Bohr's *antirealism* (cf. *Appendix*). Later in 2004, Unruh proposed and analyzed an alternative interferometric experiment, which is equivalent to Afshar's setup but simpler for comprehension. Regretfully, Unruh defended the principle of complementarity (as well as Bohr's antirealism) via inconsistent mathematical reasoning, which lead to a heated debate both in Internet and academic journals.

Before we discuss the different viewpoints expressed in the complementarity debate, we would like to point out that under *standard QM postulates* are understood the unitary evolution of the wavefunction ψ of a given quantum system described by the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{1.4}$$

and the Born rule:

$$\mathbf{P}(\mathbf{r}, t) = \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2 \tag{1.5}$$

according to which the *probability density function* $\mathbf{P}(\mathbf{r}, t)$ for observing a quantum system at position $\mathbf{r}$ and at time t is obtained from the square of its wavefunction $\psi(\mathbf{r}, t)$ (cf. [7]).

All other *auxiliary assumptions* that make it possible to attribute some physical meaning to the performed calculations should be considered *interpretation dependent* and could potentially lead to inconsistencies (paradoxes). Two problematic *auxiliary assumptions* present in the Copenhagen Interpretation of QM and Bohr's *antirealism*, which should not be classified as standard QM viewpoint, are the following.

(A1) Results or conclusions derived from investigation of two alternative single-slit setups could be extrapolated to a coherent version of the setup in which both slits are open [8–13].

(A2) Introduction of an obstacle (absorber) at a place where the wavefunction of given quantum system is zero ($\psi = 0$) could fundamentally change the results or the conclusions derived from the experimental setup [11–14].

The problems stemming from assumptions (A1) and (A2) will be analyzed in detail in Sections 3.1 and 3.2.

Some of the most notable positions expressed in the complementarity debate are as follows.

(P1) Quantum complementarity is violated; there are both which-way information in Afshar's setup and detectable interference fringes, hence $V^2 + D^2 \leqslant 2$ [6, 8–10, 15–17]. Afshar's setup is fundamentally different from Unruh's setup.

(P2) Quantum complementarity is not violated in Afshar's setup; it is true that $V^2+D^2 \leqslant 2$; however Englert-Greenberger duality relation has not been derived for Afshar's setup [18, 19].

(P3) Quantum complementarity is not violated in Afshar's setup, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is which-way information in Afshar's setup; however putting the wire grid at the dark interference fringes (partially) erases the which-way information [11, 12, 14].

(P4) Quantum complementarity is not violated in Afshar's setup, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is which-way information in Afshar's setup, putting the wire grid simply does not detect interference pattern; a photon detected at the image plane of the lens reveals the which-way information and a photon detected at the focal plane of the lens reveals the interference pattern; however since a photon cannot be detected twice there is no paradox to be solved [20].

(P5) Quantum complementarity is not violated in Afshar's setup, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is which-way information in Afshar's setup and also there is destructive interference at the dark fringes provided that the interference is only calculated but not measured; however putting the wire grid measures the interference and therefore (partially) erases the which-way information. Afshar's setup is equivalent to Unruh's setup [13].

(P6) Quantum complementarity is not violated in Afshar's setup, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is no which-way information in Afshar's setup, putting the wire grid at the dark interference fringes is irrelevant for the existence of which-way information. Afshar's setup is completely different from Unruh's setup [21].

(P7) Quantum complementarity is not violated, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is no which-way information in Afshar's setup, and putting the wire grid at the dark interference fringes is irrelevant for the existence of which-way information [22, 23].

(P8) Quantum complementarity is not violated, $V^2 + D^2 \leqslant 1$, without wire grid on the photon path there is no which-way information in Afshar's setup, and putting the wire grid at the dark interference fringes is irrelevant for the existence of which-way information. Afshar's setup is equivalent to Unruh's setup. The pseudo-paradox requires at least 8 alternative quantum histories and troubles result from inconsistent calculation of interferences among these 8 quantum histories [24–26].

As it can be seen the various positions might agree on some statements, but disagree in the details. In this work, we will try to address each of the positions (P1)–(P8) expressed in the complementarity debate. We will provide complete description of Unruh's and Afshar's setups using Feynman's sum-over-histories approach, and we will show that the quantum complementarity results from the requirement for *mathematical consistency* of the theory. In Section 2 we will briefly introduce the reader to three important concepts, these of *quantum history*, *history Hilbert space,* and *branch vectors* in standard Hilbert space, in order to be able to tackle the problem of mathematical consistency in the context of QM experiments. In Section 3 we will describe Unruh's setup and will provide a geometrically intuitive description of the mutually exclusive possible ways in which the 8 quantum histories can interfere among each other. In Section 4 we will provide complete mathematical description of Unruh's setup using Feynman's sum over histories. In Section 5 we will introduce Afshar's setup and will prove the equivalence between Afshar's setup and Unruh's setup. In Section 6 we will solve numerically the Fresnel integrals for Afshar's setup and will show that the interference pattern in coherent setup is present regardless of the presence or absence of wire grid on photon paths. In Section 7 we will compute the visibility and distinguishability for Unruh's and Afshar's setups in two mutually exclusive cases: coherent setup versus incoherent setup. Then we will compare the proposed consistent mathematical treatment of those two setups with the inconsistent antirealist solution of the problem advocated by Unruh [13]. In Section 8 we will show that because the philosophical troubles with the notion of which-way information result from the Schrödinger equation, the which-way problem can be formulated even in Bohmian QM where the particle hidden trajectories through the slits can be easily inferred from the so-called Guiding equation (in other words the Guiding equation in Bohmian QM cannot resolve quantum complementarity issues). At the end we will briefly discuss what has been gained from the analysis of Afshar experiment using Feynman's sum-over-histories approach and will explain why Englert-Greenberger duality relation cannot be violated experimentally if the standard QM postulates are accepted.

2. Quantum Histories

Before we discuss Unruh's and Afshar's setups we will introduce several basic concepts, which will be used throughout our exposition. A central concept in the definition of which-way information is the *sequence of events* or *quantum history*. The concept of a *quantum history* originates from Feynman's path integral (sum-over-histories) formulation of QM [27, 28]. Modern interpretation of QM based on Feynman's work and utilizing the notion of *decoherent quantum histories* was further developed by Griffiths [29–32], Hartle [33, 34], Isham [35], Omnès [36], and others. In order to avoid confusion we would like to point out that the current work is based upon Feynman's sum-over-histories approach and does not require *decoherent quantum histories.*

In order to be able to meaningfully express in mathematical form the possibility for a quantum particle to pass through one slit, then to pass through region with interference fringes and finally to be registered by a detector, we need something more than just writing ordinary *state vector* $|\psi\rangle = \sum_i c_i|\psi_i\rangle$ in the *standard Hilbert space* $\mathcal{H}$ of the usual quantum formulation (cf. [37]). For example, let us discuss the tossing of a quantum coin which can end up either *tails* $|\psi_0\rangle$ or *heads* $|\psi_1\rangle$. The probability to obtain each observable outcome (tails or heads) is given by the Born rule $|\langle\psi_i|\psi\rangle|^2$, where $i = 0, 1$. Still, however we do not have

a *vocabulary* to talk about a sequence of n outcomes obtained at different times $t_1, t_2, \ldots, t_n$. We might want to discuss two different sequences of coin tossing, in which we have obtained, for example, (1) tails, tails, tails, heads, versus (2) tails, heads, tails, heads. Looking only at the last outcome (i.e., the state vector at a given time) we do not have sufficient information to distinguish between sequence (1) and (2). In addition, there is no single *observable* within the *standard Hilbert space* $\mathcal{H}$, which corresponds to any sequence of n coin tosses.

2.1. History Hilbert Space and History Projection Operators

Let us take the closed quantum system to be described by a quantum state $|\psi\rangle$ in $\mathcal{H}$. Each *quantum history* could be described by giving a sequence of alternatives $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ at a series of times $t_0, t_1, t_2, \ldots, t_n$. Alternatives at a moment of time t_k are presented by an exhaustive set of *orthogonal projection operators* (or simply *projectors*) $\{P^k_{\alpha_k}(t_k)\}$, $\alpha_k = 0, 1, 2, \ldots, n$. Each projector satisfies $P^\dagger = P = P^2$, where $P^\dagger$ denotes the conjugate transpose of P. Any nonzero ket $|\phi\rangle$ generates a one-dimensional subspace $\mathcal{P}$, often called a *ray* or a *pure state*, consisting of all scalar multiples of $|\phi\rangle$, that is to say, the collection of kets of the form $\alpha|\phi\rangle$, where α is any complex number. The *projector* onto $\mathcal{P}$ is the dyad:

$$P = [\phi] = \frac{|\phi\rangle\langle\phi|}{\langle\phi \mid \phi\rangle}. \tag{2.1}$$

In the following discussion we will assume that each ket $|\phi\rangle$ is normalized (which means $\langle\phi|\phi\rangle = 1$), therefore $P = [\phi] = |\phi\rangle\langle\phi|$. The symbol $[\phi]$ for the *projector* projecting onto the ray generated by $|\phi\rangle$ is not part of standard Dirac notation, but it is very convenient, and will be used throughout this exposition (cf. [32, page 29]).

Because the statement that a certain quantum history is realized is itself a proposition, it follows that the set of all such quantum histories should possess a lattice structure analogous to the lattice of single-time propositions in standard quantum logic [38]. In particular, a quantum history proposition could be represented by a *history projection operator* in a new type of Hilbert space called the *history Hilbert space* $\breve{\mathcal{H}}$ defined as a tensor product:

$$\breve{\mathcal{H}} = \mathcal{H}_{t_0} \odot \mathcal{H}_{t_1} \odot \mathcal{H}_{t_2} \odot \cdots \mathcal{H}_{t_n}, \tag{2.2}$$

where $\mathcal{H}_{t_n}$ is a copy of the Hilbert space $\mathcal{H}$ used to describe the system at time t_n and $\odot$ is a variant of the tensor product symbol $\otimes$. We can equally well write $\breve{\mathcal{H}} = \mathcal{H}_{t_0} \otimes \mathcal{H}_{t_1} \otimes \cdots \mathcal{H}_{t_n}$ but it is helpful to have a distinctive notation (i.e., $\odot$) for a tensor product when the factors in it refer to different times $t_0, t_1, t_2, \ldots, t_n$, and reserve the symbol $\otimes$ for a tensor product of spaces at a single time (cf. [32, page 96]).

If the initial state at t_0 of the studied quantum system is $|\psi_0\rangle$ in $\mathcal{H}$, the history of alternatives in $\breve{\mathcal{H}}$ is represented by the corresponding *chain of projections* called a *history projection operator*:

$$Y_\alpha = [\psi_0] \odot P^1_{\alpha_1}(t_1) \odot \cdots \odot P^{n-1}_{\alpha_{n-1}}(t_{n-1}) \odot P^n_{\alpha_n}(t_n). \tag{2.3}$$

Note that the superscripts in (2.3) are labels, not powers. This usage need not cause any confusion, since the square of a projector is the projector itself, and thus there is never any need to raise it to some power.

Although we have abbreviated the whole chain $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ by a single index α in the left-hand side of the equation (cf. [34]), in the following exposition it will be impractical to use Greek letters for denoting different *history projection operators* Y_α, Y_β, $Y_\gamma, \ldots$, and so forth, because we have to memorize what $\alpha, \beta, \gamma, \ldots$ stand for. Therefore it would be much easier if we have each *history projection operator* written in full notation $Y_{\alpha_1\alpha_2\cdots\alpha_n}$ instead of simply Y_α:

$$Y_{\alpha_1\alpha_2\cdots\alpha_n} \equiv Y_\alpha = [\psi_0] \odot P^1_{\alpha_1}(t_1) \odot \cdots \odot P^{n-1}_{\alpha_{n-1}}(t_{n-1}) \odot P^n_{\alpha_n}(t_n). \tag{2.4}$$

In order to complete the sample space we could add one more history:

$$Y_0 = (I - [\psi_0]) \odot I \odot \cdots \odot I \odot I \tag{2.5}$$

to which could be assigned *zero* probability because we required that the initial state of the studied quantum system is $|\psi_0\rangle$.

2.2. Chain Operators on the Standard Hilbert Space

The time development of a quantum system in the histories perspective is given by the time-dependent Schrödinger equation (1.4), which is used as a tool to calculate the probabilities of different histories. If we integrate the Schrödinger equation from time t_1 to t_2 starting from an arbitrary initial state $|\psi_{t_1}\rangle$, due to the fact the equation is linear, the dependence of the state $|\psi_{t_2}\rangle$ at time t_2 upon the initial state $|\psi_{t_1}\rangle$ can be written in the form

$$|\psi_{t_2}\rangle = T(t_2, t_1)|\psi_{t_1}\rangle. \tag{2.6}$$

The operator $T(t_2, t_1)$ is unitary operator known as *time development operator* [29–31]. It can be easily shown that $T(t_1, t_2)T(t_2, t_1) = I$, where I is the identity operator and $T(t_2, t_1) = T(t_1, t_2)^\dagger$.

With the use of time development operators, for the history Y_α in (2.3) can be defined a *chain operator* (also known as *class operator*) that operates on the *standard Hilbert space* $\mathscr{H}$:

$$K_\alpha = P^n_{\alpha_n}(t_n)T(t_n, t_{n-1})P^{n-1}_{\alpha_{n-1}}(t_{n-1})T(t_{n-1}, t_{n-2}) \cdots P^1_{\alpha_1}(t_1)T(t_1, t_0)[\psi_0], \tag{2.7}$$

and its adjoint is given by the expression

$$K^\dagger_\alpha = [\psi_0]T(t_0, t_1)P^1_{\alpha_1}(t_1) \cdots T(t_{n-2}, t_{n-1})P^{n-1}_{\alpha_{n-1}}(t_{n-1})T(t_{n-1}, t_n)P^n_{\alpha_n}(t_n). \tag{2.8}$$

Notice that the adjoint is formed by replacing each $\odot$ in (2.3) separating $P^j_{\alpha_j}$ from $P^{j+1}_{\alpha_{j+1}}$ by $T(t_j; t_{j+1})$. Therefore there is a one-to-one mapping between each Y_α that is an operator on the *history Hilbert space* $\breve{\mathscr{H}}$ and each K_α that is an operator on the *standard Hilbert space* $\mathscr{H}$ [32, page 120]. In the subsequent discussion chain operators could also be written in full notation $K_{\alpha_1\alpha_2\cdots\alpha_n}$ instead of simply K_α.

It is more natural to use Y_α when we refer to each quantum history, whereas it is more effective to use K_α when we calculate the quantum amplitudes associated with each quantum

history. However, *the main purpose of using both history projection operators Y_α and chain operators K_α is to stress on the fact that our arguments could be represented both in $\breve{\mathcal{H}}$ and $\mathcal{H}$, from which follows that the validity of our results does not depend on the introduction of history Hilbert space $\breve{\mathcal{H}}$ and history projection operators Y_α.*

2.3. Branch Vectors in the Standard Hilbert Space

With the use of the *chain operator* for each history K_α we can define *branch vectors (*also known as *chain kets* or *chain vectors)* in $\mathcal{H}$ [32–35, 38]:

$$|\psi_{\alpha_1\alpha_2\cdots\alpha_n}\rangle = K_\alpha|\psi_0\rangle. \tag{2.9}$$

Due to the one-to-one mapping between Y_α and K_α, to each quantum history from the *history Hilbert space* $\breve{\mathcal{H}}$ can be assigned a *branch vector* $|\psi_{\alpha_1\alpha_2\cdots\alpha_n}\rangle$ in the *standard Hilbert space* $\mathcal{H}$. Insofar the *history projection operators* Y_α in $\breve{\mathcal{H}}$ and the branch vectors $|\psi_{\alpha_1\alpha_2\cdots\alpha_n}\rangle$ in $\mathcal{H}$ represent the same underlying quantum reality; it is not surprising that one can construct one-to-one correspondence between them. Such one-to-one correspondence between Y_α and $|\psi_{\alpha_1\alpha_2\cdots\alpha_n}\rangle$ implies that *the validity of our results should not depend on the choice of QM interpretation, because branch vectors are defined in $\mathcal{H}$ and thus present in all QM interpretations*.

2.4. Feynman's Sum-over-Histories versus Decoherent Histories

In the usual *consistent histories* approach advocated by Griffiths [32] it is required that one works within a *consistent family* of quantum histories (also known as *decoherent histories*). The consistency (decoherence) condition for a set of quantum histories requires that the chain operators K_α and K_β in $\mathcal{H}$ associated with any two different quantum histories Y_α and Y_β in the sample space be mutually orthogonal in terms of the operator inner product:

$$\langle K_\alpha, K_\beta\rangle = \mathrm{Tr}\left[K_\alpha^\dagger, K_\beta\right] = 0. \tag{2.10}$$

Therefore one is limited to work only with sample spaces of quantum histories for which the consistency (decoherence) condition is fulfilled. According to Griffiths discussing a family of quantum histories which are not decoherent is *meaningless* [32, page 123].

Noteworthy, the decoherence condition is absent in Feynman's sum-over-histories approach where the quantum probability amplitude for an event is given by adding together the contributions of all the histories in configuration space leading to the event in question. The contribution of each history to the amplitude is proportional to $e^{iS/\hbar}$, where $\hbar$ is reduced Planck constant and S is the *action* of that history. The summation over all histories builds the concept of superposition, and thus the possibility of quantum interference, directly into the formulation of the theory. Because thorough elucidation of the origin of Afshar's pseudo-paradox requires discussion of quantum interferences among a family of 8 non-decoherent quantum histories that could interfere in two mutually exclusive ways, in the following exposition we will adopt Feynman's original approach. In this latter case the requirement for *mathematical consistency* should not be confused with the decoherence condition given by (2.10).

2.5. Coarse Graining

The *coarse graining* in the sum-over-histories approach consists in partitioning the fine-grained particle paths $x_i(t)$, $i = 1, \ldots, N$ into an exhaustive set of exclusive classes corresponding to regions of the configuration space $\{\Delta_{\alpha_k}(t_k)\}$, $\alpha_k = 1, 2, \ldots$ at a sequence of times t_k, $k = 1, 2, \ldots$. Each quantum history defined in the above manner is *coarse-grained* because alternatives are not specified at every time but only at some times $t_1, t_2, \ldots, t_n$ (cf. [34]).

In order to clarify the coarse-graining procedure we will provide as an example the construction of a coarse-grained position basis in one dimension. Let us first consider one-dimensional *fine-grained* wavefunction $\psi(x)$ that is acceptable in QM, namely, single valued, continuous, nowhere infinite and with piecewise continuous first derivative. Such a function can be normalized so that $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1$. Since the position x is a *continuous variable*, we should write uncountable number of position kets $|x\rangle$ in rigged Hilbert space $\mathscr{H}^\star$ [39]. For each position ket $|x\rangle$ we have

$$|\psi\rangle = \int_{-\infty}^{\infty} dx|x\rangle \ \langle x||\psi\rangle. \tag{2.11}$$

Further, it can be seen that

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1. \tag{2.12}$$

In order to normalize the position kets, we might plug in arbitrary position ket $|x'\rangle$ instead of $|\psi\rangle$

$$|x'\rangle = \int_{-\infty}^{\infty} dx|x\rangle\langle x||x'\rangle \tag{2.13}$$

which implies $\langle x\|x'\rangle = \delta(x - x')$. Thus if $x \neq x'$ it follows that $\langle x\|x'\rangle = 0$, while for $x = x'$ one obtains the δ-infinity, which is removable after integration [40, pages 149–151]. The complex number $\langle x\|\psi\rangle \equiv \psi(x)$ is the amplitude to find a particle in a state $|\psi\rangle$ at position x.

With these preliminary notes in mind, for a *fine-grained* wavefunction $\psi(x)$ confined within the interval $[-n\Delta x, n\Delta x]$:

$$\int_{-n\Delta x}^{n\Delta x} \psi^*(x)\psi(x)dx = 1 \tag{2.14}$$

we could introduce a *coarse-grained* basis $|x_{\pm i}\rangle$, $i = \pm 1, \pm 2, \ldots, \pm n$ and finite-dimensional *coarse-grained* Hilbert space $\mathscr{H}$ such that, for arbitrary position ket $|x''\rangle$ in $\mathscr{H}$,

$$\langle x'' \mid x_i\rangle = \operatorname{sgn}(i)\left[\theta\left(x'' - \left(i - \operatorname{sgn}(i)\right)\Delta x\right) - \theta\left(x'' - i\Delta x\right)\right], \tag{2.15}$$

where Δx is the size of the "grain" in the coarse graining, and the *Heaviside theta function* $\theta(x)$ is defined as

$$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{2.16}$$

Thus (2.15) is just a neat way to summarize the following cases:

$$\begin{aligned} &\text{for } i > 0 \begin{cases} \langle x'' \mid x_i \rangle = 1 & \text{if } x'' \in [(i-1)\Delta x, i\Delta x), \\ \langle x'' \mid x_i \rangle = 0 & \text{if } x'' \notin [(i-1)\Delta x, i\Delta x), \end{cases} \\ &\text{for } i < 0 \begin{cases} \langle x'' \mid x_i \rangle = 1 & \text{if } x'' \in [i\Delta x, (i+1)\Delta x), \\ \langle x'' \mid x_i \rangle = 0 & \text{if } x'' \notin [i\Delta x, (i+1)\Delta x). \end{cases} \end{aligned} \tag{2.17}$$

The quantum amplitudes α_i for each vector $|x_i\rangle$ in the *coarse-grained* Hilbert space $\mathscr{H}$ are computed from the *fine-grained* wavefunction $\psi(x)$:

$$\alpha_i = \int_{[i-\theta(i)]\Delta x}^{[i+\theta(-i)]\Delta x} \psi(x)dx. \tag{2.18}$$

The *coarse-grained* wavefunction $|\psi''\rangle$ in $\mathscr{H}$ can be expressed in the coarse-grained position basis $|x_i\rangle$ as a sum of finite number of elements:

$$\left|\psi''\right\rangle = \sum_{i=-n}^{n} \alpha_i |x_i\rangle. \tag{2.19}$$

The possible countably infinite number of coarse-grained vectors for $i < -n$ or $i > n$ are discarded from the coarse-grained description due to the fact that they contribute to the sum in $|\psi''\rangle$ with amplitudes $\alpha_i = 0$. This is our rationale for describing the subsequent interferometer setup with a finite coarse-grained position basis. Simply, we do not need to account for the huge number of coarse-grained positions filling the space outside the interferometer arms or filling the space throughout the Universe, because we assume an ideal case where the probability for the photon to be detected outside of the interferometer is *zero*, or in other words we assume a standard *particle in a box* description with an infinite potential at the walls of the box (interferometer). The above one-dimensional description can be easily generalized to 3 dimensions.

Because for the construction of the coarse-grained Hilbert space $\mathscr{H}$ one starts from a fine-grained rigged Hilbert space $\mathscr{H}^\star$, the final result is to *produce a simplified description of the same physical reality*. The construction of the coarse-grained function leads to a reduction in the number of dimensions of the Hilbert space and the coarse-grained wavefunction $\psi''(x_i)$ contains less information compared to the fine-grained wavefunction $\psi(x)$ because the mapping $\mathscr{H}^\star \to \mathscr{H}$ is many to one. Indeed distinct fine-grained wavefunctions $\psi_1(x)$, $\psi_2(x)$, $\ldots, \psi_n(x)$ may produce the same coarse-grained wavefunction if they satisfy the equation:

$$\int_{[i-\theta(i)]\Delta x}^{[i+\theta(-i)]\Delta x} \psi_1(x)dx = \int_{[i-\theta(i)]\Delta x}^{[i+\theta(-i)]\Delta x} \psi_2(x)dx = \int_{[i-\theta(i)]\Delta x}^{[i+\theta(-i)]\Delta x} \psi_n(x)dx. \tag{2.20}$$

The benefits of the coarse-grained description are that *it highlights the important features of the underlying physics and removes the inessential details* that may distract or even frustrate the reader.

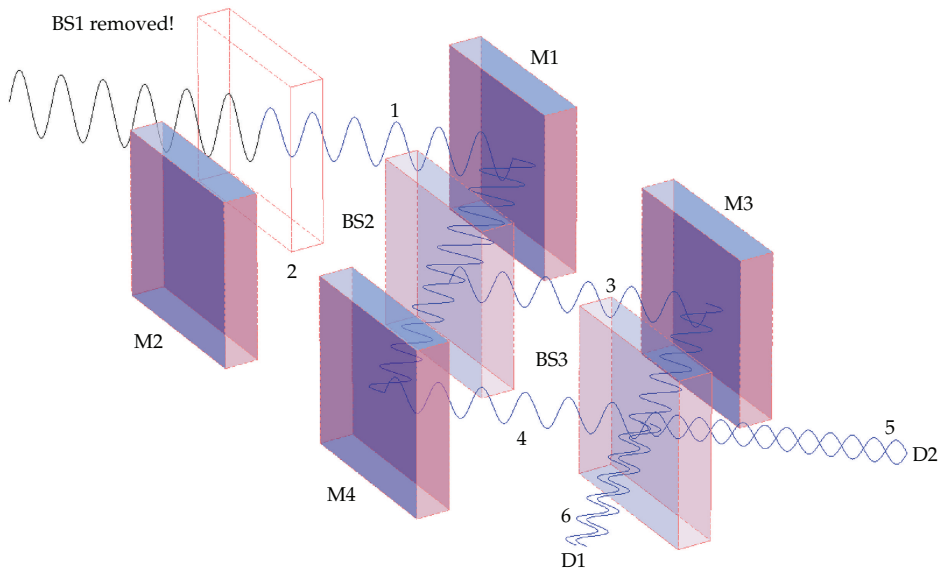

Figure 1: If the photon goes along path 1 then it always comes out at path 6. Destructively interfering branch vectors at path 5 are shown explicitly because these destructively interfering branch vectors are essential for arising of Afshar's pseudo-paradox.

3. Unruh's Setup

Unruh proposed a double Mach-Zehnder interferometric setup as a formal (and much easier to discuss) one-to-one model of Afshar's setup [13]. In Unruh's setup we arrange 3 half-silvered mirrors (beam splitters) and 4 fully silvered mirrors in such a fashion so that we obtain two Mach-Zehnder interferometers in a sequence (see Figures 1–6). Mathematically the action of the fully silvered mirrors is to reflect an incoming photon at $\pi/4$ angle with a phase delay of $\pi/2$. With the use of Euler's formula $e^{\imath(\pi/2)} = \imath$ it can be shown that in the complex plane the action of the fully silvered mirror amounts to multiplication by $\imath$ of the incoming quantum amplitude. The action of the half-silvered mirror is to coherently reflect $1/\sqrt{2}$ of the incoming photon amplitude with phase delay of $\pi/2$ and at the same time transmit without phase delay another $1/\sqrt{2}$ of the photon amplitude (such a treatment is standard, cf. [24, 25], and acknowledged to be correct by Unruh). The *coarse graining* of the setup that is proposed here is one interferometer arm and takes into account the events at each fully silvered mirror, whereas Unruh used a coarser coarse graining where he considered just interferometer segments between beam splitters and did not distinguish between states before or after the mirror reflections [13]. The photons that exit from the interferometer arms are assumed to be "detected" by detectors D_1 or D_2, which just trap the photon within a QED cavity. This assumption is introduced so that we can describe the process of photon detection with a projector onto path 5 or path 6 (see Figures 1–6 for labeling of the interferometer paths).

In his original exposition Unruh investigates two cases in which there are obstacles (absorbers) either on path 1 or path 2 of the interferometer. Unruh's intention is to produce two incoherent setups, which are taken together in coherent superposition to reproduce the situation in which both paths are free from obstacles. It can be shown however that Unruh's intention is to be correctly mathematically modeled by either removing the first beam splitter (Figure 1) or replacing the first beam splitter with fully silvered mirror (Figure 2) [26].

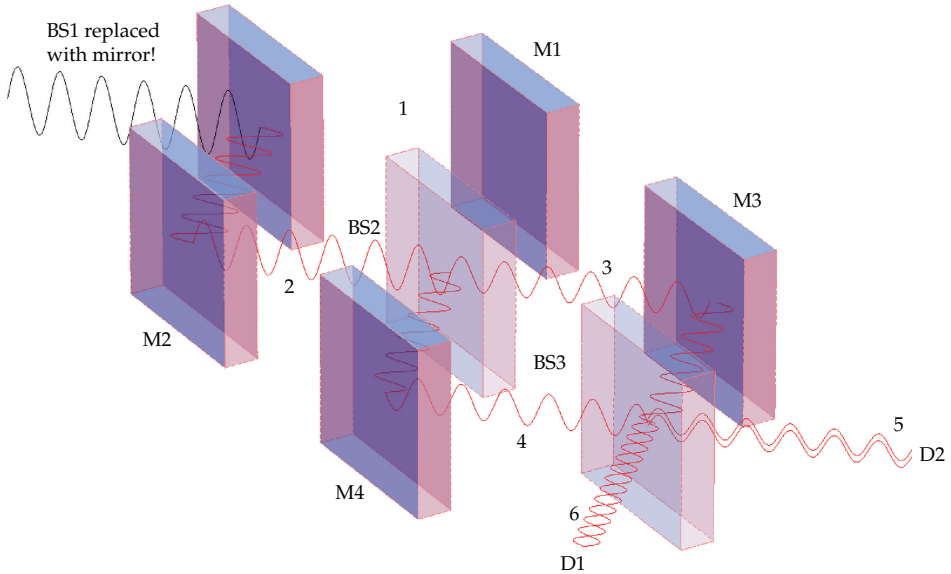

Figure 2: If the photon goes along path 2 then it always comes out at path 5. Destructively interfering branch vectors at path 6 are shown explicitly because these destructively interfering branch vectors are essential for arising of Afshar's pseudo-paradox.

The advantage of the latter proposal is that one can directly make quantum coherent superposition of the two single-path setups, while in Unruh's original proposal (putting obstacles on photon paths) one has to artificially "eliminate" from the superposition the excited electron states in the obstacles (which result from photon absorption by the obstacles).

In the first setup (Figure 1) we remove beam splitter 1 (BS1) and see that the photon exits at path 6 and is thus detected by detector 1 (D_1). This is because the beam passing via path 1 produces two branches at paths 3 and 4, which destructively interfere at path 5 and constructively interfere at path 6. We will see later that it is the existence of destructive interference at arm 5 that is important for arising of Afshar's pseudo-paradox.

In the second setup (Figure 2) we replace beam splitter 1 (BS1) with fully silvered mirror which reflects the photon and see that the photon exits at path 5 and is thus detected by detector 2 (D_2). This is because the beam passing via path 2 also produces two branches at paths 3 and 4, which destructively interfere at path 6 and constructively interfere at path 5. Again, we note that the existence of destructive interference at arm 6 is essential for arising of Afshar's pseudo-paradox.

If we now open both arms (Figure 3) there will be $2^3 = 8$ branch vectors (corresponding to 8 alternative quantum histories) which can destructively interfere in two exclusive ways. At each detector there are exactly 3 branch vectors, which are with the same phase and a single branch vector which is out of phase by π radians. The question is how we decide which branch vectors cancel out and which branch vectors do not? The answer lies within the claim for *presence* or *absence* of destructive interference at arm 4 of the interferometer. Moreover, whether we put obstacles on arm 4 is irrelevant from mathematical viewpoint as we will see in Section 3.2. One can now understand why Afshar's pseudo-paradox occurs—if the branch vectors at arm 4 are assumed to destructively interfere (see Figure 4), then the destructive interferences at path 5 and path 6 (shown in Figures 1 and 2) cannot be inferred any more.

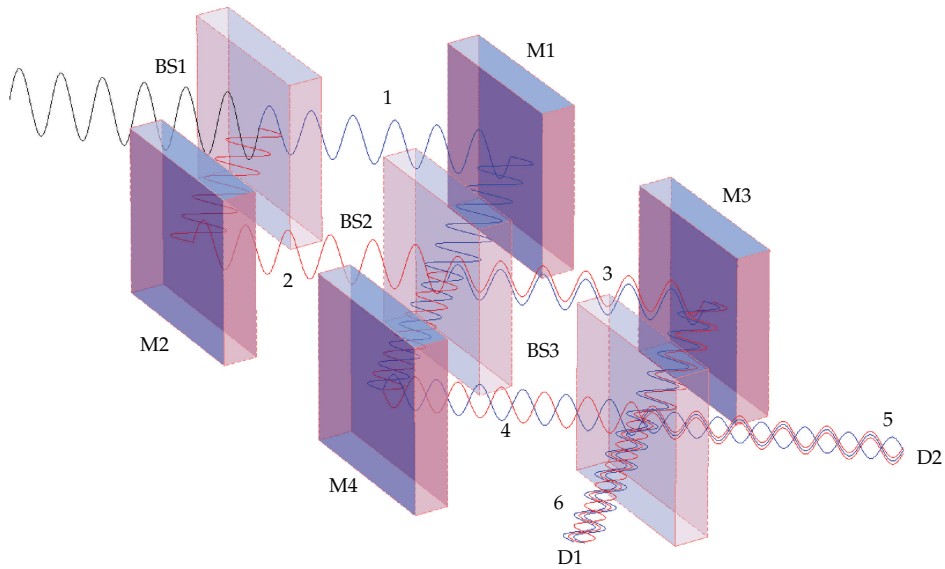

Figure 3: Coherent version of the setup with photon propagating along both paths 1 and 2. None of the interfering branch vectors is erased in order to illustrate the fact that there are 3 branch vectors constructively interfering and one branch vector destructively interfering at each of the detectors located at paths 5 and 6. The question is which two branch vectors from the quadruple set do annihilate each other and which two branch vectors remain? Branch vectors in blue come from path 1, while branch vectors in red come from path 2.

If there are no labels on the photon paths 1 and 2 (e.g., there are no different polarization filters) straightforward calculations yield destructive interference at path 4 and no which-way information (Figure 4). As explained in [24, 25] one can simply think that the recombined beam is now located at path 3 and the which-way information is erased similarly to a single Mach-Zehnder setup. Then beam splitting of this beam located at path 3 cannot produce any which-way information (Figure 4), and this will be verified by the sum-over-histories calculation in Section 4.

Notice that if there are, say, two different polarization filters on paths 1 and 2 (e.g., right circular $|R\rangle$ and left circular $|L\rangle$), due to entanglement of the photon paths 1 and 2 with the photon polarizations there will be no quantum interference at path 4 and there will be which-way information at paths 5 and 6 (Figure 5). In such a case the *standard Hilbert space* $\mathscr{H}$ describing the photon state cannot have a single dimension for passage through path 1, $|x_1\rangle$, and a single dimension for passage through path 2, $|x_2\rangle$, instead one needs a tensor product Hilbert space, with doubled number of dimensions in order to account for the photon polarizations, which are entangled with the photon paths. If the two different polarizations are right circular $|R\rangle$ and left circular $|L\rangle$, then the newly produced tensor product Hilbert space will have doubled number of dimensions: $|x_1\rangle|R\rangle$, $|x_1\rangle|L\rangle$, $|x_2\rangle|R\rangle$, $|x_2\rangle|L\rangle$,..., $|x_6\rangle|R\rangle$, $|x_6\rangle|L\rangle$. In this higher-dimensional space there is no overlap and no destructive interference at path 4. This is the only mechanism possible by which one could avoid destructive interference and preserve the which-way information, namely, to increase the dimensions of the Hilbert space (via entanglements) in such a manner that there are at least two orthogonal possibilities for passage along $|x_1\rangle$, $|x_2\rangle$, $|x_3\rangle$, $|x_4\rangle$, $|x_5\rangle$, and $|x_6\rangle$.

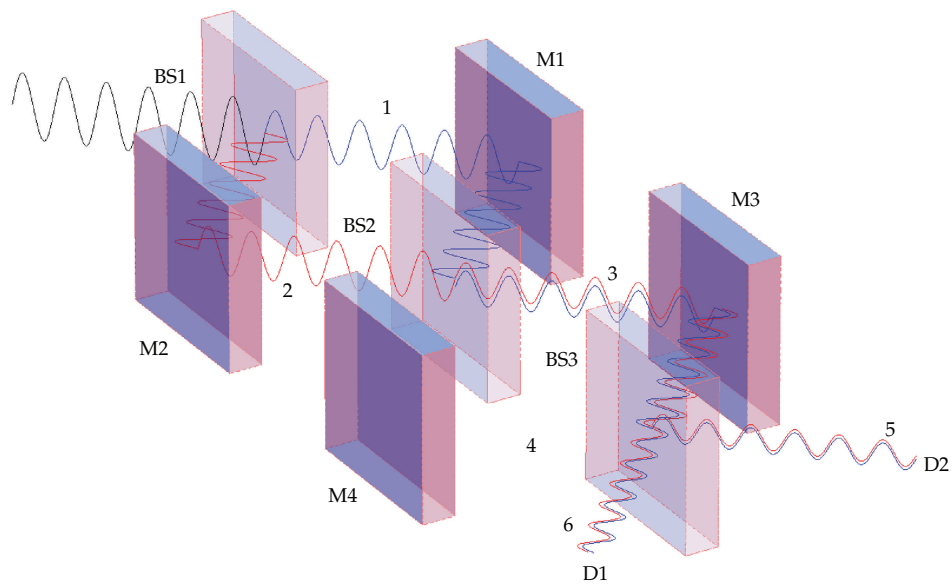

Figure 4: If one infers that there is destructive interference at path 4, then only one mathematical possibility for the output at path 5 and 6 is consistent. There is no which-way information. Branch vectors in blue come from path 1, while branch vectors in red come from path 2.

3.1. Rebuttal of Auxiliary Assumption (A1)

A central feature of both Unruh's argument [13] and Afshar's argument [8–10] is the *auxiliary assumption* (A1) formulated in Section 1, that is, both authors rely on preliminary analysis of single-slit setups with either slit 1 (path 1) open or slit 2 (path 2) open. Only after they do this preliminary analysis with either one of the two slits open, but not both, Unruh and Afshar extrapolate to setup with both slits open. However it was noted in [24, 25] that the thought experiments with only one slit open implicitly assume occurrence at different thought times. First, at time t_1 one discusses what will happen if only slit 1 is open. Then one goes on to discuss another time t_2 in which one has only slit 2 open. The central point, which we would like to stress upon, is that these thought times t_1 and t_2 effectively play the role of polarization filters positioned at the slits, or in other words one mathematically produces a tensor product Hilbert space with doubled number of dimensions $|x_1\rangle|t_1\rangle$, $|x_1\rangle|t_2\rangle$, $|x_2\rangle|t_1\rangle$, $|x_2\rangle|t_2\rangle$,..., $|x_6\rangle|t_1\rangle$, $|x_6\rangle|t_2\rangle$, where the different times t_1 and t_2 are entangled with the photon paths. As it is accepted in physics, *time* in the current exposition stands for the reading of a clock, that is, one can easily imagine an entangled clock located near the interferometric setup, which reads two different times for each single-slit case.

In both ways (either inserting different polarization filters or thinking of two different times with only one slit open at a time) one has a *mixed setup* and mixed photon density matrix. That is why there is no destructive interference at path 4 (Figure 5) and photons from path 1 destructively interfere at path 5, and end at path 6 (D_1), and photons from path 2 destructively interfere at path 6, and end at path 5 (D_2).

However, the logic that holds for mixed setups is not directly applicable to quantum coherent setups in which both paths are open and in which nothing forbids the destructive interference from happening. Mixed setups (incoherently superposed setups) are not

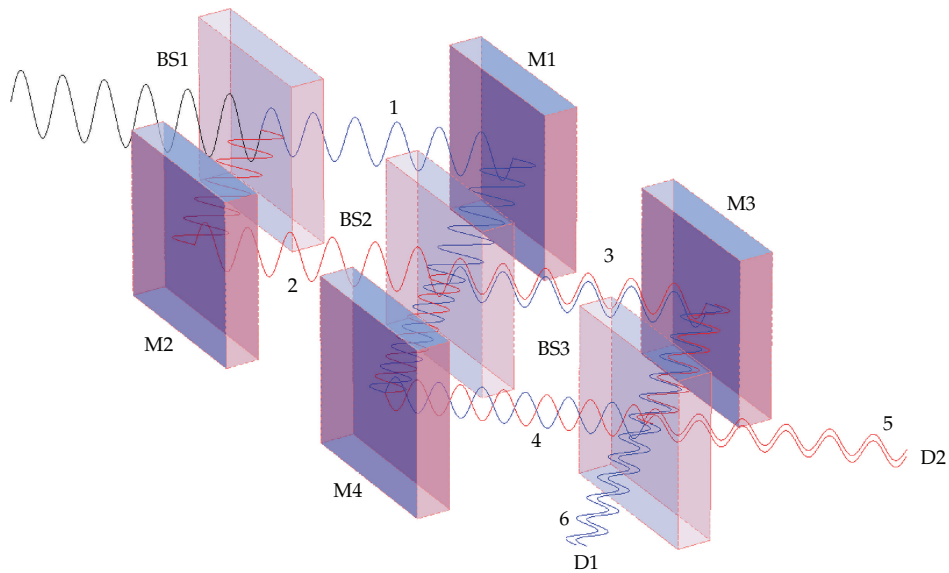

Figure 5: If one records the which-way information by entangling the photon paths with another degree of freedom there will be no destructive interference of the branch vectors at path 4 and the which-way information claims from the two single-path setups can be applied to this mixed setup also. In such a scenario the photon density matrix is not that of a pure state but mixed one. Red and blue branch vectors denote either photons with orthogonal polarizations (such as $|R\rangle$ and $|L\rangle$) or photons propagating at different times (t_1 and t_2), in either case destructive interference at arm 4 cannot occur.

equivalent to quantum coherent setups, and these two types of setups can be distinguished by the presence/absence of interference between the two paths or slits.

Surprisingly, for a coherent setup with both paths open Unruh claims that despite the fact one *knows* that there is interference at path 4, still photons coming at path 5 and 6 do have which-way information from the corresponding slits [13]. According to Unruh, only if there is obstacle on path 4 to *measure* the interference the which-way information will be lost. Thus Unruh insists that only *measurement* of the quantum interference destroys the which-way information at paths 5 and 6, while inferring (calculating) that there is interference can be peacefully done along with the claim for existent which-way information. In Unruh's anti-realistic philosophy *inferring* something is not *measuring* something (and this presumably is the essence of quantum weirdness). According to Unruh one can *know* that there *is* destructive interference along with the which-way information claim, but in this case one cannot measure the interference. The ground for the which-way information claim is the incoherent mixture of the single-path setups described in Figures 1 and 2. From existent which-way information ("which-way correlation" in Unruh's words) in the single-path setups Unruh infers (extrapolates) that this should hold in coherent setup with both paths open due to the linearity of QM. Some standard textbooks claim that due to linearity of QM the waves superimpose "without permanently destroying or disrupting either wave" [41], yet such statement seems to ignore the fact that destructive interference means exactly *permanent destroying or disrupting of some waves (branch vectors)*. The linearity of QM cannot justify the usage of the *auxiliary assumption* (A1), because incoherent quantum superpositions are not equivalent to coherent superpositions. What we proved in [24–26] based on Feynman's sum-over-histories approach is that quantum complementarity is enforced at the level of addition or subtraction of

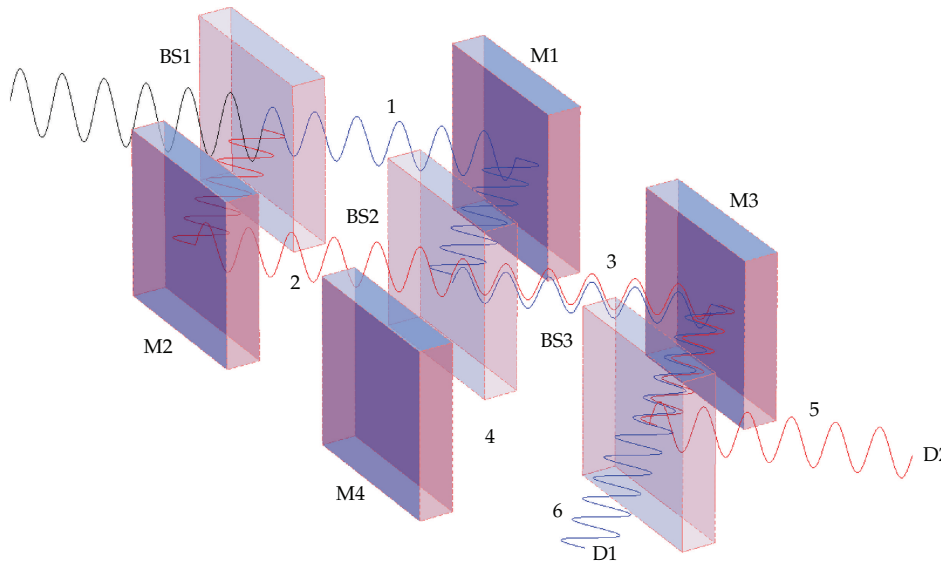

Figure 6: According to Unruh there is both which-way information in the coherent setup and one might infer existent destructive interference at path 4 if the destructive interference is not measured. Thus according to Unruh one can consistently infer the destructive interference, but not measure it. The antirealist logic goes by extrapolation from single-path setups to coherent double-path setup like this: verify that closing path 2 leads the photon to emerge always at path 6. This means that the photon path 6 is correlated with passage along path 1. Similarly one closes path 1 and verifies that photon passage along path 2 is correlated with emerging photon at path 5. Then according to Unruh the which-way information is guaranteed by the linearity of QM, and only trying to measure the interference by putting obstacle at path 4 destroys the which-way information. In other words one cannot measure both which-way information and destructive interference, yet one can infer the interference provided that it is not measured. This however implies that BS3 can distinguish the past of branch vectors coming along path 3, which is inconsistent with the postulates of standard QM.

individual branch vectors in QM and importantly that it matters, which branch vectors one adds and which branch vectors one cancels from the calculation (complete outline of the calculation will be provided in Section 4). Thus, quantum complementarity results from the requirement for mathematical consistency of branch vector addition and subtraction in cases where there are more than two exclusive ways to do the addition/subtraction of branch vectors. The *linearity of QM* cannot be used to infer conclusions from single-path experiments (incoherent superpositions) to coherent double-path experiments, especially if one has to decide existence/nonexistence of which-way information. Furthermore, our arguments do not violate the *superposition principle* because the existence of more than two exclusive ways to do the addition/subtraction of branch vectors does not change the value of the quantum amplitudes resulting from the summation. Instead, Afshar's pseudo-paradox is born exactly because the superposition principle cannot differentiate between the two exclusive ways to do the addition/subtraction of branch vectors.

Here, we defend the viewpoint that *knowledge of interference pattern derived from calculation is sufficient to erase the which-way information*. In other words, the issue of mathematical consistency is not subject to experimental measurement. If the calculation shows that there is a destructive interference, then we do not need to measure it in order to prove it. Calculated interference is inconsistent with the which-way information claim. Even if we only calculate

but do not perform measurement of the interference it will be inconsistent to claim that there is which-way information because the interference is not measured experimentally. Our opinion is thus contrasting Unruh's claim that we can calculate the interference and still consistently claim that there is which-way information because the interference is not measured experimentally [13].

If one infers the existent interference at path 4 and still claims that there is which-way information (as Unruh did in [13]) then the only way is to have beam splitter BS3, which is able to violate QM principles and separate path 1 from path 2, which at path 3 are indistinguishable [24, 25]. In other words BS3 needs to be able to "distinguish" the past of the branch vectors which at the moment of arrival at the beam splitter BS3 are "indistinguishable" in the standard Hilbert space $\mathscr{H}$ (Figure 6).

One of Unruh's objections to the above argument was that quantum states propagating along definite trajectories do not exist in any standard QM interpretation [13]. This is wrong, because quantum states propagating along a given history are well-defined mathematical constructions known as *branch vectors* and could be found in all modern QM texts (cf. [32, 34, 36]). In [26] it was explained that each *quantum history* must be represented by a sequence of events (operators) and a *branch vector* evolving in time along a given quantum history could be represented as a sequence of state vectors. In addition, Unruh claimed that which-way information is present at the detectors even though we know destructive interference occurs at path 4. Such a claim implicitly assumes at least 3 events separated in time: (1) passage through one of the interferometer arms in Unruh's setup, (2) destructive interference at an intermediate time point at path 4, and (3) detection at the detectors. Therefore if Unruh is to be consistent, he should not talk only about state vectors [42], instead the discussion should be based on the notions of *history projection operators* in history Hilbert space or *chain operators* and *branch vectors* in standard Hilbert space, as explained in Section 2.

3.2. Rebuttal of Auxiliary Assumption (A2)

Lastly, we will show that putting obstacles at path 4 could not change the which-way information of the setup [24–26]. This is necessary because all researchers who believe in (A1) are eventually forced to accept (A2) in order to cover up lurking paradoxes.

The first reason in favor of the conclusion that putting obstacles at path 4 could not change the which-way information of the setup is that mathematical theorems follow analytically from the axioms, and they are *true* if the axioms are *true*. Because the calculation of the interference pattern is a mathematical theorem, it must be true if the axioms of standard QM are true. The truth (or validity) of calculated interference pattern cannot be further proved by measurement. Indeed, if the measurement does not agree with the calculation and one believes in the truth of the axioms of standard QM, then one should conclude that either the measurement apparatus is broken or corrupt or the noise in the measurement was intolerably high.

The second reason is a corollary of Reininger's negative measurement in QM.

> *Reininger's Negative Measurement*. If one observes that a quantum particle is absent at place where the wavefunction amplitude was *nonzero*, the observation collapses the wavefunction, so that now the quantum particle is with probability 1 at all other locations, except the observed empty one.
>
> *Corollary*. It is possible to perform a nondestructive measurement without collapsing the quantum particle wavefunction only in regions where the quantum particle

wavefunction amplitude is *zero*, because both *before* and *after* the measurement the quantum particle is with probability 1 at all other locations, except the observed empty one, that is, there has been no change of the quantum state. Thus, QM permits measurements of the quantum particle, which do not collapse the quantum particle wavefunction, and these could be done where the quantum particle wavefunction amplitude is *zero.*

To measure the wavefunction where the amplitude is *zero* is, for example, somewhere outside of the interferometer behind mirrors 1–4. If we put obstacle outside of the interferometer we do not expect to change the which-way information of the photon. It is just plain obvious. The fact that the space of path 4 happens to be inside the interferometer does not make it privileged in this respect. If the wavefunction there is *zero*, putting obstacle does not collapse the wavefunction and does not change anything to the which-way information. Such a philosophical argument might be met with suspicion by antirealists, but it is necessarily (mathematically) true. In the next section we will prove rigorously that the which-way information claims follow from the mathematics of the setup and particularly depends on whether there is inferred destructive interference at path 4 or not. This effectively rebuts the *auxiliary assumption* (A2) formulated in Section 1.

4. Complementarity in Unruh's Setup

Since we have 3 beam splitters in Unruh's setup it is easy to be seen that there are exactly $2^3 = 8$ alternative quantum histories for a photon to travel from $|x_0\rangle$ to one of the two detectors $|D_1\rangle$ or $|D_2\rangle$. We assume that the detectors are located within the corresponding path 6 and path 5 and trap the photon within a QED cavity. Thus the detection itself *will not* require separate projector different from $[x_5]$ or $[x_6]$, since the coarse graining is such that $\langle x_6|D_1\rangle = 1$ and $\langle x_5|D_2\rangle = 1$. Using Unruh's original coarse graining [13] with an orthonormal position basis $\{x_k\rangle\}$, $k = 1, 2, \ldots, 6$ we can write the 8 possible quantum histories as *history projection operators* in *history Hilbert space* $\breve{\mathscr{H}}'$:

$$\begin{aligned}
Y_{135} &= [\psi_0] \odot [x_1] \odot [x_3] \odot [x_5],\\
Y_{235} &= [\psi_0] \odot [x_2] \odot [x_3] \odot [x_5],\\
Y_{145} &= [\psi_0] \odot [x_1] \odot [x_4] \odot [x_5],\\
Y_{245} &= [\psi_0] \odot [x_2] \odot [x_4] \odot [x_5],\\
Y_{136} &= [\psi_0] \odot [x_1] \odot [x_3] \odot [x_6],\\
Y_{236} &= [\psi_0] \odot [x_2] \odot [x_3] \odot [x_6],\\
Y_{146} &= [\psi_0] \odot [x_1] \odot [x_4] \odot [x_6],\\
Y_{246} &= [\psi_0] \odot [x_2] \odot [x_4] \odot [x_6].
\end{aligned} \tag{4.1}$$

If we work in a finer coarse graining, where we consider each interferometer arm, we can split each position vector $|x_1\rangle, |x_2\rangle, \ldots, |x_4\rangle$ into two halves a and b (before and after the mirror reflection, resp.), so that we obtain higher-dimensional Hilbert spaces describing the system at each time t_n, which contain basis vectors $|x_{1a}\rangle, |x_{1b}\rangle, \ldots, |x_{4a}\rangle, |x_{4b}\rangle$ defined by

3-dimensional generalization of (2.15). From the newly defined finer coarse-grained Hilbert spaces we can construct new history Hilbert space $\check{\mathscr{H}}''$, where we can discuss finer coarse-grained histories:

$$\begin{aligned}
Y_{1a1b3a3b5} &= [\psi_0] \odot [x_{1a}] \odot [x_{1b}] \odot [x_{3a}] \odot [x_{3b}] \odot [x_5],\\
Y_{2a2b3a3b5} &= [\psi_0] \odot [x_{2a}] \odot [x_{2b}] \odot [x_{3a}] \odot [x_{3b}] \odot [x_5],\\
Y_{1a1b4a4b5} &= [\psi_0] \odot [x_{1a}] \odot [x_{1b}] \odot [x_{4a}] \odot [x_{4b}] \odot [x_5],\\
Y_{2a2b4a4b5} &= [\psi_0] \odot [x_{2a}] \odot [x_{2b}] \odot [x_{4a}] \odot [x_{4b}] \odot [x_5],\\
Y_{1a1b3a3b6} &= [\psi_0] \odot [x_{1a}] \odot [x_{1b}] \odot [x_{3a}] \odot [x_{3b}] \odot [x_6],\\
Y_{2a2b3a3b6} &= [\psi_0] \odot [x_{2a}] \odot [x_{2b}] \odot [x_{3a}] \odot [x_{3b}] \odot [x_6],\\
Y_{1a1b4a4b6} &= [\psi_0] \odot [x_{1a}] \odot [x_{1b}] \odot [x_{4a}] \odot [x_{4b}] \odot [x_6],\\
Y_{2a2b4a4b6} &= [\psi_0] \odot [x_{2a}] \odot [x_{2b}] \odot [x_{4a}] \odot [x_{4b}] \odot [x_6].
\end{aligned} \tag{4.2}$$

Mathematically the different coarse grainings require introduction of different standard Hilbert spaces to describe the system at each time t_n and require different history Hilbert spaces. However since the different coarse grainings refer to the same underlying quantum reality and there is a one-to-one mapping between histories with different coarse grainings (e.g., $Y_{135} \rightarrow Y_{1a1b3a3b5}$), hereafter we will refer to such corresponding histories with the same name (e.g., we will call both Y_{135} and $Y_{1a1b3a3b5}$ simply "Y_{135}"). *This will not cause confusion because the use of different coarse-grainings does not lead to differences in the calculated quantum amplitudes for the different quantum histories*.

In order to easily calculate the quantum amplitudes contributed by each of the 8 quantum histories, we can use the chain operators K_α given by (2.7) instead of the expressions for Y_α. If we consider times $t_0, t_1, \ldots, t_5$ for which the photon travels: (t_0) from the laser apparatus to BS1, (t_1) from BS1 to M1 or M2, (t_2) from M1 or M2 to BS2, (t_3) from BS2 to M3 or M4, (t_4) from M3 or M4 to BS3, and (t_5) from BS3 to D_1 or D_2, we can write *time development operators* constructed from mirror reflection or beam splitting as follows:

$$\begin{aligned}
T(t_1, t_0) &= \frac{1}{\sqrt{2}}(|x_{1a}\rangle\langle x_0| + \imath|x_{2a}\rangle\langle x_0|),\\
T(t_2, t_1) &= \imath(|x_{2b}\rangle\langle x_{2a}| + |x_{1b}\rangle\langle x_{1a}|),\\
T(t_3, t_2) &= \frac{1}{\sqrt{2}}(|x_{3a}\rangle\langle x_{2b}| + \imath|x_{4a}\rangle\langle x_{2b}| + |x_{4a}\rangle\langle x_{1b}| + \imath|x_{3a}\rangle\langle x_{1b}|),\\
T(t_4, t_3) &= \imath(|x_{4b}\rangle\langle x_{4a}| + |x_{3b}\rangle\langle x_{3a}|),\\
T(t_5, t_4) &= \frac{1}{\sqrt{2}}(|x_5\rangle\langle x_{4b}| + \imath|x_6\rangle\langle x_{4b}| + |x_6\rangle\langle x_{3b}| + \imath|x_5\rangle\langle x_{3b}|).
\end{aligned} \tag{4.3}$$

After applying the rule given by (2.7) and after simplification we obtain the following chain operators in $\mathscr{H}$:

$$K_{1a1b3a3b5} = \imath\frac{1}{\sqrt{2}}|x_5\rangle\langle x_{3b}|\imath|x_{3b}\rangle\langle x_{3a}|\imath\frac{1}{\sqrt{2}}|x_{3a}\rangle\langle x_{1b}|\imath|x_{1b}\rangle\langle x_{1a}|\frac{1}{\sqrt{2}}|x_{1a}\rangle\langle x_0|, \tag{4.4}$$

$$K_{2a2b3a3b5} = \imath\frac{1}{\sqrt{2}}|x_5\rangle\langle x_{3b}|\imath|x_{3b}\rangle\langle x_{3a}|\frac{1}{\sqrt{2}}|x_{3a}\rangle\langle x_{2b}|\imath|x_{2b}\rangle\langle x_{2a}|\imath\frac{1}{\sqrt{2}}|x_{2a}\rangle\langle x_0|, \tag{4.5}$$

$$K_{1a1b4a4b5} = \frac{1}{\sqrt{2}}|x_5\rangle\langle x_{4b}|\imath|x_{4b}\rangle\langle x_{4a}|\frac{1}{\sqrt{2}}|x_{4a}\rangle\langle x_{1b}|\imath|x_{1b}\rangle\langle x_{1a}|\frac{1}{\sqrt{2}}|x_{1a}\rangle\langle x_0|, \tag{4.6}$$

$$K_{2a2b4a4b5} = \frac{1}{\sqrt{2}}|x_5\rangle\langle x_{4b}|\imath|x_{4b}\rangle\langle x_{4a}|\imath\frac{1}{\sqrt{2}}|x_{4a}\rangle\langle x_{2b}|\imath|x_{2b}\rangle\langle x_{2a}|\imath\frac{1}{\sqrt{2}}|x_{2a}\rangle\langle x_0|, \tag{4.7}$$

$$K_{1a1b3a3b6} = \frac{1}{\sqrt{2}}|x_6\rangle\langle x_{3b}|\imath|x_{3b}\rangle\langle x_{3a}|\imath\frac{1}{\sqrt{2}}|x_{3a}\rangle\langle x_{1b}|\imath|x_{1b}\rangle\langle x_{1a}|\frac{1}{\sqrt{2}}|x_{1a}\rangle\langle x_0|, \tag{4.8}$$

$$K_{2a2b3a3b6} = \frac{1}{\sqrt{2}}|x_6\rangle\langle x_{3b}|\imath|x_{3b}\rangle\langle x_{3a}|\frac{1}{\sqrt{2}}|x_{3a}\rangle\langle x_{2b}|\imath|x_{2b}\rangle\langle x_{2a}|\imath\frac{1}{\sqrt{2}}|x_{2a}\rangle\langle x_0|, \tag{4.9}$$

$$K_{1a1b4a4b6} = \imath\frac{1}{\sqrt{2}}|x_6\rangle\langle x_{4b}|\imath|x_{4b}\rangle\langle x_{4a}|\frac{1}{\sqrt{2}}|x_{4a}\rangle\langle x_{1b}|\imath|x_{1b}\rangle\langle x_{1a}|\frac{1}{\sqrt{2}}|x_{1a}\rangle\langle x_0|, \tag{4.10}$$

$$K_{2a2b4a4b6} = \imath\frac{1}{\sqrt{2}}|x_6\rangle\langle x_{4b}|\imath|x_{4b}\rangle\langle x_{4a}|\imath\frac{1}{\sqrt{2}}|x_{4a}\rangle\langle x_{2b}|\imath|x_{2b}\rangle\langle x_{2a}|\imath\frac{1}{\sqrt{2}}|x_{2a}\rangle\langle x_0|. \tag{4.11}$$

In (4.4)–(4.11) we use a *coarse graining* such that each path is divided into a and b halves, respectively before and after the corresponding mirror, for example, path 1 is considered as path $1a$ before the mirror M1 and as path $1b$ after the mirror M1. It is easy to verify that the above chain operators indeed propagate an input $|x_0\rangle$ along the specified histories. One has to read the operators from right to left and keep in mind that the operator $O = |x_j\rangle\langle x_i|$ transforms input $|x_i\rangle$ into output $|x_j\rangle$, that is, $O|x_i\rangle = |x_j\rangle\langle x_i||x_i\rangle = |x_j\rangle$ since $\langle x_i|x_i\rangle = 1$. However it might be necessary to prove that these chain operators are indeed equivalent to the chain operators constructed by (2.7) for each history given by (2.3).

Full derivation will be provided only for the first chain operator given by (4.4):

$$K_{1a1b3a3b5} = [x_5]T(t_5,t_4)[x_{3b}]T(t_4,t_3)[x_{3a}]T(t_3,t_2)[x_{1b}]T(t_2,t_1)[x_{1a}]T(t_1,t_0)[x_0], \tag{4.12}$$

$$\begin{aligned} K_{1a1b3a3b5} &= |x_5\rangle\langle x_5|T(t_5,t_4)|x_{3b}\rangle\langle x_{3b}|T(t_4,t_3)|x_{3a}\rangle\langle x_{3a}| \\ &\quad\cdot T(t_3,t_2)|x_{1b}\rangle\langle x_{1b}|T(t_2,t_1)|x_{1a}\rangle\langle x_{1a}|T(t_1,t_0)|x_0\rangle\langle x_0|. \end{aligned} \tag{4.13}$$

Substitution of the time development operators given by (4.3) for each "sandwiched" part within the long equation (4.13) leads to

$$\begin{aligned} \langle x_5|T(t_5,t_4)|x_{3b}\rangle &= \langle x_5|\frac{1}{\sqrt{2}}(|x_5\rangle\langle x_{4b}| + \imath|x_6\rangle\langle x_{4b}| + |x_6\rangle\langle x_{3b}| + \imath|x_5\rangle\langle x_{3b}|)|x_{3b}\rangle \\ &= \frac{1}{\sqrt{2}}\imath\langle x_{3b}||x_{3b}\rangle, \end{aligned}$$

$$\begin{aligned}
\langle x_{3b}|T(t_4,t_3)|x_{3a}\rangle &= \langle x_{3b}|\imath(|x_{4b}\rangle\langle x_{4a}| + |x_{3b}\rangle\langle x_{3a}|)|x_{3a}\rangle = \imath\langle x_{3a}||x_{3a}\rangle,\\
\langle x_{3a}|T(t_3,t_2)|x_{1b}\rangle &= \langle x_{3a}|\frac{1}{\sqrt{2}}(|x_{3a}\rangle\langle x_{2b}| + \imath|x_{4a}\rangle\langle x_{2b}| + |x_{4a}\rangle\langle x_{1b}| + \imath|x_{3a}\rangle\langle x_{1b}|)|x_{1b}\rangle\\
&= \frac{1}{\sqrt{2}}\imath\langle x_{1b}||x_{1b}\rangle,\\
\langle x_{1b}|T(t_2,t_1)|x_{1a}\rangle &= \langle x_{1b}|\imath(|x_{2b}\rangle\langle x_{2a}| + |x_{1b}\rangle\langle x_{1a}|)|x_{1a}\rangle = \imath\langle x_{1a}||x_{1a}\rangle,\\
\langle x_{1a}|T(t_1,t_0)|x_0\rangle &= \langle x_{1a}|\frac{1}{\sqrt{2}}(|x_{1a}\rangle\langle x_0| + \imath|x_{2a}\rangle\langle x_0|)|x_0\rangle = \frac{1}{\sqrt{2}}.
\end{aligned} \tag{4.14}$$

Further substitution of (4.14) into (4.13) results into (4.4), which we wanted to prove. The derivation of the remaining chain operators given by (4.5)–(4.11) is analogous.

Since we are investigating setup in which we initially plug in the pure state $|\psi_0\rangle = |x_0\rangle$, the quantum amplitudes of the 8 quantum histories are easy to calculate using (4.4)–(4.11): $Y_{135}, Y_{235}, Y_{245} \to 1/\sqrt{8}$, $Y_{145} \to -1/\sqrt{8}$, $Y_{136}, Y_{236}, Y_{146} \to -\imath(1/\sqrt{8})$, $Y_{246} \to \imath(1/\sqrt{8})$.

Here we remind that instead of using a single Greek subscript $\alpha, \beta, \gamma, \ldots$ for each *history projection operator* (quantum history) we use chain of subscripts (e.g., Y_{246} describes the history of a photon passing via path 2, then path 4 and emerging at path 6). To the history Y_{246} can be ascribed *branch vector* ψ_{246}.

One can multiply the operators within each chain operator K_α and obtain just a single operator that transforms the input state $|x_0\rangle$ directly into output state at $|D_1\rangle \equiv |x_6\rangle$ or $|D_2\rangle \equiv |x_5\rangle$. The obtained operator however describes photon "jump" from $|x_0\rangle$ to $|D_1\rangle$ or $|D_2\rangle$ and therefore cannot capture the concept of photon path (history). The latter operator could be useful only if we are interested to get the final output of the operator without being interested in intermediate events such as passage along the interferometer arms.

In the single-path experiment with beam splitter BS1 removed (Figure 1) the branch vectors travel along path 1 and there is destructive interference of branch vectors ψ_{135} and ψ_{145} at D_2 and constructive interference of branch vectors ψ_{136} and ψ_{146} at D_1:

$$\psi_{135} + \psi_{145} = 0, \qquad \left|\psi_{136} + \psi_{146}\right| = 1. \tag{4.15}$$

Similarly in the single-path experiment with beam splitter BS1 replaced by fully silvered mirror (Figure 2) the branch vectors travel along path 2 and there is destructive interference of branch vectors ψ_{236} and ψ_{246} at D_1 and constructive interference of branch vectors ψ_{235} and ψ_{245} at D_2:

$$\left|\psi_{235} + \psi_{245}\right| = 1, \qquad \psi_{236} + \psi_{246} = 0. \tag{4.16}$$

In the quantum coherent version of the setup without having polarization filters on paths 1 and 2 (Figure 4), the destructive interference at path 4 implies

$$\psi_{145} + \psi_{245} = 0, \qquad \psi_{146} + \psi_{246} = 0, \qquad \left|\psi_{135} + \psi_{235}\right| = \frac{1}{\sqrt{2}}, \qquad \left|\psi_{136} + \psi_{236}\right| = \frac{1}{\sqrt{2}}\,. \tag{4.17}$$

Because of destructive interference at path 4, the constructive interference at both detectors D_1 and D_2 is due to branch vectors coming from both slits. The calculated interference (even without measuring it by wire grid) erases the which-way information exactly as the complementarity principle requires.

If we put right or left circular polarization filters ($|R\rangle$ or $|L\rangle$) on path 1 and path 2 labeling the which-way information (Figure 5), then the equations for each single slit setup hold:

$$\psi_{135} + \psi_{145} = 0, \qquad \psi_{236} + \psi_{246} = 0, \qquad \left|\psi_{235} + \psi_{245}\right| = \frac{1}{\sqrt{2}}, \qquad \left|\psi_{136} + \psi_{146}\right| = \frac{1}{\sqrt{2}}\,. \tag{4.18}$$

Due to the which-way information labels however, there is no interference at path 4 (i.e., there is nonzero probability to detect photon at path 4). What this analysis shows is that it is impossible both to have satisfied destructive quantum interference at path 4 and to have which-way information, which requires destructive self-interferences of branch vectors coming from path 1 at the opposite detector D_2 (path 5) and from path 2 at the opposite detector D_1 (path 6). For example, the branch vector ψ_{145} cannot annihilate both vectors ψ_{135} and ψ_{245} at once and this follows from the fact that

$$\psi_{145} + \psi_{135} + \psi_{245} \neq 0. \tag{4.19}$$

This exhausts the proof of the statement that there is no which-way information if one infers (calculates) quantum interference at path 4 and decides the Georgiev-Unruh debate [13, 24–26]. *Knowing* (mathematically proving) that there is destructive interference destroys the which-way information even without measuring the destructive interference. Mathematical knowledge (theorems) follows via rigorous deduction from the QM postulates (axioms). Thus Unruh ignored the fact that knowledge can be not only *synthetic* (i.e., experimental), but can be also *analytic* (true by virtue of mathematical proof and therefore not subject to experimental testing once one accepts the standard QM postulates).

5. Afshar's Setup

In Afshar's setup, light generated by a laser passes through two closely spaced circular pinholes. After the dual pinholes, a lens refocuses the light so that *each image of a pinhole* is received by a separate photodetector. A wire grid could be placed at the dark fringes before the lens (or at the focal plane of the lens after the lens), where the photon wavefunction is *zero* due to destructive interference. The corollary from the Reininger's negative measurement however states that placing the wire grid at a place where the photon wavefunction is *zero* cannot alter the photon wavefunction and cannot change the results from the setup. Therefore, we will consider only cases without wire grid on the photon path.

Considering a mixture of single-pinhole setups Afshar argues that a photon that goes through *pinhole 1* impinges only on detector D_1 and, similarly, if it goes through *pinhole 2* impinges only on detector D_2 (Figures 7–9). Exactly as in Unruh's setup, Afshar investigates a statistical mixture of single-pinhole setups and after that draws *nonsequitur* conclusions for the quantum coherent setup with both pinholes open. According to Afshar, there is an one-to-one correspondence between pinholes and the corresponding images even when the light coherently passes through both pinholes (notice the complete analogy with Unruh's setup).

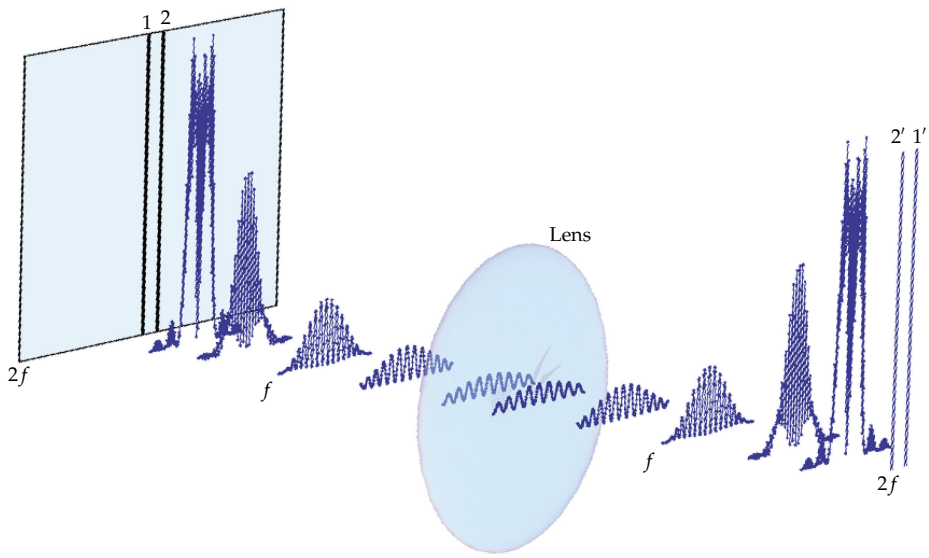

Figure 7: Action of a lens in a dual-slit setup—slits 1 and 2 create a two-slit image 1′ and 2′, denoted are the focal plane of the lens $f = 2.25$ m and the image plane of the lens $2f$. The wire grid that could be used to verify the existence of an interference pattern in the coherent setup when both slits are open is *not* shown because this is analytically true statement (provable as theorem) and we do not need to verify the presence of interference experimentally in order to know that the theorem is true. If we accept the standard QM postulates, then we must accept the interference as true statement as well, even though we have not measured it. Plotted are the probability density functions at the level of the optical axis ($y = 0$) for photon with $\lambda = 650$ nm passing through square slits with side of 250 μm separated by distance of 2 mm.

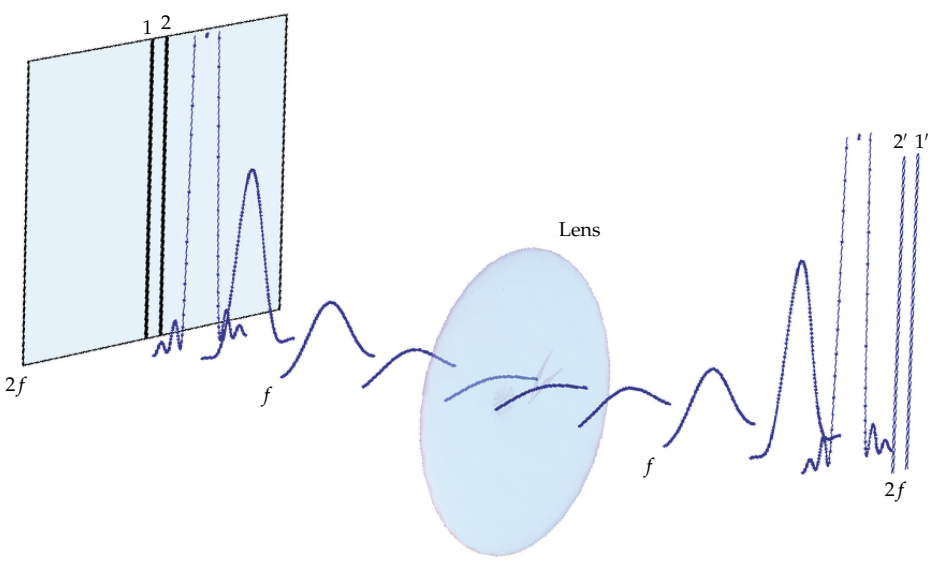

Figure 8: Action of a lens in a dual-slit setup with different polarization filters (R and L) at the slits—slits 1 and 2 create a two-slit image 1′ and 2′, denoted are the focal plane of the lens $f = 2.25$ m, the image plane of the lens $2f$. Plotted are the probability density functions at the level of the optical axis ($y = 0$) for photon with $\lambda = 650$ nm passing through square slits with side of 250 μm separated by distance of 2 mm. There is perfect which-way information due to entanglement of photon polarization with photon path, and there is no interference pattern at the focal plane of the lens.

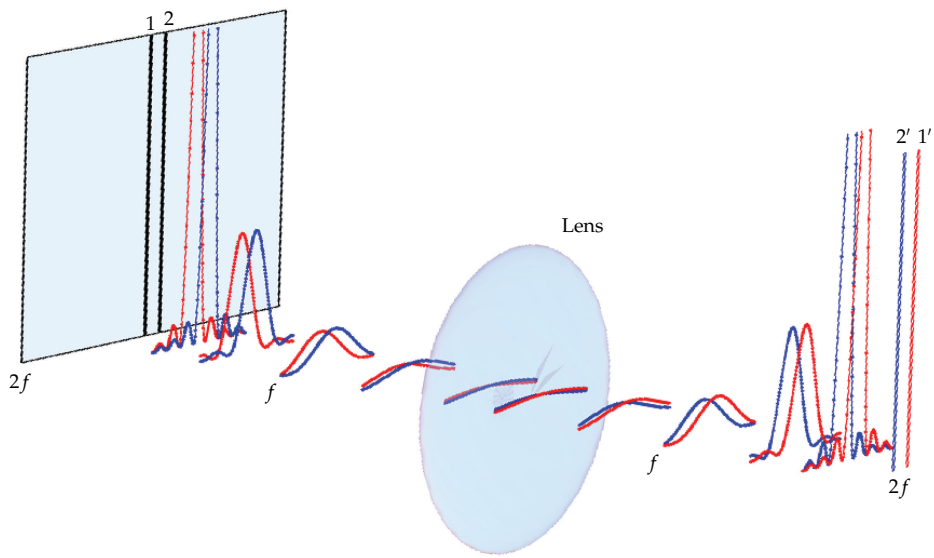

Figure 9: Action of a lens in a dual-slit setup with different polarization filters (R and L) at the slits—slits 1 and 2 create a two-slit image 1′ and 2′, denoted are the focal plane of the lens $f = 2.25$ m, the image plane of the lens $2f$. Plotted are the probability density functions for each slit separately (R in red and L in blue) at the level of the optical axis ($y = 0$) for photon with $\lambda = 650$ nm passing through square slits with side of 250 μm separated by distance of 2 mm. There is perfect which-way information due to entanglement of photon polarization with photon path, and there is no interference pattern at the focal plane of the lens. The two different probability density functions (R in red and L in blue) can be separated my simultaneous measurement of the photon polarization, and this is in sharp contrast with the coherent setup, where we do not have means to distinguish the photon history after the interference has occurred.

While in classical optics this is a straightforward conclusion, in quantum coherent setups we will shortly prove that each image of a pinhole in the coherent dual pinhole setup is counterintuitively assembled by light coming from both pinholes at once.

First, we will explain generally why Afshar's setup is equivalent to Unruh's setup. In Afshar's setup we have two pinholes, a lens, and detectors that record photons streaming away from the pinhole images created at the image plane of the lens. If one opens only *pinhole 1* the light will go only to *image 1*, and if one opens only *pinhole 2* the light will go only to *image 2*. One may, analogously to Unruh's setup, inconsistently postulate which-way information in the quantum coherent setup. However, one should note that, in the single pinhole experiments, at the image plane of the lens the *zero* light intensity outside the central Airy disc of the pinhole image is a result of *destructive quantum interference*. Thus in the case of open *pinhole 1* the branch vectors from *pinhole 1* will constructively interfere at *image 1*, whereas there will be destructively interfering branch vectors contributed by *pinhole 1* at *image 2* because *image 2* resides outside the central Airy maximum of *image 1*. The absolute value for each of these destructively interfering branch vectors contributed by *pinhole 1* at *image 2* will be further calculated to be exactly $1/\sqrt{8}$ when both pinholes are open or $1/2$ if only *pinhole 1* is open (the difference is due to different normalization procedures for the single- and double-pinhole setups). The situation with only *pinhole 1* open is therefore analogous to the case in which photons travel only along path 1 in Unruh's setup (Figure 1). Similar argument shows that the situation with only *pinhole 2* open is analogous to the case in which photons travel only along path 2 in Unruh's setup (Figure 2). However, if both pinholes are open and some of the branch vectors coming from *pinhole 2* cross-interfere with branch vectors coming from

pinhole 1 at the dark fringes at the focal plane of the lens, there will remain a contribution by *pinhole 2* at *image 1* that will compensate exactly the decrease in the quantum probability amplitude contributed by *pinhole 1* at *image 1* caused by the destructive interference at the dark fringes. Thus arises the very same problem as in Unruh's setup: how do we decide which branch vectors will annihilate, and which will remain to contribute to the detected image intensity? If one postulates the existent interference at the focal plane of the lens then the annihilation between ψ_1 and ψ_2 at the dark fringes (where ψ_1 denotes branch vectors passing through pinhole 1 and ψ_2 denotes branch vectors passing through pinhole 2) will be equivalent to the interference at path 4 of Unruh's setup, and the final observed intensities at the detectors cannot be claimed to come only from one of the pinholes. Instead the branch vectors constructively interfering at the bright fringes come from both pinholes 1 and 2 and go to both images 1 and 2. In the case with both pinholes open there is no which-way information in Afshar experiment. Counterintuitively *each image of a pinhole* is assembled from light coming by half from both pinholes (passing through the bright fringes).

Afshar argued that there is a major difference between Unruh's setup and Afshar's setup and compared *"Unruh's experiment to one in which the wings are removed from an airplane. The experimenter might find that a wingless airplane cannot fly, but such an experiment would not prove anything about the flight capabilities of an intact plane."* In the following exposition however we will prove mathematically that Afshar's thesis is wrong and that Unruh's and Afshar's setups are indeed equivalent. An exact calculation for Afshar's setup is adduced by Qureshi for spin 1/2 particle [22]. Qureshi shows that the quantum state at the overlap region where the dark interference fringes occur can be written as

$$\begin{aligned}\psi(x,t) = {} & aC(t)e^{-(x^2+x_0^2)/\Omega(t)}\left[\cosh\left(\frac{2xx_0}{\Omega(t)}\right)+\sinh\left(\frac{2xx_0}{\Omega(t)}\right)\right] \\ & + bC(t)e^{-(x^2+x_0^2)/\Omega(t)}\left[\cosh\left(\frac{2xx_0}{\Omega(t)}\right)-\sinh\left(\frac{2xx_0}{\Omega(t)}\right)\right],\end{aligned} \tag{5.1}$$

where m is the mass of the particle, $C(t) = 1/((\pi/2)^{1/4}\sqrt{\epsilon + 2\imath\hbar t/m\epsilon})$, $\Omega(t) = \epsilon^2 + 2\imath\hbar t/m$, a is the amplitude contribution from *pinhole 1*, b is the amplitude contribution from pinhole 2, ϵ is the width of the wave packets, and $2x_0$ is the slit separation.

For Afshar's setup $a = b = 1/\sqrt{2}$ so the sinh terms cancel out at the dark fringes and what is left at the bright fringes is

$$\psi(x,t) = \frac{1}{2}aC(t)\left[e^{-(x-x_0)^2/\Omega(t)} + e^{-(x+x_0)^2/\Omega(t)}\right] + \frac{1}{2}bC(t)\left[e^{-(x-x_0)^2/\Omega(t)} + e^{-(x+x_0)^2/\Omega(t)}\right]. \tag{5.2}$$

If a lens is used, after the interference has occurred, to direct the $e^{-(x-x_0)^2/\Omega(t)}$ part into detector D_1 and the part $e^{-(x+x_0)^2/\Omega(t)}$ into detector D_2, one easily sees that the *branch vectors* from each slit evolve into a superposition of two parts that go to both detectors.

The branch vectors that will be responsible for which-way information in *mixed setups* and make possible the one-to-one correspondence $a \to D_1$, $b \to D_2$ are hidden in the erased sinh terms. If we express the hyperbolic functions with the help of exponential ones, namely, $\cosh x = (1/2)(e^x + e^{-x})$ and $\sinh x = (1/2)(e^x - e^{-x})$, after simple algebraic work one may recover the four zeroed sinh components at the dark fringes in the form

$$0 = \frac{1}{2}aC(t)\left[e^{-(x-x_0)^2/\Omega(t)} - e^{-(x+x_0)^2/\Omega(t)}\right] + \frac{1}{2}bC(t)\left[-e^{-(x-x_0)^2/\Omega(t)} + e^{-(x+x_0)^2/\Omega(t)}\right]. \tag{5.3}$$

Note that the absolute value of the coefficients in front of each *branch vector* in (5.2) and (5.3) (going from one of the slits to bright or dark fringe and then to one of the detectors) becomes exactly $1/\sqrt{8}$ as obtained in the analysis of Unruh's setup [24–26].

If the eight interfering branch vectors from Qureshi's calculation are denoted as Q_i, where Q_{1-4} arise from the cosh terms and Q_{5-8} arise from the sinh terms, then the *one-to-one mapping* with the eight branch vectors defined in Section 4 is

$$\begin{aligned}
&\cosh\begin{cases} Q_1 = \dfrac{1}{2}aC(t)e^{-(x-x_0)^2/\Omega(t)} \longrightarrow \psi_{136}, \\ Q_2 = \dfrac{1}{2}aC(t)e^{-(x+x_0)^2/\Omega(t)} \longrightarrow \psi_{135}, \\ Q_3 = \dfrac{1}{2}bC(t)e^{-(x-x_0)^2/\Omega(t)} \longrightarrow \psi_{236}, \\ Q_4 = \dfrac{1}{2}bC(t)e^{-(x+x_0)^2/\Omega(t)} \longrightarrow \psi_{235}, \end{cases} \\
&\sinh\begin{cases} Q_5 = \dfrac{1}{2}aC(t)e^{-(x-x_0)^2/\Omega(t)} \longrightarrow \psi_{146}, \\ Q_6 = -\dfrac{1}{2}aC(t)e^{-(x+x_0)^2/\Omega(t)} \longrightarrow \psi_{145}, \\ Q_7 = -\dfrac{1}{2}bC(t)e^{-(x-x_0)^2/\Omega(t)} \longrightarrow \psi_{246}, \\ Q_8 = \dfrac{1}{2}bC(t)e^{-(x+x_0)^2/\Omega(t)} \longrightarrow \psi_{245}. \end{cases}
\end{aligned} \tag{5.4}$$

Here each Q_i is given with the output state only; however it should be understood as a *branch vector* propagating along certain *quantum history* and hence is mathematically defined by a sequence of states. This is implied in the requirement for a or b and sinh or cosh *origin* of each Q_i. Purely algebraic reasoning (e.g., $a = b$, etc.) treats each Q_i *only* as a *complex number* [42] and fails to grasp the quantum history concept defined in Section 2. Treating each Q_i as a *branch vector* associated with a certain quantum history explicitly verifies that a and b terms in Qureshi's calculation have the same meaning as path 1 and path 2 in Unruh's setup; cosh and sinh terms have the meaning of path 3 and path 4, and $e^{-(x-x_0)^2/\Omega(t)}$ and $e^{-(x+x_0)^2/\Omega(t)}$ terms have the meaning of detection at D_1 or D_2.

6. Fresnel Diffraction Integrals for Coherent and Incoherent Afshar's Setups

In the previous sections we have presented a general description of Afshar's setup and have shown its equivalence with Unruh's setup. Furthermore, we have resolved the pseudo-paradox using a thorough mathematical description using Feynman's sum over histories. Here, we will provide exact description of Afshar's setup using Fresnel diffraction integrals and will calculate the resulting probability density functions using computer calculations with *Wolfram's Mathematica 7*. The rationale is to address some of the misconceptions propagated in the work of Drezet, who defended the complementarity principle using a trivially true fact that one cannot detect the photon twice [20]. In Drezet's opinion if one detects a photon at the image plane, one cannot infer for the same photon what the probability density function was at the focal plane of the lens. Further, Drezet gives examples of various wavefunctions at the focal plane, which do have interference minima and argues that one needs the full

information at the focal plane in order to discriminate between the various alternatives. Such a naive argument is nevertheless erroneous. Since one has the axioms of standard QM as given knowledge, the issue of Unruh's and Afshar's arguments is not to experimentally prove/disprove the axioms of standard QM. Instead *the debate is focused on the question whether the principle of complementarity fails provided that the standard axioms of QM are true*.

If one accepts the standard QM axioms as *true* then Drezet argument appears to be a straw man argument. One can calculate the photon probability density functions both at the focal plane and at the image plane using Fresnel diffraction integrals. Moreover, one can argue that some of the wavefunctions provided by Drezet for the focal plane of the lens will not evolve into two resolved peaks at the image plane (they are not solutions of the Fresnel diffraction integrals). Thus they are ruled out *a priori* due to theoretical considerations (i.e., the geometry of the lens setup), and no measurement is needed. Instead, one should be concerned with the consistency between the calculated probability density functions and the inferred which-way information claims. The fact that the calculated probability density functions will be exactly the ones that could be measured follows directly from the premise that QM axioms are *true*. Therefore further discussion on the possible discrepancy between calculated and measured probability density functions is relevant only if one wants to disprove standard QM as adequate description of the world.

After clarifying that all philosophical issues on the complementarity debate presume that standard QM axioms are true, we will go on and calculate the photon probability density functions for coherent or incoherent double-slit setups. Using the results from the computer calculations we will illustrate the which-way information claims and link them with the quantum histories discussed in Sections 4 and 5.

The photon quantum probability amplitude behind a slit (aperture) could be calculated with the use of Huygens-Fresnel principle:

$$\psi(x,y) = \frac{z}{\imath\lambda}\iint_{-\infty}^{\infty} \psi(\xi,\eta)\frac{\exp[\imath k r]}{r^2}\,d\xi\,d\eta, \tag{6.1}$$

where the distance $r = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}$, the wave number $k = 2\pi/\lambda$, ξ, η are the coordinates at the plane of the slit, $\psi(\xi,\eta)$ is the aperture function (describing the transmission through the slit), x, y are the coordinates at the detection plane, and z is the distance behind the slit. The previous expression is quite general and is based on the approximation inherent in the scalar theory and the assumption $r \gg \lambda$ (cf. [43]).

For the terms in the exponent we can further approximate

$$r \approx z\left[1 + \frac{1}{2z}(x-\xi)^2 + \frac{1}{2z}(y-\eta)^2\right] \tag{6.2}$$

while for the denominator we can use just $r \approx z$:

$$\psi(x,y) = \frac{\exp[\imath k z]}{\imath z\lambda}\iint_{-\infty}^{\infty} \psi(\xi,\eta)\exp\left\{\frac{\imath k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\}d\xi\,d\eta. \tag{6.3}$$

With the use of the Heaviside theta function $\theta(x)$ defined by (2.16) we can compactly express $\psi(\xi,\eta)$ for a double slit as follows:

$$\psi(\xi,\eta) = [\theta(\xi-\xi_1) - \theta(\xi-\xi_2) + \theta(\xi-\xi_3) - \theta(\xi-\xi_4)] \times [\theta(\eta-\eta_1) - \theta(\eta-\eta_2)], \tag{6.4}$$

where both slits extend from η_1 to η_2 along η-axis (height), and slit 1 extends from ξ_1 to ξ_2 and slit 2 extends from ξ_3 to ξ_4 along ξ-axis (width). Essentially this is just a neat way to say that the function $\psi(\xi,\eta) = 1$ for any region within one of the slits and $\psi(\xi,\eta) = 0$ outside the slits.

The photon *probability density function* **P** was numerically calculated according to the Born rule using *Wolfram's Mathematica 7*. In our calculations we have used laser emitting photons with $\lambda = 650\,\text{nm}$ and two square pinholes (with side of $250\,\mu\text{m}$) separated by distance of 2 mm. Further we have positioned the slit at a distance of $2f$ from a convex lens with *focal length* $f = 2.25\,\text{m}$. The image plane will thus be also located at a distance $2f$ behind the lens. The parameters of the setup closely resemble the setup reported by Afshar [8–10].

Here we report the calculation of photon probability density function as the photon passes through the double slit, refracts in the lens, and reproduces the double-slit image at the image plane in two scenarios: (1) coherent setup—there are no polarization filters at the slits—and (2) incoherent setup—there are different polarization filters at each slit (e.g., R and L).

For the coherent setup the Born rule requires that we add each slit wavefunction and then square the sum in order to obtain the probability density function:

$$\mathbf{P_1} = (\psi_1 + \psi_2)(\psi_1 + \psi_2)^* = |\psi_1 + \psi_2|^2. \tag{6.5}$$

For the wavefunction in the coherent setup we obtain the general expression given in (6.3), which can be also written as

$$\begin{aligned}&\psi_1(x,y) + \psi_2(x,y)\\ &\quad = \frac{\exp[\imath kz]}{\imath z\lambda}\iint_{-\infty}^{\infty}[\psi_1(\xi,\eta) + \psi_2(\xi,\eta)]\exp\left\{\frac{\imath k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\}d\xi\,d\eta,\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}\psi_1(\xi,\eta) &= [\theta(\xi-\xi_1) - \theta(\xi-\xi_2)] \times [\theta(\eta-\eta_1) - \theta(\eta-\eta_2)],\\ \psi_2(\xi,\eta) &= [\theta(\xi-\xi_3) - \theta(\xi-\xi_4)] \times [\theta(\eta-\eta_1) - \theta(\eta-\eta_2)].\end{aligned} \tag{6.7}$$

The probability to find a photon either at image $1'$ or at image $2'$ cannot be ascribed to the contribution of branch vectors coming from only one of the slits (Figure 7). Instead we have equal contributions from branch vectors passing through both slits and a bright fringe in accordance with the analysis presented in Section 5. Because the quantum histories that pass through a dark fringe cancel each other, there is no one-to-one correspondence between slits 1 or 2 and images $1'$ or $2'$.

In the incoherent setup if we put different polarization filters at each slit, we will entangle the photon path through a slit with the photon polarization. Thus we will have new higher dimensional Hilbert space (as discussed in Section 3), and in the region behind the slit

the photon amplitudes passing through each slit will not interfere. For the incoherent setup the Born rule requires that we square the slit wavefunctions before we add them:

$$\mathbf{P_2} = \psi_1\psi_1^* + \psi_2\psi_2^* = |\psi_1|^2 + |\psi_2|^2. \tag{6.8}$$

Each slit wavefunction is given by

$$\begin{aligned}\psi_1(x,y) &= \frac{\exp[\imath kz]}{\imath z\lambda}\iint_{-\infty}^{\infty}\psi_1(\xi,\eta)\exp\left\{\frac{\imath k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi\,d\eta,\\ \psi_2(x,y) &= \frac{\exp[\imath kz]}{\imath z\lambda}\iint_{-\infty}^{\infty}\psi_2(\xi,\eta)\exp\left\{\frac{\imath k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi\,d\eta.\end{aligned} \tag{6.9}$$

The results for the incoherent setup are shown in Figure 8. There are no interference maxima or minima behind the slits or the focal plane of the lens. If we plot the evolution of the probabilities due to each polarized branch vector separately, we can see that each branch vector goes from slit 1 or slit 2 to the corresponding image 1′ or image 2′ and there is complete which-way information (Figure 9). Actually the plot shows the resultant squared sum of two branch vectors for each slit: one branch vector that corresponds to region where dark fringe occurs in the coherent setup, and one branch vector that corresponds to region where bright fringe occurs in the coherent setup. The destructively interfering branch vectors from each slit at the opposite image are not shown, analogously to Figure 5.

After calculation of the probability density functions for the coherent and incoherent setups, we see that at the image plane there could be formed two types of images. In incoherent setup the images at the image plane contain which-way information; however there is no interference before or after the lens (Figures 8 and 9). In contrast, in coherent setup there is interference before and after the lens and the images do not contain which-way information. We have calculated that the interference is there due to standard QM postulates, and we have shown that this interference implies that the images at the image plane are assembled from branch vectors coming from both slits. The central issue is that the erasure of which-way information happens even without inserting wire grid in front of the photons. We have proved this rigorously using Feynman's sum-over-histories approach. It is equivalent to standard QM formalism, but has the advantage to visualize things so that they can be explained even to a reader who does not want to follow in detail the mathematical equations but who has some geometrical intuition.

Here we point out that other commentators on Afshar's setup claimed that only inserting the wire grid erases the which-way information through diffraction, otherwise without wire grid there is which-way information at the image [11, 12, 14]. Furthermore, they construct various arguments in order to explain why there is no interference at the focal plane (i.e., why the visibility of the interference is $V = 0$). Here we have proved that the interference in the coherent setup is a consequence of the standard QM axioms and that even without wire grids on the photon path $V = 1$ exactly as Afshar insists. However contrary to Afshar's beliefs, the mere fact that there is interference (no matter if it is measured or not) is sufficient to erase the which-way information. The mechanism is not associated with putative diffraction from the wire grid that one needs to detect the interference, instead the mechanism is inherent in the way one sums and annihilates the branch vectors coming from the two slits.

7. Computing Visibility and Distinguishability

We have shown that a proper understanding of QM cannot be reached by relying on classical intuition such as *auxiliary assumption* (A1). It is exactly the case that mathematically one cannot extrapolate conclusions from incoherent setups to coherent setups even in cases when the observable intensities did not differ between the incoherent and the coherent setup. In incoherent setups the thing that does not happen is the cross-interference of branch vectors passing through the first slit with branch vectors passing through the second slit, which matters if one makes which-way information claims. Therefore to start from existent which-way information in coherent version of Afshar's setup is to start from inconsistent position and then try to defend the principle of complementarity with fallacious arguments [12–14, 18–20]. Whatever the details of the arguments are, they are all mistaken for the fact that they accept the existence of which-way information in the coherent lens setup with no wire grid on the photon path.

One of the most interesting things in the debate is the proper computation of the distinguishability and visibility in the setup. Afshar claimed he has violated the Englert-Greenberger duality relation $V^2 + D^2 \leqslant 1$. Because ψ_1 and ψ_2 are *pure states* in Afshar's and Unruh's setups, *visibility* V and *distinguishability* D are defined by (1.2) and (1.3). Simple arithmetic work gives us

$$\begin{aligned} V^2 + D^2 &= \left(\frac{2|\psi_1||\psi_2|}{|\psi_1|^2 + |\psi_2|^2}\right)^2 + \left(\frac{\left||\psi_1|^2 - |\psi_2|^2\right|}{|\psi_1|^2 + |\psi_2|^2}\right)^2 \\ &= \frac{4|\psi_1|^2|\psi_2|^2}{\left(|\psi_1|^2 + |\psi_2|^2\right)^2} + \frac{|\psi_1|^4 - 2|\psi_1|^2|\psi_2|^2 + |\psi_2|^4}{\left(|\psi_1|^2 + |\psi_2|^2\right)^2} \\ &= \frac{|\psi_1|^4 + 2|\psi_1|^2|\psi_2|^2 + |\psi_2|^4}{\left(|\psi_1|^2 + |\psi_2|^2\right)^2} = \frac{\left(|\psi_1|^2 + |\psi_2|^2\right)^2}{\left(|\psi_1|^2 + |\psi_2|^2\right)^2} = 1. \end{aligned} \tag{7.1}$$

Since the duality relation is a mathematically true statement (theorem) then it cannot be disproved by experiment and certainly means that Afshar's arguments, through which he violates the duality relation, are inconsistent.

From this does not follow that any argument that defends complementarity is correct. The wrong proof of correct theorem is still wrong and even worse. Kastner went even further and agreed that $V^2 + D^2 = 2$ but insisted that we are summing from two different planes—focal plane and image plane [18, 19]. In contrast, the current work clearly shows that if $V = 1$ and $D = 0$ at the focal plane of the lens, then the same $V = 1$ and $D = 0$ holds at the image plane of the lens, and one can never get $V^2 + D^2 = 2$.

Indeed the calculation of V and D depends on distinguishability of the states ψ_1 and ψ_2 and the existence of interference pattern in case of indistinguishable states ψ_1 and ψ_2. The calculation of V and D in Unruh's and Afshar's setup is different for quantum coherent setup (indistinguishable states ψ_1 and ψ_2) and incoherent (mixed state) setup (distinguishable states ψ_1 and ψ_2).

In both coherent and incoherent setups one can consider ψ_{136}, ψ_{236}, ψ_{146}, ψ_{246} at D_1 (keep in mind that for Afshar's setup subscripts 1 and 2 denote passage through slits 1 and

2, subscripts 3 and 4 denote passage through bright fringe or dark fringe, and subscripts 5 and 6 denote detection at D_2 or D_1). The most tricky part of the computation is to be aware that three histories ψ_{136}, ψ_{236}, and ψ_{146} contribute exactly $-\imath(1/\sqrt{8})$ amplitude to detector D_1 and only ψ_{246} contributes $+\imath(1/\sqrt{8})$. In the coherent setup the amplitude contributed by the history ψ_{236} can be incorrectly attributed to the history ψ_{146} and hence one can erroneously compute $|\psi_1| = 1/\sqrt{2}$ and $|\psi_2| = 0$ instead of the correct $|\psi_1| = 1/\sqrt{8}$ and $|\psi_2| = 1/\sqrt{8}$. The possible error for the coherent setup is easy to make if we forget that destructive interference occurs at path 4 and ψ_{146} annihilates with ψ_{246}. In incoherent setup there is no interference at path 4, therefore ψ_{236} and ψ_{246} annihilate at path 6, and only vectors ψ_{136} and ψ_{146} from slit 1 contribute to the final amplitude at D_1.

Similarly, in both coherent and incoherent setups one can consider ψ_{135}, ψ_{235}, ψ_{145}, ψ_{245} at D_2. In coherent setup ψ_{145} annihilates with ψ_{245} at path 4 and both ψ_{135} and ψ_{235} contribute to the amplitude left at D_2. In incoherent setup ψ_{135} and ψ_{145} annihilate at path 5, and only vectors ψ_{235} and ψ_{245} from slit 2 contribute to the final amplitude at D_2.

With the above remarks in mind, the correct calculation of D and V is straightforward. In incoherent setups it is exactly $D = 1$ and $V = 0$. At detector D_1 we have $|\psi_1| = 1/\sqrt{2}$ and $|\psi_2| = 0$ and at detector D_2 we have $|\psi_1| = 0$ and $|\psi_2| = 1/\sqrt{2}$. In coherent setups the existence of interference in the overlap region makes $D = 0$ and $V = 1$ for all times after the interference has occurred. At detector D_1 we have $|\psi_1| = 1/\sqrt{8}$ and $|\psi_2| = 1/\sqrt{8}$, and at detector D_2 we have $|\psi_1| = 1/\sqrt{8}$ and $|\psi_2| = 1/\sqrt{8}$. This happens because at the dark fringes some of the 8 branch vectors annihilate each other; however these annihilated branch vectors are needed in order to obtain which-way information. It is now clear why even though there are no wire grids on photon paths, the which-way information is erased. Unruh wrongly believed that BS3 or the lens can "undo" the interference [13]. Instead, if interference has occurred it cannot be "undone." One can restore only the appearance of the two-slit image (the probability density functions at the image plane might look similar in coherent and incoherent setups) but cannot restore the which-way information.

The reader is referred to [12] for a typical example how for coherent setup instead of the correct $D = 0$ and $V = 1$ one starts with D and V inverted, that is, $D = 1$ and $V = 0$. Then one *needs* to "invent" some arguments to invert D and V values in case when there are wires on the photon paths.

Afshar's pseudo-paradox results from quantum coherent overlap of 2 branch vectors from one path that destructively self-interferes with 2 branch vectors from another path that constructively self-interferes [24–26]. The smallest possible number of alternative histories for arising of such "paradox" is probably $2^3 = 8$ and the proof is purely combinatorial. Therefore Unruh's setup is the simplest possible way to formalize Afshar's pseudo-paradox. In Afshar's setup there are infinite number of quantum histories and at the first glance the discussion of which-way information may seem more complicated. The solution is to use very coarse coarse graining leaving only 8 quantum histories and then to construct one-to-one mapping between branch vectors in Unruh's setup and branch vectors in Afshar's setup as was done in Section 5.

8. Bohmian QM and Complementarity

At the end we would like to point out that the problems with the existence of which-way information can be also formulated even in Bohmian QM, although in this case the problematic statement should not be formulated as related to the photon hidden trajectories, which are calculated from (8.1). In Bohmian QM the state of a system of N particles is

described by its wavefunction $\psi = \psi(q_1, \ldots, q_N) = \psi(q)$, a complex (or spinor) valued function on the space of possible configurations q of the system, together with its actual configuration Q defined by the actual positions $Q_1, \ldots, Q_N$ of its particles (the actual positions are hidden). The theory is then defined by two evolution equations.

The first one is the Schrödinger equation (1.4) for $\psi(t)$, where H is the nonrelativistic (Schrödinger) Hamiltonian, containing the masses of the particles and a potential energy term.

The second one is called the Guiding Equation:

$$\frac{dQ_k}{dt} = \frac{\hbar}{m_k} \operatorname{Im}\left[\frac{\psi^* \partial_k \psi}{\psi^* \psi}\right](Q_1, \ldots, Q_N) \tag{8.1}$$

for $Q(t)$, the simplest first-order evolution equation for the positions of the particles that is compatible with the Galilean (and time reversal) covariance of the Schrödinger evolution [44, 45].

The essence of the above equations is that one first needs to solve the Schrödinger equation and then obtain the probability distribution using the Born rule (e.g., as we did in Figure 7 for coherent version of Afshar's setup). Only after having this information one can plug in the data into the Guiding equation (see the occurrence of the Born rule $\psi^*\psi$ in the denominator). In the coherent setup the plotted hidden trajectories will have to obey the condition that they do not cross each other, and therefore as correctly described by Drezet the hidden trajectories of a photon passing through slit 1 will always go to the opposite detector D_2, and the hidden trajectories of a photon passing slit 2 will always go to the opposite detector D_1 [20].

However we can easily reformulate the main problem so that it has nothing to do with the hidden trajectories, that is, we may ask through which slit pass the branch vectors that contribute to the Bohmian quantum potentials at each detector? For example, suppose that Unruh was correct and each quantum amplitude at one of the detectors was contributed only from branch vectors coming from the corresponding slit. Then in Bohmian QM it would appear that the photon passing through slit 1 is guided solely by the Bohmian quantum potential contributed by branch vectors coming from slit 2 at D_2 at the moment when the photon is absorbed by D_2, and analogously it would appear that the photon passing through slit 2 is guided solely by the Bohmian quantum potential contributed by branch vectors coming from slit 1 at D_1 at the moment when the photon is absorbed by D_1. The original problem that leads to the pseudo-paradox is thus present also in Bohmian QM: which branch vectors are responsible for the photon to be able to arrive at D_1 or D_2? If, for example, only ψ_1 was present at D_1 then the photon guided through slit 2 would arrive at D_1 due to the fact that ψ_1 was present at D_1. In such case the presence of ψ_1 would be the only cause for the photon to enter D_1. The Guiding equation did not solve the problem; just we had to formulate the problem as related to the Bohmian quantum potential and not related to the hidden trajectories. Contrary to Drezet's beliefs (cf. [20]), the problem is not changed at all in Bohmian QM, what is changed is the *language* that we use in order to formulate the problem.

Thus the essence of the debate is solely based on studying the solutions of the Schrödinger equation and the way we manipulate the branch vectors. It is irrelevant whether we will add Guiding equation to the QM formalism or not, hence Drezet's discussion on Bohmian QM did not help clarify anything. Moreover, if we could derive a moral, it is that in order to be mathematically rigorous we should always be careful if somewhere along the argument we have not changed fundamentally the meaning of what is understood as a photon path.

9. Discussion

In this work we have concisely summarized all notable positions discussing the principle of complementarity in Unruh's and Afshar's setups. Then we have provided complete mathematical description of those two setups using Feynman's sum-over-histories approach. A major result that we have obtained is that the image plane of a lens does not always provide the which-way information, contrary to widespread belief in the opposite (cf. [13, 14, 20, 46]). If the photon passes coherently through both slits, the destructive interference of branch vectors at the focal plane of the lens erases the which-way information. With the use of Feynman's sum-over-histories approach we have shown that the principle of complementarity is derived from the requirement for mathematical consistency in the process of summation/annihilation of branch vectors. A photon that has pure state density matrix in coherent double-slit setup and forms interference pattern at the focal plane of the lens, has the corresponding which-way information erased (branch vectors from both slits contribute half the intensity at each detector at the image plane). In contrast, a photon that has mixed state density matrix in incoherent double-slit setup (with different polarization filters at the slits) does not form interference at the focal plane of the lens and keeps the which-way information from the slits. This implies that $V^2 + D^2 \leqslant 1$, even if we consider measurements of identically prepared photons both at the focal plane and at the image plane of the lens.

Indeed, philosophically it should not matter whether we detect at the image plane or at the focal plane of the lens. Let us consider Drezet's argument that we cannot detect a photon twice in the context of *single-photon* double-slit experiment. If one does not believe in a principle known as *contrafactual definiteness*, then there is no reason for one to believe that each photon detected at the focal plane of a lens forms interference pattern. A single photon forms a spot at the photographic film and not interference pattern. In order to obtain interference pattern one needs to run many single-photon trials and detect many photons prepared in the same way. Only after that using the axioms of standard QM one could argue that all these photons *were* in the same quantum state and each formed interference pattern *individually*. This assumes *contrafactual definiteness* because even if a single photon is detected as a spot, the other spots that comprise the interference pattern are considered to be made by wavefunction collapses of identical photons to the one we study. Only after following the reasoning outlined above, one can say that a single photon detected at the focal plane is passing through both slits at once (or needs Bohmian quantum potential contributed from both slits) in order to form interference pattern *individually*.

The very same principle of *contrafactual definiteness* however allows us to argue that a photon arriving at the image plane of a lens has formed interference pattern at the focal plane of the lens even if the interference has not been measured. This is because we might perform the measurement on identically prepared photon and conclude that the photon of interest would have had the predicted behavior. In other words, the *contrafactual definiteness* is manifestation of our belief that QM calculations never fail in their predictions and describe real probability density functions of the studied physical systems. The latter statement should be understood with the more general belief that the ultimate *physical laws* should never fail in their description of the Universe, or conversely that the Universe is describable by such physical laws.

The proposed explanations by [11–14] are flawed because they contradict the corollary derived from the Reininger's negative measurement experiment. All of the above viewpoints require radical change of the which-way information claim even though Afshar puts wire grids at positions where the wavefunction amplitude is *zero*. Such an intervention at place

with *zero* quantum amplitude should not have effect on the photon wavefunction (should not "collapse" the wavefunction), and this could be easily proved as a mathematical theorem (actually this is the only point on which our analysis agrees with some of Afshar's claims [8–10]). Although for Afshar's setup the *zero* amplitude is only approximate at the dark fringes, for Unruh's setup the zero amplitude at path 4 might be considered absolute. Due to the equivalence of Afshar's and Unruh's setups the solution of the pseudo-paradox should be applicable to both cases, which means that one should not base the argument on the fact that the dark fringes in Afshar's setup might not be absolutely dark or equivalently that the wire grid due to its nonzero size occupies region from the nearby semidark region of the dark fringe as argued by Motl [14]. Motl's argument can be theoretically challenged by scaling the Afshar's setup in astronomic proportions so that for any size of the wire grid it fits arbitrarily well in the absolutely dark region of the fringe at the focal plane of the lens.

Due to the general applicability of Feynman's sum-over-histories approach we consider the current work superior to previous work [11–14, 20] not only because it is a consistent description, but also because one can geometrically see the mechanism producing the pseudo-paradox. Furthermore, it can be easily proved through combinatorial mathematical argument that *the pseudo-paradox cannot occur if there are less than 8 alternative quantum histories*.

At the end, we will briefly assess each of the positions (P1)–(P8) formulated in Section 1.

Positions (P1) [6, 8–10, 15–17] and (P2) [18, 19] are fundamentally flawed. Surprisingly, those researchers conclude that $V^2 + D^2 \leqslant 2$ despite that simple arithmetic calculation (7.1) shows that $V^2 + D^2 = 1$.

Position (P3) [11, 12, 14] cannot explain why measuring the wavefunction at place where it is *zero* leads to wavefunction collapse or modifies the wavefunction in such a way that the which-way information is erased. Also some of the researchers supporting (P3) claimed that there is no interference pattern at the focal plane if it is not measured, but this contradicts the probability density function obtained by solving the Fresnel diffraction integrals (see Figure 7).

Position (P4) [20] is unsatisfying because it altogether denies that the wire grid detects interference pattern and ignores the necessity to explain why all of the photons arriving at the image plane have managed to avoid the wire grid.

Position (P5) [13] is correct on the equivalence of Afshar's and Unruh's setups, however inconsistently claims that without wire grid one can infer the existence of interference pattern and claim that photons arriving at the detectors possess which-way information. It is concluded that only measuring the interference pattern destroys the which-way information, which is surprising in view that *measuring* the interference pattern does not provide further truth to the claim that there is an interference pattern, which is already proved (calculated) as a theorem. Mathematical theorems are true once they are proved and do not require further measurement to convince us that they are true.

Position (P6) [21] in its first half is essentially equivalent to our sum over history description and correctly recognizes the importance of Feynman's sum-over-histories approach. The error is in the second half of (P6) in the claim that Unruh's setup is not equivalent to Afshar's setup, which implies that the authors did not properly understand the origin of the pseudo-paradox from the mutually exclusive ways in which one can sum 8 alternative quantum histories corresponding to Unruh's setup. Indeed, Afshar's pseudo-paradox arises because one can consider a coarse graining of Afshar's setup in which there are exactly 8 alternative quantum histories that can be mapped in one-to-one fashion with the 8 alternative quantum histories in Unruh's setup.

Position (P7) [22, 23] is completely correct, but the authors did not provide explanation for the origin of the pseudo-paradox. Qureshi's work [22] is almost complete, in a sense that one can explicitly write the canceled sinh terms (5.3) and derive the one-to-one mapping with the Unruh's setup.

The position (P8) [24–26] is essentially a brief summary of the current work, and we believe we have provided complete mathematical derivation of each of the basic statements contained within (P8). A novelty, compared to previous publications [24–26], is the detailed illustration of various claims concerning Unruh's setup (Figures 1–6) and the computer calculation of Fresnel diffraction integrals for Afshar's setup (Figures 7–9). To our knowledge this is the first work that besides general theoretical arguments also provides numerical calculations of the Fresnel diffraction integrals for both coherent and incoherent Afshar's setups.

Appendix

Quantum Antirealism versus Quantum Realism

N. Bohr was the most prominent proponent of a philosophical position known as *antirealism*, according to which measurement of a quantum observable within a pair of complementary observables (e.g., position) necessarily forces the complementary observable from the pair (e.g., momentum) to be undefined, unreal or one that cannot be meaningfully discussed. Bohr believed that quantum phenomena are *unanschaulich* (unvisualizable) and that quantum particles exist only when we measure them, whereas between each two consecutive measurements the quantum particles are nonexistent and their properties cannot be meaningfully discussed [47]. Bohr insisted that if some physical observable is not measured, it does not exist, for example, if someone is not looking at the moon to verify its position, it is meaningless to say that the moon is there. Therefore *"the moon is not there if nobody looks at it"* [48, 49]. Philosophically such *antirealism* is not a pleasing description of physical reality and not surprisingly physicists were trying to find a way to restore the scientific realism and more importantly to find a way to illustrate quantum phenomena.

In 1926, Schrödinger developed a wave equation that describes the evolution of quantum probability amplitudes in space and time [50]. Thus although the detection of a quantum particle is a probabilistic event, the probabilities for that event are propagated causally in space and time in accordance with the Schrödinger equation. In 1948, Feynman showed how one can recover the Schrödinger equation from a path integral formulation of QM in which a quantum particle evolving between two points in space and time is allowed to travel along all possible histories between those two points [27]. In this way Feynman provided us with a theory that is able to visualize the behavior of quantum particles. Furthermore, the sum-over-histories approach restored a form of quantum *realism*, according to which each complementary observable within a pair of complementary observables is considered to be real, even though it might be in a *quantum superposition*. The difference between antirealism and realism as philosophical viewpoints is huge, because from undefined things one cannot draw conclusions or if one concludes something it is likely to arrive at paradoxes, whereas from well-defined but possibly quantum superposed states one can draw conclusions without arriving at paradoxes. Such a philosophical position in which one is ready to accept the strange world of QM as *real*, without trying to fit the quantum world in the framework of classical Newtonian mechanics, is based on the following argument: if we are really seeking the truth of what the world *is*, then we should be ready to accept whatever we find by experiment as

real, without trying to fit the results in some already preconceived version of what the real world *should be* [51].

Al-Khalili reported [52] that when the young Feynman presented for first time his geometrization of quantum electrodynamics (QED) (so-called Feynman diagrams) at the first Shelter Island Conference on the Foundations of QM, N. Bohr jumped out from his chair in disgust and along with other scientists present (including P. Dirac) considered Feynman *an idiot, who did not understand the first thing in QM, namely, that quantum trajectories cannot be geometrically represented*. Later Feynman's approach was proved correct winning him the Nobel Prize in 1965, and the sum-over-histories formulation of QM is now known to be equivalent to the standard formulation of QM. Yet even at present many physicists, including Unruh among others, stick to Bohr's antirealism and avoid Feynman's sum-over-histories treatment of quantum problems as extravagant. Unfortunately sticking to antirealist position makes it easy for one to slip a mathematical error in the chain of reasoning and does not help in clarifying the mathematical mechanism by which the Afshar's pseudo-paradox is created.

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