Research Article

On the Adjacent Cycle Derangements

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A derangement, that is, a permutation without fixed points, of a finite set is said to be an adjacent cycle when all its cycles are formed by a consecutive set of integers. In this paper we determine enumerative properties of these permutations using analytical and bijective proofs. Moreover a combinatorial interpretation in terms of linear species is provided. Finally we define and investigate the case of the adjacent cycle derangements of a multiset.

1. Introduction

Let $[n] = \{1, 2, \dots, n\}$, and let S_n denote the set of permutations of $[n]$. We will use both the one-line notation of a permutation of S_n and its decomposition as a product of cycles. It will be convenient to write each cycle ending with its smallest element and order cycles by increasing smallest elements.

For example $\alpha = 3176452 = (3721)(654)$.

In [1] the authors define an adjacent cycle of a permutation to be a cycle in which the elements form a consecutive set of integers.

We generalize the notion defining a permutation of S_n to be *adjacent cycle* (a.c. for short) when in every cycle the elements form a consecutive set of integers. In other words every cycle of k elements, having i as the smallest element, is represented by the sequence $(i + 1, i + 2, \dots, i + k, i)$.

In [2] it is proved that the adjacent cycle permutations of $[n]$ turn out to be fixed by the bijection that associates to a permutation π of $[n]$ the permutation π' whose one-line form is obtained from the canonical form of π by removing all inner parentheses.

For example, let $\pi = 23146785 = (231)(4)(6785)$.

Notice that we omit commas between the integers in a cycle or in a sequence, unless it leads to ambiguity.

We will denote by P_n^a and D_n^a the set of adjacent cycle permutations and the set of adjacent cycle derangements of $[n]$, respectively, and $p_n^a = |P_n^a|$ and $d_n^a = |D_n^a|$ their cardinalities; moreover, $D_{k,n}^a$ represents the set of adjacent cycle derangements having k cycles and $d_{k,n}^a = |D_{k,n}^a|$ its cardinality.

We say that an adjacent cycle permutation α of $[n]$ is *extended* to $n+1, \dots, n+h$, where $h \geq 1$, when $n+1, \dots, n+h$ are inserted in this order in the last cycle of α before its last element. Clearly we obtain an adjacent cycle permutation of $[n+h]$.

For example, the extension of the permutation $\alpha = (21)(453)$ of $[5]$ to $6, 7, 8$ determines the new permutation $(21)(456783)$ of $[8]$.

Let F_n denote the n th Fibonacci number, with initial conditions $F_0 = 0, F_1 = 1$.

We now outline the contents of this paper. In Section 2 we determine properties of the adjacent cycle permutations and adjacent cycle derangements of $[n]$, using analytical and bijective proofs.

In Section 3 we prove a relation involving the Fibonacci numbers, which seems new. In Section 4 we present a combinatorial interpretation of the adjacent cycle permutations in terms of linear species; as a consequence we are able to calculate the enumerative results previously obtained in a direct way. In the last section we define and investigate the derangements of a multiset.

2. Adjacent Cycle Permutations and Adjacent Cycle Derangements

In this section we investigate the adjacent cycle permutations and adjacent cycle derangements of the set $[n]$, for a positive integer n . First we determine the number of adjacent cycle derangements having a fixed number of cycles.

Theorem 2.1. *For positive integers n and k such that $1 \leq k \leq n$ one has*

$$d_{k,n}^a = \binom{n-k-1}{k-1}. \quad (2.1)$$

Proof. An adjacent cycle derangement π of the set $[n]$ with k cycles is defined by inserting $k-1$ dividers in the positions between two consecutive integers of the sequence $(1, 2, \dots, n)$, so that the numbers of integers before the first divider, between two consecutive dividers, and after last divider are greater than 1. As a consequence we have to exclude the positions between 1 and 2 and between $n-1$ and n and the $k-2$ positions that immediately follow every divider, but the last one; in other words, we have to exclude k among the $n-1$ possible positions. Thus the number turns out to be equal to

$$\binom{n-k-1}{k-1}. \quad (2.2)$$

□

In the more general case of adjacent cycle permutations of $[n]$, the number $p_{k,n}$ of the adjacent cycle permutations having k cycles is obtained by choosing $k-1$ among the $n-1$ possible positions. See also [2].

Proposition 2.2. *The number of adjacent cycle permutations of $[n]$ having k cycles, where $1 \leq k \leq n$, is $\binom{n-1}{k-1}$, and the number of all adjacent cycle permutations is $\sum_{k=1}^n \binom{n-1}{k-1} = 2^{n-1}$.*

Proposition 2.3. *For every positive integer n one has*

$$d_n^a = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} = F_{n-1}. \quad (2.3)$$

Proof. Note that the maximum number of cycles of an adjacent cycle derangement of $[n]$ is obtained by setting the length of all cycles equal to 2, with possibly one cycle of length 3, if n is odd. Then the maximum value of k is $\lfloor n/2 \rfloor$, and therefore we obtain

$$d_n^a = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1}. \quad (2.4)$$

Moreover

$$\begin{aligned} d_n^a &= \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\binom{n-1-k-1}{k-1} + \binom{n-1-k-1}{k-2} \right) \\ &= \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k-1}{k-1} + \sum_{h=1}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-h-1}{h-1} \\ &= d_{n-1}^a + d_{n-2}^a. \end{aligned} \quad (2.5)$$

Thus d_n^a satisfies the same recursive relation of the Fibonacci numbers. It is immediate to verify that $d_1^a = 0$ and $d_2^a = 1$; then $d_n^a = F_{n-1}$. $\square$

Another proof of (2.4) may be obtained in a bijective way.

Theorem 2.4. *For every integer $n > 2$ one has*

$$D_n^a \cong D_{n-1}^a + D_{n-2}^a. \quad (2.6)$$

Hence $d_n^a = d_{n-1}^a + d_{n-2}^a$, where the initial conditions satisfy $d_1^a = 0$ and $d_2^a = 1$; thus $d_n^a = F_{n-1}$.

Proof. Let π be an element of D_n^a . We have two cases to consider.

- (1) n belongs to a cycle of π with more than 2 elements. Then π is the extension to n of a permutation $\pi' \in D_{n-1}^a$.
- (2) n belongs to a cycle ρ having 2 elements. The π is the product of an a.c. derangement $\alpha \in D_{n-2}^a$ and the cycle $\rho = (n, n-1)$.

The sets D_{n-1}^a and D_{n-2}^a are clearly disjoint; then $D_n^a \cong D_{n-1}^a + D_{n-2}^a$. By equating the cardinalities we obtain $d_n^a = d_{n-1}^a + d_{n-2}^a$, where the initial conditions hold $d_1^a = 0$ and $d_2^a = 1$. Then $d_n^a = F_{n-1}$. $\square$

Example 2.5. Let us consider the case of the set $[4] = \{1, 2, 3, 4\}$.

The a.c. permutations are the following 2^3 permutations represented in canonical form:

$$(1)(2)(3)(4), (1)(3\ 2)(4), (1)(2)(4\ 3), (1)(3\ 4\ 2), (2\ 1)(3)(4), (2\ 1)(4\ 3), (2\ 3\ 1)(4), (2\ 3\ 4\ 1). \quad (2.7)$$

The a.c. derangements are $(2\ 1)(4\ 3), (2\ 3\ 4\ 1)$. Their number d_4^a coincides with the Fibonacci number F_3 .

The a.c. permutations having a fixed number of cycles, for instance 2, are $p_{4,2}^a = \binom{3}{1} = 3$, while the number of a.c. derangements having 2 cycles is $d_{4,2}^a = \binom{1}{1} = 1$.

Another bijective proof of the results of Theorem 2.4 is obtained in the following theorem.

Proposition 2.6. *Let n be a positive integer. Then*

$$\begin{aligned} D_n^a &\cong D_{n-1}^a + D_{n-2}^a + \cdots + D_2^a + D_1^a; \\ d_n^a &\cong d_{n-1}^a + d_{n-2}^a + \cdots + d_2^a + 1. \end{aligned} \quad (2.8)$$

Moreover $d_{n+1}^a = d_n^a + d_{n-1}^a$, with $d_2^a = d_3^a = 1$.

Proof. Let $\alpha \in D_n^a$, and let $\alpha(n) = i$, where $3 \leq i \leq n-1$. Then α is the product of an a.c. derangement of $\{1, 2, \dots, i-1\}$ and the cycle $(i+1, i+2, \dots, n, i)$. The cases $i = 2$ and $i = n$ are excluded because they imply a cycle of length 1. In particular $D_1^a = \{\gamma\}$, where $\gamma = (2, 3, \dots, n, 1)$.

The sets $D_j^a, 2 \leq j \leq n-1$ and $\{\gamma\}$ are disjoint; then equating the cardinalities we obtain $d_{n+1}^a = d_{n-1}^a + d_{n-2}^a + \cdots + d_2^a + 1$.

In a similar way we have d_n^a and, by subtraction, last result. $\square$

3. Fibonacci Numbers

In this section we determine a relation involving the Fibonacci numbers.

Denote by A_i the set of a.c. permutations of $[n]$ fixing the integer i .

Proposition 3.1. *For every positive n ,*

$$|A_i| = \begin{cases} 2^{n-2} & \text{for } i \in \{1, n\} \\ 2^{n-3} & \text{for } i \in \{2, \dots, n-1\}. \end{cases} \quad (3.1)$$

Proof. For i equal to 1 or to n an a.c. permutation π of A_i is the product of the unitary cycle (i) and an a.c. permutation of a set of cardinality $n-1$. Then by Proposition 2.2 the number of these permutations is 2^{n-2} .

For $i \notin \{1, n\}$ the permutation π is the product of an a.c. permutation of the set $[i-1]$ by the cycle (i) and an a.c. permutation of the set $\{i+1, \dots, n\}$ having cardinality $n-i$. Then the number of these permutations is $2^{i-2} \cdot 2^{n-i-1} = 2^{n-3}$. $\square$

Denote

$$s_h = \sum_{1 \leq i_1 \leq \dots \leq i_h \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_h}|. \quad (3.2)$$

Theorem 3.2. *For every positive integer n one has the identity*

$$F_{n-1} = s_0 - s_1 + s_2 - \dots + (-1)^n s_n, \quad (3.3)$$

where s_0 denotes the number of all the adjacent cycle permutations of $[n]$.

Proof. The identity follows from the inclusion-exclusion principle, the relations (3.1) and (3.2), and the condition that $s_0 = 2^{n-1}$, by Proposition 2.2. $\square$

As an example consider the case $n = 4$. Then $S_0 = 2^3$, $S_1 = |A_1| + |A_2| + |A_3| + |A_4| = 2^2 + 2 + 2 + 2^2 = 12$, $S_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| = 2 + 1 + 2 + 1 + 1 + 2 = 9$, $S_3 = 4$, $S_4 = 1$.

Then $F_3 = 8 - 12 + 9 - 4 + 1 = 2$.

4. A Combinatorial Interpretation

In this section we give an interpretation of the adjacent cycle permutations and the adjacent cycle derangements of a finite set in terms of linear species of particular linear partitions.

See [3] for the fundamental notions of the theory of species.

We recall that a *linearly ordered set* or a *linear order* $\langle L, \leq \rangle$ is a set L together with a linear order relation $\leq$. We will write simply L for $\langle L, \leq \rangle$ and $|L|$ for its cardinality.

A k -interval of L is a subset of the form

$$[x; y] = \{u \in L : x \leq u \leq y\} \quad (4.1)$$

having exactly k elements.

Let **Lin** be the category of finite linearly ordered sets and order bijections, and let **Set** be the category of finite sets and functions.

A *linear species of combinatorial structures* is a functor $S : \mathbf{Lin} \rightarrow \mathbf{Set}$.

The *cardinality* of a linear species S is the *formal geometric series*

$$|S| = \sum_{n \geq 0} |S[n]| t^n, \quad (4.2)$$

where $[0] = \emptyset$ and $[n] = \{1, 2, \dots, n\}$ for $n > 0$.

A *linear partition* π of a finite linear order L is a family of nonempty disjoint intervals, called blocks, whose union is L . For an integer h , we will say that a block B is an h -block of π when B has cardinality h .

Let S and T be two linear species, with $T[\emptyset] = \emptyset$. The composition $S \circ T$ is the species equivalent to giving a partition π of L , a structure of species T on each block of π , and a structure of species S on π .

The *geometric species* or *uniform species* G is the species defined by $G[L] = \{*\}$ (a singleton), for all finite linear orders L .

The cardinality of G is the geometric series

$$G(t) = \sum_{n \geq 0} t^n = \frac{1}{1-t}. \quad (4.3)$$

The linear species $G - 1$ is defined by

$$(G - 1)[L] = \begin{cases} \emptyset & \text{for } L = \emptyset \\ \{*\} & \text{for } L \neq \emptyset \end{cases} \quad (4.4)$$

for every $L \in \mathbf{Lin}$. Its cardinality is the series

$$(G - 1)(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}. \quad (4.5)$$

For an integer h , the h -geometric species G_h is defined as the singleton on every linear order L of cardinality h and as the empty set in all other cases:

$$G_h[L] = \begin{cases} \{*\} & \text{for } |L| = h \\ \emptyset & \text{for } |L| \neq h. \end{cases} \quad (4.6)$$

Thus the cardinality of G_h is the series $G_h(t) = t^h$.

4.1. The Linear Species of Adjacent Cycle Permutations

Let P be the linear species of the linear partitions. Giving a linear partition of a finite linear order L is equivalent to giving a partition π of L , a structure of a nonempty linear order on each block of π , and a geometric structure on π . Therefore we have the isomorphism

$$P = G \circ (G - 1). \quad (4.7)$$

Then, passing to the cardinalities and setting $|P| = P(t) = \sum p_n t^n$, the relation (4.7) becomes

$$P(t) = \frac{1}{1-t} \circ \frac{t}{1-t} = \frac{1-t}{1-2t} = 1 + t + 2t^2 + 2^2 t^3 + \dots. \quad (4.8)$$

Expanding the series, we obtain

$$p_n = \begin{cases} 2^{n-1} & \text{for } n > 0 \\ 1 & \text{for } n = 0. \end{cases} \quad (4.9)$$

From the definition it follows that an adjacent cycle permutation of $[n]$ corresponds to a linear partition of the linear order $L = [1, 2, \dots, n]$ in which the blocks are nonempty.

For example, the linear partitions of $[1, 2, 3]$ are

$$\{[1], [2], [3]\}, \{[1], [2, 3]\}, \{[1, 2], [3]\}, \{[1, 2, 3]\}. \quad (4.10)$$

The corresponding a.c. permutations of $\{1, 2, 3\}$ in cyclic form are

$$\{(1)(2)(3), (1)(32), (2, 1)(3), (2, 3, 1)\}. \quad (4.11)$$

Then the number of a.c. permutations coincides with the number of linear partitions of $[1, 2, \dots, n]$.

4.2. The Linear Species of the Adjacent Cycle Permutations with h Cycles

Let P_h be the linear species of the linear partitions with h blocks, and let $p_{h,n}$ be the cardinality of $P_h[L]$ when $|L| = n$.

Giving a structure of species P_h on a finite linear order L is equivalent to giving a partition π of L , a structure of linear order with at least 1 element on each block of π , and a structure of a linear order with h elements on π . Therefore we have

$$P_h = G_h \circ (G - G_1). \quad (4.12)$$

Passing to the cardinalities we have

$$\begin{aligned} t^h \circ \left(\frac{1}{1-t} - 1 \right) &= t^h \cdot \left(\frac{1}{1-t} \right)^h \\ &= t^h \cdot \sum_{k \geq 0} \binom{h+k-1}{k} t^k \\ &= \sum \binom{h+k-1}{k} t^{k+h}. \end{aligned} \quad (4.13)$$

For $k + h = n$, we obtain

$$p_{h,n} = \binom{n-1}{h-1}. \quad (4.14)$$

In the previous section we interpreted P as the linear species of a.c. permutations; now we interpret P_h as the linear species of a.c. permutations with h cycles.

From Proposition 2.2 we have $p_{h,n} = \binom{n-1}{h-1}$. This value coincides with the coefficient of t^{k+h} in the previous generating function when $k + h = n$.

4.3. The Linear Species of Adjacent Cycle Derangements

An adjacent cycle derangement of $[n]$ is an a.c. permutation without fixed points.

The linear species Δ^a of the a.c. derangements of a finite set corresponds to giving a finite linear order L , a partition π of L , a structure of linear order of cardinality at least 2 on each block of π of L , and a geometric structure on π . Therefore

$$\Delta^a = (G - 1) \circ (G - G_1 - G_2). \quad (4.15)$$

Passing to the cardinalities we obtain

$$\begin{aligned} \sum_{n \geq 0} d_n^a t^n &= \frac{t}{1-t} \circ \left(\frac{1}{1-t} - 1 - t \right) \\ &= \frac{t^2}{1-t-t^2}. \end{aligned} \quad (4.16)$$

As $t/(1-t-t^2) = \sum_{k \geq 0} F_k \cdot t^k$ is the series of the Fibonacci numbers with initial conditions $F_0 = 0, F_1 = 1$, then

$$d_n^a = F_{n-1}. \quad (4.17)$$

4.4. The Linear Species of the Adjacent Cycle Derangements with h Cycles

Let Δ_h^a be the linear species of the a.c. derangements with h cycles, and let $d_{h,n}^a$ be the cardinality of $\Delta_h^a[L]$ when $|L| = n$.

In order to give a structure of species Δ_h^a on a finite linear order L one has to give a partition π of L , a structure of linear order with at least 2 elements on each block of π , and a structure of a linear order with h elements on π . Therefore we have

$$\Delta_h^a = G_h \circ (G - G_1 - G_2). \quad (4.18)$$

Passing to the cardinalities we have

$$\begin{aligned} t^h \circ \left(\frac{1}{1-t} - 1 - t \right) &= t^{2h} \cdot \left(\frac{1}{1-t} \right)^h \\ &= t^{2h} \cdot \sum_{k \geq 0} \binom{h+k-1}{k} t^k \\ &= \sum_{k \geq 0} \binom{h+k-1}{k} t^{k+2h}. \end{aligned} \quad (4.19)$$

For $k + 2h = n$, we obtain

$$d_{h,n}^a = \binom{n-h-1}{h-1}, \quad (4.20)$$

which coincides with the result of Theorem 2.1.

In [4] the above-mentioned species were investigated in a different context.

5. Adjacent Cycle Derangements of a Multiset

Let $M = \{1^{a_1} 2^{a_2} \cdots m^{a_m}\}$ be a multiset, where a_i , the multiplicity of the element i , satisfies the condition $a_i \geq 1$. Recall, [5, page 24], that the intercalation product $\alpha \top \beta$ of two permutations α and β of M is obtained by the two-line notation of α and β , setting beside these two-line representations and sorting the columns into nondecreasing order of the top line.

It is known that every permutation π of M has a unique representation as the intercalation of cycles

$$\pi = (x_{11} \cdots x_{1n_1} y_1) \top (x_{21} \cdots x_{2n_2} y_2) \top \cdots \top (x_{t1} \cdots x_{tn_t} y_t), \quad (5.1)$$

where $t \geq 0$,

$$y_1 \leq y_2 \leq \cdots \leq y_t, \quad (5.2)$$

$$y_i < x_{ij} \quad \text{for } 1 \leq j \leq n_i, 1 \leq i \leq t. \quad (5.3)$$

In other words in this representation, said *canonical*, the smallest element of each cycle is at the end of the cycle and is smaller than every other element; moreover the sequence (5.2) of last elements is in nondecreasing order.

Now if we drop the parentheses and the $\top$'s in this representation, we obtain a sequence which represents a permutation π' in one-line form. Namely the one-line representation of π' is

$$x_{11} \cdots x_{1n_1} y_1 x_{21} \cdots x_{2n_2} y_2 \cdots x_{t1} \cdots x_{tn_t} y_t. \quad (5.4)$$

This procedure determines a function $g : M \rightarrow M$ such that $g(\pi) = \pi'$, which turns out to be a bijection.

In [2] it is proved that a permutation π is g -fixed if and only if in every cycle the elements form a nondecreasing sequence of integers but the last element which is lesser than every other in the cycle. In the case in which M is a set we recognize the notion of adjacent cycle permutation.

Now we define a permutation of a multiset *adjacent cycle* when in every cycle two consecutive elements a, b satisfy the condition that either $b = a + 1$ or $b = a$.

In particular an a.c. permutation without 1-cycles is an *a.c. derangement*.

In relation to the multiset $M = \{1, 2^{a_2}, \dots, m^{a_m}\}$, where $a_i \geq 1$, for $2 \leq i \leq m$, denote by $D^a(M)$ and $D_k^a(M)$ the sets of a.c. derangements of M and a.c. derangements of M having k cycles, respectively, and $d^a(M)$ and $d_k^a(M)$ their cardinalities.

Theorem 5.1. Let $M = \{1, 2^{a_2}, \dots, m^{a_m}\}$ be a multiset, where $a_i \geq 1$, for $2 \leq i \leq m$. Then

$$D^a(M) \cong D^a(M') + D^a(M''), \quad (5.5)$$

where $M' = \{1, 2^{a_2}, \dots, (m-1)^{a_{m-1}}\}$ and $M'' = \{1, 2^{a_2}, \dots, (m-1)^{a_{m-1}-1}\}$; it implies $d^a(M) = d^a(M') + d^a(M'')$.

Proof. Let π be an element of $D^a(M)$. Denote $m_1, m_2, \dots, m_{a_m}$ the a_m elements of M coinciding with m . We have two cases to consider.

- (1) m_{a_m} belongs to a cycle of π with more than 2 distinct elements. Then π is the extension to $m_1, m_2, \dots, m_{a_m}$ of a permutation $\pi' \in D^a(M')$.
- (2) m_{a_m} belongs to a permutation ρ having 2 distinct elements. Then π is product of a cycle $\alpha \in D^a(M'')$ and the cycle $\rho = (m_1, m_2, \dots, m_{a_m}, m-1)$.

The sets $D^a(M')$ and $D^a(M'')$ are clearly disjoint; then $D^a(M) \cong D^a(M') + D^a(M'')$. Equating the cardinalities we obtain $d^a(M) = d^a(M') + d^a(M'')$. $\square$

Note that if M is the set $[n+1]$, then $M' = [n]$, $M'' = [n-1]$, and the initial conditions hold $d_1^a = 0$, $d_2^a = 1$; thus we obtain the results of Theorem 2.4.

Proposition 5.2. Let $M = \{1, 2^{a_2}, \dots, m^{a_m}\}$ be a multiset, where $a_i \geq 1$, for $2 \leq i \leq m$. Then

$$D^a(M) \cong D^a(M_{m-1}) + D^a(M_{m-2}) + \dots + D^a(M_2) + D^a(M_1), \quad (5.6)$$

$$d^a(M) = d^a(M_{m-1}) + d^a(M_{m-2}) + \dots + d^a(M_2) + d^a(M_1). \quad (5.7)$$

Proof. Clearly the sets $D^a(M_i)$ form a partition of $D^a(M)$, and there is a bijection between $D^a(M)$ and $D^a(M_{m-1}) + D^a(M_{m-2}) + \dots + D^a(M_2) + D^a(M_1)$. Equating the cardinalities we obtain the relation (5.7). $\square$

In the case of the set $[n]$ we obtain Proposition 2.6.

In Proposition 2.2 it is proved that the number of a.c. permutations of $[n]$ is 2^{n-1} ; now we prove that this number coincides with the number of a.c. derangements of particular multisets.

Theorem 5.3. Let $m \geq 2$ be an integer and $M = \{1, 2^{a_2}, \dots, m^{a_m}\}$ a multiset, where $a_i > 1$ for $2 \leq i \leq m-1$ and $a_m \geq 1$. Then

$$D^a(M) \cong P([m-1]) \quad (5.8)$$

and $d^a(M) = 2^{m-2}$.

Proof. Let α be an a.c. permutation of the ordered set $\{1, 2, \dots, m-1\}$, having k cycles; then α is obtained by inserting $k-1$ dividers in the positions between an integer and its consecutive in the sequence $(1, 2, \dots, m-1)$. Let f be the map that associates to α the a.c. permutation α' of M such that if in α there is a divider between t and $t+1$ then in α' a divider is inserted before last and last but one of the elements t . Because the multiplicity of every element of M ,

but the first and possibly the last one, is greater than 1, then between two distinct dividers there are at least two distinct elements and the permutation turns out to be a derangement.

Conversely, let β be an a.c. derangement of M having k cycles. We may represent β by inserting $k - 1$ suitable dividers in the positions before last element of the sequence $i, \dots, i$, having a_i elements, $1 \leq i \leq m$. In correspondence of every possible divider before last i we insert a divider between i and $i - 1$ in the sequence $(1, 2, \dots, m - 1)$. Thus we obtain an a.c. permutation of the set $[m - 1]$ having k cycles. This implies that there is a bijection between $D^a(M)$ and $P([m - 1])$. Passing to the cardinalities we obtain the enumerative result. $\square$

Note that the result of Theorem 5.3 does not depend on the multiplicities, but only on the condition that they are greater than 1 (but $a_1 = 1$ and possibly a_m).

For instance, for $m = 4$ the a.c. permutations of $[3]$ are

$$(1)(2)(3), (1)(32), (21)(3), (231). \quad (5.9)$$

The corresponding a.c. derangements of $M = \{1, 2^3, 3^2, 4^2\}$ are

$$(221)(32)(443), (221)(33442), (22231)(443), (22233441). \quad (5.10)$$

Proposition 5.4. Let $M_1 = \{1, 2^{a_2}, \dots, m^{a_m}\}$ and $M_2 = \{1, 2^{a_2}, \dots, (m - 1)^{a_{m-1}}, m^{b_m}\}$ be two multisets, where $a_i > 1$, for $2 \leq i \leq m - 1$, $a_m, b_m \geq 1$, $a_m \neq b_m$. Then $d^a(M_1) = d^a(M_2)$.

Proof. By Theorem 5.3 the values of $d^a(M_1)$ and $d^a(M_2)$ do not depend on the multiplicity of m . $\square$

As particular case we obtain the following result.

Proposition 5.5. Let $M = \{1, 2, \dots, n - 1, (n)^h\}$ be a multiset, where $h > 1$. Then $d_k^a(M) = \binom{n-k-1}{k-1}$ and $d^a(M) = F_{n-1}$.

Proof. An a.c. derangement α of the multiset $M = \{1, 2, \dots, n - 1, (n)^h\}$, where $h > 1$, with k cycles of length greater than 1 coincides with the number of a.c. derangement of the set $\{1, 2, \dots, n\}$ with k cycles. Then $d_k^a(M)$ and $d^a(M)$ follow from Theorem 2.1 and Proposition 2.3. $\square$

For instance, let $M = \{1, 2, 3, 4, 5, 6, 6, 6\}$ and $Q = \{1, 2, 3, 4, 5, 6\}$. In order to obtain the a.c. derangements of M we have to consider the three positions 2-3, 3-4, 4-5. Assign 1 or 0 to a position when there is or there is not a divider; we obtain that the possible sequences are $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, and $(1, 0, 1)$. The corresponding derangements are (23456661) , $(2341)(6665)$, $(231)(56664)$, $(21)(456663)$, and $(21)(43)(6665)$.

Lemma 5.6. Let $M = \{1, 2^a, b_1, b_2, \dots, b_k\}$ be a multiset, where $a > 1$.

Then $d^a(M) = F_{k+2}$.

Proof. The set of a.c. derangements of M is partitioned into the sets $P_1, \dots, P_k$, where P_i , $1 \leq i \leq k$, is the set of the derangements which contain the cycle $(2^{a-1}1)$ extended to the i elements $2, b_1, \dots, b_{i-1}$, and one of the derangements of the set $\{b_i, \dots, b_k\}$. The case $i = 1$ corresponds to the derangement $(2^a 1)$. Thus the number of a.c. derangements is $F_k + F_{k-1} + \dots + F_2 + F_1 + F_1$. Then the result follows. $\square$

Theorem 5.7. Let $M = \{1, 2^{a_2}, \dots, m^{a_m}, b_1, b_2, \dots, b_k\}$ be a multiset, where $a_i > 1$ for $2 \leq i \leq m$. Then $d^a(M) = 2^{m-2} \cdot F_{k+2}$.

Proof. The proof is by induction on m . For $m = 2$ the result follows from Lemma 5.6.

Now assume the result holds for every integer lesser than m .

Note that the elements of $D^a(M)$ can be partitioned into two sets D_1 and D_2 , where D_1 consists on the set of the derangements which are the product of the cycle formed by the elements of the multiset $M_1 = \{1, 2^{a_2-1}\}$ and the derangements of the multiset $M_2 = \{2, 3^{a_3}, \dots, m^{a_m}, b_1, b_2, \dots, b_k\}$, while D_2 is formed by the derangements obtained from the elements of M_2 , where the first cycle now contains the elements of M_1 .

It implies that $d^a(M) = 2d^a(M_2)$ and, by the induction assumption, $d^a(M) = 2 \cdot 2^{m-3} \cdot F_{k+2} = 2^{m-2} F_{k+2}$. $\square$

Theorem 5.8. Let $M = \{1, 2, \dots, k, c_1^{a_1}, \dots, c_m^{a_m}\}$ be a multiset where $a_i > 1$, $1 \leq i \leq m$, $k > 2$, and $m > 1$.

Then

$$d^a(M) = 2^{m-1} \cdot F_k. \quad (5.11)$$

Proof. The proof is by induction on m .

Let $m = 2$. Then the set of derangements is partitioned into the set of derangements of $\{1, 2, \dots, k, c_1^{a_1-1}\}$ with the addition of the cycle $(c_2^{a_2}, c_1)$ and the set of derangements obtained by extending last cycle of every derangement of $\{1, 2, \dots, k, c_1^{a_1-1}\}$ to the elements $c_2^{a_2}, c_1$.

Both the sets have cardinality F_k by Proposition 5.5; then the number of derangements is $2F_k$.

Now we prove the result assuming it holds for every integer lesser than m .

The set of derangements of M is partitioned into the set D_1 of derangements of $M_1 = \{1, 2, \dots, k, c_1^{a_1}, \dots, c_{m-1}^{a_{m-1}-1}\}$ with the addition of the cycle $(c_m^{a_m}, c_{m-1})$ and the set D_2 of derangements obtained from D_1 by the extension of last cycle of every derangement to the elements $c_{m-1}, c_m^{a_m}$. By the assumption hypothesis $|D_1| = 2^{m-2} \cdot F_k$, and, since $|D_1| = |D_2|$, we obtain the result. $\square$

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References

- [1] R. A. Brualdi and E. Deutsch, "Adjacent q -cycles in permutations," *Annals of Combinatorics*, vol. 16, no. 2, pp. 203–213, 2012.
- [2] N. Zagaglia Salvi, "Adjacent-cycle permutations of a multiset," *Advances and Applications in Discrete Mathematics*, vol. 8, no. 2, pp. 65–74, 2011.
- [3] A. Joyal, "Une théorie combinatoire des séries formelles," *Advances in Mathematics*, vol. 42, no. 1, pp. 1–82, 1981.
- [4] O. M. D'Antona and E. Munarini, "A combinatorial interpretation of punctured partitions," *Journal of Combinatorial Theory, Series A*, vol. 91, no. 1-2, pp. 264–282, 2000.
- [5] D. E. Knuth, *The Art of Computer Programming: Volume 3, Sorting and searching*, Addison-Wesley Series in Computer Science and Information Processing, Addison-Wesley, Reading, Mass, USA, 1973.