Research Article

The Discretization for a Special Class of Ideal Projectors

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We focus on a special class of ideal projectors, subspaces, which possesses two classes of D -invariant polynomial subspaces. The first is a classical type, while the second is a new class. With matrix computation, we discretize this class of ideal projectors into a sequence of Lagrange projectors.

1. Introduction

Polynomial interpolation is to construct a polynomial p belonging to a finite-dimensional subspace of $\mathbb{F}[x]$ from a set of data that agrees with a given function f at the data set, where $\mathbb{F}[x] := \mathbb{F}[x_1, \dots, x_d]$ denotes the polynomial ring in d variables over the field $\mathbb{F}$. It is important to make the comment that $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ in this paper. Univariate polynomial interpolation has a well-developed theory, while the multivariate one is very problematic since a multivariate interpolation polynomial is determined not only by the cardinal but also by the geometry of the data set, cf. [1, 2].

Recently, more and more people are getting interested in ideal interpolation, which is defined by an *ideal projector* on $\mathbb{F}[x]$, namely, a linear idempotent operator on $\mathbb{F}[x]$ whose kernel is an ideal, cf. [3]. When the kernel of an ideal projector P is the vanishing ideal of certain finite set Ξ in $\mathbb{F}^d$, P is a *Lagrange projector* which provides the Lagrange interpolation on Ξ .

It is well known that an ideal projector can be characterized completely by the range of its dual projector, cf. [4–7]. Given a finite-rank linear projector P on $\mathbb{F}[x]$, the kernel of P is ideal if and only if the range of its dual projector is of the form:

$$\bigoplus_{\xi \in \Xi} \delta_{\xi} Q_{\xi}(D), \quad (1.1)$$

with some finite point set $\Xi \subset \mathbb{F}^d$, D -invariant finite-dimensional polynomial subspace $Q_{\xi} \subset \mathbb{F}[\mathbf{x}]$ for each $\xi \in \Xi$. δ_{ξ} denotes the evaluation functional at the point ξ , and $Q_{\xi}(D)$ will be explained in the next section.

In the univariate case, for an integer n , there is only one D -invariant polynomial subspace of degree less than n , which implies that every univariate ideal projector can be viewed as a limiting case of Lagrange projectors, cf. [8]. This prompted de Boor to define the *Hermite projector* as the pointwise limit of Lagrange projectors and conjecture that every ideal projector in $\mathbb{C}[\mathbf{x}]$ is an Hermite projector in [9].

However, Shekhtman [10] constructed a counterexample to this conjecture for every $d \geq 3$. In the same paper, Shekhtman also showed that the conjecture is true for bivariate complex projectors with the help of Fogarty Theorem (see [11]). Later, using linear algebra, de Boor and Shekhtman [12] reproved the same result. Specifically, Shekhtman [13] completely analyzed the bivariate ideal projectors which are onto the space of polynomials of degree less than n over real or complex field and verified the conjecture in this particular case.

In this paper, we consider the converse problem: how to generate a sequence of Lagrange projectors practically such that this sequence of Lagrange projectors converges pointwise to a given Hermite projector. We focus on a special class of ideal projectors, which possesses two classes of D -invariant polynomial subspaces. Sections 3 and 4 discuss these two classes of D -invariant subspaces, respectively. In Section 5, we discretize this class of ideal projectors into a sequence of Lagrange projectors. The next section is devoted a preparation for the paper.

2. Preliminaries

In this section, we will introduce some notation used throughout the paper and give some background results. For more details, we refer the reader to [9, 14].

We use $\mathbb{N}$ to stand for the monoid of nonnegative integers and boldface type for tuples with their entries denoted by the same letter with subscripts, for example, $\alpha = (\alpha_1, \dots, \alpha_d)$. For arbitrary $\alpha \in \mathbb{N}^d$, we define $\alpha! = \alpha_1! \cdots \alpha_d!$.

Henceforward, $\leq$ will denote the usual product order on $\mathbb{N}^d$, that is, for arbitrary $\alpha, \beta \in \mathbb{N}^d$, $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i, i = 1, \dots, d$. A finite subset $\mathcal{A} \subset \mathbb{N}^d$ is *lower* if for every $\alpha \in \mathcal{A}$, $0 \leq \beta \leq \alpha$ implies $\beta \in \mathcal{A}$.

A monomial $\mathbf{x}^\alpha \in \mathbb{F}[\mathbf{x}]$ is a power product of the form $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with $\alpha \in \mathbb{N}^d$. Thus, a polynomial p in $\mathbb{F}[\mathbf{x}]$ can be expressed as

$$p = \sum_{\alpha \in \mathbb{N}^d} \hat{p}(\alpha) \mathbf{x}^\alpha, \quad (2.1)$$

with $\hat{p}(\alpha) \in \mathbb{F}$ nonzero. For a polynomial $p(\mathbf{x})$ as in (2.1), we write

$$p(D) : \mathbb{F}[\mathbf{x}] \longrightarrow \mathbb{F}[\mathbf{x}]$$

$$f \longmapsto \sum_{\alpha \in \mathbb{N}^d} \hat{p}(\alpha) \frac{\partial^{|\alpha|_1} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad (2.2)$$

for its associated differential operator. For a finite-dimensional polynomial subspace Q , we define

$$Q(D) := \text{span}_{\mathbb{F}}\{q(D) : q \in Q\}. \quad (2.3)$$

If the polynomial subspace Q is closed under differentiation, then we call it a *D-invariant polynomial subspace*.

Let P be a finite-rank ideal projector on $\mathbb{F}[\mathbf{x}]$. The range and kernel of P will be described as

$$\begin{aligned} \text{ran } P &:= \{g \in \mathbb{F}[\mathbf{x}] : g = Pf \text{ for some } f \in \mathbb{F}[\mathbf{x}]\}, \\ \ker P &:= \{g \in \mathbb{F}[\mathbf{x}] : Pg = 0\}. \end{aligned} \quad (2.4)$$

Let P' be the dual projector of P , then $\text{ran } P'$ is the set of interpolation conditions matched by P . If $\mathbf{q} = (q_1, \dots, q_s)$ is an $\mathbb{F}$ -basis for $\text{ran } P$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ an $\mathbb{F}$ -basis for $\text{ran } P'$, then their Gram matrix:

$$\boldsymbol{\lambda}^T \mathbf{q} := (\lambda_i q_j)_{1 \leq i, j \leq s} \quad (2.5)$$

is invertible.

Lemma 2.1 (see [15, page 90]). *Let n and m be arbitrary nonnegative integers satisfying $n \leq m$. Then*

$$\sum_{r=0}^m (-1)^{m-r} \binom{m}{r} r^n = \begin{cases} m!, & n = m; \\ 0, & 0 \leq n < m. \end{cases} \quad (2.6)$$

3. The First Class of D-Invariant Polynomial Subspaces

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d) \in (\mathbb{F}^d)^d$ in which

$$\rho_1 = (\rho_{1,1}, \dots, \rho_{1,d}), \quad \rho_2 = (\rho_{2,1}, \dots, \rho_{2,d}), \dots, \quad \rho_d = (\rho_{d,1}, \dots, \rho_{d,d}) \quad (3.1)$$

are $\mathbb{F}$ -linearly independent unit vectors. Then we define a differential operator $D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{F}[\mathbf{x}]$ by the formula:

$$D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} f = \frac{\partial^{\|\boldsymbol{\alpha}\|_1} f}{\partial \rho_1^{\alpha_1} \dots \partial \rho_d^{\alpha_d}}. \quad (3.2)$$

Lemma 3.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d) \in (\mathbb{F}^d)^d$ be as above, and let*

$$p = \prod_{i=1}^d (\rho_i \cdot \mathbf{x})^{\alpha_i}, \quad (3.3)$$

where $\cdot$ denotes the Euclidean product. Then $p(D) = D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}}$.

Given a lower set $\mathfrak{d} \subset \mathbb{N}^d$ and $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d) \in (\mathbb{F}^d)^d$ as above, then the subspace:

$$Q = \text{span}_{\mathbb{F}} \left\{ \prod_{i=1}^d (\rho_i \cdot \mathbf{x})^{\alpha_i} : \boldsymbol{\alpha} \in \mathfrak{d} \right\} \quad (3.4)$$

is obviously D -invariant. The following proposition discretizes this functional subspace $\delta_{\xi} Q(D)$ into a sequence of evaluation functional subspaces.

Proposition 3.2. *Let $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d) \in (\mathbb{F}^d)^d$ be as above, h a nonzero number in $\mathbb{F}$, and let $\boldsymbol{\alpha} \in \mathbb{N}^d$, $\boldsymbol{\xi} \in \mathbb{F}^d$. Then for arbitrary $p \in \mathbb{F}[\mathbf{x}]$*

$$\frac{1}{h^{\|\boldsymbol{\alpha}\|_1}} \sum_{0 \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} (-1)^{\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_1} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} p \left(\boldsymbol{\xi} + h \sum_{i=1}^d \beta_i \rho_i \right) = (D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} p)(\boldsymbol{\xi}) + O(h), \quad (3.5)$$

where $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$, and the remainder $O(h)$ is a polynomial in h .

Proof. For convenience, let

$$\varphi = \sum_{0 \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} (-1)^{\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_1} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} p \left(\boldsymbol{\xi} + h \sum_{k=1}^d \beta_k \rho_k \right). \quad (3.6)$$

Then we can obtain that for $m = 0, \dots, \|\boldsymbol{\alpha}\|_1$:

$$\begin{aligned} \frac{d^m \varphi(0)}{dh^m} &= \sum_{\|\boldsymbol{\gamma}_1\|_1 + \dots + \|\boldsymbol{\gamma}_d\|_1 = m} \rho_{1,1}^{\gamma_{1,1}} \dots \rho_{d,1}^{\gamma_{d,1}} \dots \rho_{1,d}^{\gamma_{1,d}} \dots \rho_{d,d}^{\gamma_{d,d}} \frac{\partial^m p(\boldsymbol{\xi})}{\partial x_1^{\|\boldsymbol{\gamma}_1\|_1} \dots \partial x_d^{\|\boldsymbol{\gamma}_d\|_1}} \\ &\quad \times \left(\begin{matrix} m \\ \gamma_{1,1} \dots \gamma_{1,d}, \dots, \gamma_{d,1}, \dots, \gamma_{d,d} \end{matrix} \right) \prod_{j=1}^d \left(\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} \right), \end{aligned} \quad (3.7)$$

with $\boldsymbol{\gamma}_i = (\gamma_{i,1}, \dots, \gamma_{i,d}) \in \mathbb{N}^d$, $1 \leq i \leq d$.

In the following, we will simplify the right-hand side of the above equality. There are two cases which must be examined.

Case 1 ($0 \leq m \leq \|\boldsymbol{\alpha}\|_1 - 1$). In this case, there must exist some $1 \leq j \leq d$ such that $\sum_{i=1}^d \gamma_{i,j} < \alpha_j$. By Lemma 2.1, it follows for such j :

$$\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} = 0. \quad (3.8)$$

Thus, for all $0 \leq m \leq \|\boldsymbol{\alpha}\|_1 - 1$, $d^m \varphi(0) / dh^m = 0$.

Case 2 ($m = \|\alpha\|_1$). In this case, if there exists some $1 \leq j \leq d$ such that $\sum_{i=1}^d \gamma_{i,j} > \alpha_j$, then there must exist another $1 \leq j' \leq d$, such that $\sum_{i=1}^d \gamma_{i,j'} < \alpha_{j'}$. Hence, when $m = \|\alpha\|_1$,

$$\prod_{j=1}^d \left(\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j-\beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} \right) \neq 0 \quad (3.9)$$

if and only if for all $1 \leq j \leq d$, $\sum_{i=1}^d \gamma_{i,j} = \alpha_j$. Combining this with the fact:

$$\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j-\beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\alpha_j} = \alpha_j!, \quad 1 \leq j \leq d, \quad (3.10)$$

we can conclude that

$$\begin{aligned} \frac{d^{\|\alpha\|_1} \varphi(0)}{dh^{\|\alpha\|_1}} &= \|\alpha\|_1! \sum_{\gamma_{1,1}+\dots+\gamma_{d,1}=\alpha_1} \cdots \sum_{\gamma_{1,d}+\dots+\gamma_{d,d}=\alpha_d} \rho_{1,1}^{\gamma_{1,1}} \cdots \rho_{d,1}^{\gamma_{d,1}} \cdots \rho_{1,d}^{\gamma_{1,d}} \cdots \rho_{d,d}^{\gamma_{d,d}} \\ &\times \binom{\alpha_1}{\gamma_{1,1}, \dots, \gamma_{d,1}} \cdots \binom{\alpha_d}{\gamma_{1,d}, \dots, \gamma_{d,d}} \frac{\partial^{\|\alpha\|_1} p(\xi)}{\partial x_1^{\gamma_{1,1}} \cdots \partial x_d^{\gamma_{d,d}}}. \end{aligned} \quad (3.11)$$

That is

$$\frac{d^{\|\alpha\|_1} \varphi(0)}{dh^{\|\alpha\|_1}} = \|\alpha\|_1! (D_{\rho}^{\alpha} p)(\xi). \quad (3.12)$$

In sum, we have deduced that

$$\frac{d^m \varphi(0)}{dh^m} = \begin{cases} 0, & 0 \leq m \leq \|\alpha\|_1 - 1; \\ \|\alpha\|_1! (D_{\rho}^{\alpha} p)(\xi), & m = \|\alpha\|_1, \end{cases} \quad (3.13)$$

which implies that

$$\varphi = h^{\|\alpha\|_1} (D_{\rho}^{\alpha} p)(\xi) + O(h^{\|\alpha\|_1+1}). \quad (3.14)$$

□

4. The Second Class of D -Invariant Polynomial Subspaces

Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$ with $n \geq 1$ be an $n+1$ -tuple of positive integers satisfying

$$a_0 = 1, \quad a_1 > \cdots > a_n \geq 2, \quad (4.1)$$

and let

$$\mathbf{c}_1 = (c_{1,0}, c_{1,1}, \dots, c_{1,n}), \dots, \quad \mathbf{c}_d = (c_{d,0}, c_{d,1}, \dots, c_{d,n}) \in \mathbb{F}^{n+1} \quad (4.2)$$

satisfying that $c_{1,0}, c_{2,0}, \dots, c_{d,0}$ are not all zero. Then it is clear that

$$\tau(\gamma_1, \dots, \gamma_d) = \sum_{j=0}^n a_j \sum_{i=1}^d \gamma_{i,j}, \quad (4.3)$$

with $\gamma_i = (\gamma_{i,0}, \gamma_{i,1}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}$ defines a map $\tau : (\mathbb{N}^{n+1})^d \rightarrow \mathbb{N}$.

Proposition 4.1. *Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$, $\mathbf{c}_i = (c_{i,0}, c_{i,1}, \dots, c_{i,n})$, $1 \leq i \leq d$, and the map τ be as above. Let $q_{n,m}$, $m = 0, 1, \dots, a_1$ be polynomials defined by*

$$q_{n,m} = \sum_{\tau(\gamma_1, \dots, \gamma_d) = m} \frac{\mathbf{c}_1^{\gamma_1} \cdots \mathbf{c}_d^{\gamma_d}}{\gamma_1! \cdots \gamma_d!} x_1^{\|\gamma_1\|_1} \cdots x_d^{\|\gamma_d\|_1}, \quad (4.4)$$

with $\gamma_i = (\gamma_{i,0}, \gamma_{i,1}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}$, $\mathbf{c}_i^{\gamma_i} = \prod_{j=0}^n c_{i,j}^{\gamma_{i,j}}$, and let Q be a polynomial subspace defined by

$$Q = \text{span}_{\mathbb{F}} \{q_{n,m} : 0 \leq m \leq a_1\}. \quad (4.5)$$

Then, the following hold:

- (i) $\dim Q = a_1 + 1$;
- (ii) Q is a D -invariant polynomial subspace.

Proof. (i) We assume that there exist $k_0, k_1, \dots, k_{a_1-1}, k_{a_1} \in \mathbb{F}$ such that

$$k_0 q_{n,0} + k_1 q_{n,1} + \cdots + k_{a_1-1} q_{n,a_1-1} + k_{a_1} q_{n,a_1} = 0. \quad (4.6)$$

Since there exists some $1 \leq i_0 \leq d$ such that $c_{i_0,0} \neq 0$, then $x_{i_0}^m$ must belong to the support of $q_{n,m}$. Using this together with the definition of τ , we conclude that the degree of $q_{n,m}$ is m .

Specifically, $x_{i_0}^{a_1}$ belongs to the support of q_{n,a_1} , while for all $0 \leq m \leq a_1 - 1$, $x_{i_0}^{a_1}$ cannot belong to the support of $q_{n,m}$. Therefore $k_{a_1} = 0$, which implies that

$$k_0 q_{n,0} + k_1 q_{n,1} + \cdots + k_{a_1-1} q_{n,a_1-1} = 0. \quad (4.7)$$

Arguing for $m = a_1 - 1, a_1 - 2, \dots, 1, 0$ as for $m = a_1$, we get $k_{a_1-1} = 0, k_{a_1-2} = 0, \dots, k_1 = 0, k_0 = 0$, successively. That is to say, the family of polynomials $q_{n,m}$, $0 \leq m \leq a_1$ is $\mathbb{F}$ -linearly independent. So $\dim Q = a_1 + 1$.

(ii) If $1 \leq m < a_n$, then $\tau(\gamma_1, \dots, \gamma_d) = m$ implies that $\gamma_{i,j} = 0$ with $1 \leq i \leq d, 1 \leq j \leq n$. Thus, for all $1 \leq m < a_n$:

$$\begin{aligned} q_{n,m} &= \sum_{\gamma_{1,0} + \dots + \gamma_{d,0} = m} \frac{c_{1,0}^{\gamma_{1,0}} \dots c_{d,0}^{\gamma_{d,0}}}{\gamma_{1,0}! \dots \gamma_{d,0}!} x_1^{\gamma_{1,0}} \dots x_d^{\gamma_{d,0}} \\ &= \frac{1}{m!} (c_{1,0}x_1 + c_{2,0}x_2 + \dots + c_{d,0}x_d)^m. \end{aligned} \quad (4.8)$$

Consequently,

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0} q_{n,m-1}, \quad 1 \leq m < a_n. \quad (4.9)$$

Next, let m be an arbitrary integer satisfying $a_n \leq m \leq a_1$, and s the minimum integer between 1 and n such that $m - a_s \geq 0$. We claim

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0} q_{n,m-1} + \sum_{j=s}^n c_{i,j} q_{n,m-a_j}, \quad a_n \leq m \leq a_1. \quad (4.10)$$

This claim together with equality (4.9) immediately means that Q is D -invariant.

To prove our claim, we will use induction on the number n . When $n = 1$, our claim can be easily verified. Now, assume that our claim is true for $n - 1$. To prove that it holds for n , let k be the maximum nonnegative integer such that $ka_n \leq m$, that is,

$$ka_n \leq m, \quad (k+1)a_n > m \quad (4.11)$$

and k' the maximum nonnegative integer such that $k'a_n \leq m - 1$. From the definition of $q_{n,m}$, we obtain that

$$\begin{aligned} q_{n,m} &= \sum_{l=0}^k \left(\sum_{\tau((\gamma_{1,0}, \dots, \gamma_{1,n-1}), \dots, (\gamma_{d,0}, \dots, \gamma_{d,n-1})) = m - la_n} \frac{c_{1,0}^{\gamma_{1,0}} \dots c_{1,n-1}^{\gamma_{1,n-1}} \dots c_{d,0}^{\gamma_{d,0}} \dots c_{d,n-1}^{\gamma_{d,n-1}}}{\gamma_{1,0}! \dots \gamma_{1,n-1}! \dots \gamma_{d,0}! \dots \gamma_{d,n-1}!} \right. \\ &\quad \left. \times x_1^{\gamma_{1,0} + \dots + \gamma_{1,n-1}} \dots x_d^{\gamma_{d,0} + \dots + \gamma_{d,n-1}} \right) \left(\sum_{\gamma_{1,n} + \dots + \gamma_{d,n} = l} \frac{c_{1,n}^{\gamma_{1,n}} \dots c_{d,n}^{\gamma_{d,n}}}{c_{1,n}! \dots c_{d,n}!} x_1^{\gamma_{1,n}} \dots x_d^{\gamma_{d,n}} \right). \end{aligned} \quad (4.12)$$

That is

$$q_{n,m} = \sum_{l=0}^k \frac{1}{l!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^l q_{n-1,m-la_n}, \quad (4.13)$$

which plays an important role in what follows. At this point, we have two cases to consider.

Case 1 ($a_n \leq m < a_{n-1}$). In this case, $s = n$ and $0 \leq m - la_n < a_{n-1}$ for all $0 \leq l \leq k$. Hence,

$$q_{n-1, m-la_n} = \frac{1}{(m-la_n)!} (c_{1,0}x_1 + \cdots + c_{d,0}x_d)^{m-la_n}, \quad 0 \leq l \leq k. \quad (4.14)$$

By (4.13), we get

$$q_{n,m} = \sum_{l=0}^k \frac{1}{l!(m-la_n)!} (c_{1,n}x_1 + \cdots + c_{d,n}x_d)^l (c_{1,0}x_1 + \cdots + c_{d,0}x_d)^{m-la_n}. \quad (4.15)$$

So we obtain that

$$\begin{aligned} \frac{\partial q_{n,m}}{\partial x_i} &= c_{i,0} \sum_{l=0}^{k'} \frac{1}{l!(m-1-la_n)!} (c_{1,n}x_1 + \cdots + c_{d,n}x_d)^l (c_{1,0}x_1 + \cdots + c_{d,0}x_d)^{m-1-la_n} \\ &\quad + c_{i,n} \sum_{l=1}^k \frac{1}{(l-1)!(m-la_n)!} (c_{1,n}x_1 + \cdots + c_{d,n}x_d)^{l-1} (c_{1,0}x_1 + \cdots + c_{d,0}x_d)^{m-la_n} \\ &= c_{i,0}q_{n,m-1} + c_{i,n}q_{n,m-a_n}. \end{aligned} \quad (4.16)$$

Case 2 ($a_{n-1} \leq m \leq a_1$). In this case, $0 \leq s \leq n-1$.

By (4.11) and (4.1), it follows that $m - ka_n - a_{n-1} < 0$. Thus, there exists some $0 \leq l_0 \leq k-1$ such that

$$m - l_0 a_n - a_{n-1} \geq 0, \quad m - (l_0 + 1)a_n - a_{n-1} < 0. \quad (4.17)$$

As a result, we have that $0 \leq m - la_n < a_{n-1}$ for all $l_0 + 1 \leq l \leq k$, which implies that

$$q_{n-1, m-la_n} = \frac{1}{(m-la_n)!} (c_{1,0}x_1 + \cdots + c_{d,0}x_d)^{m-la_n}, \quad l_0 + 1 \leq l \leq k. \quad (4.18)$$

Therefore,

$$\frac{\partial q_{n-1, m-la_n}}{\partial x_i} = \begin{cases} c_{i,0}q_{n-1, m-1-la_n}, & l_0 + 1 \leq l \leq k'; \\ 0, & k' + 1 \leq l \leq k. \end{cases} \quad (4.19)$$

For all $0 \leq l \leq l_0$, we have $a_{n-1} \leq m - la_n \leq a_1$. Then our inductive hypothesis implies that

$$\frac{\partial q_{n-1, m-la_n}}{\partial x_i} = c_{i,0}q_{n-1, m-1-la_n} + \sum_{j=s_l}^{n-1} c_{i,j}q_{n-1, m-la_n-a_j}, \quad 0 \leq l \leq l_0, \quad (4.20)$$

where s_l is the minimal integer between 1 and $n-1$ such that $m - la_n - a_{s_l} \geq 0$. It should be noticed that $s_0 = s$.

By (4.19) and (4.20), we can deduce that

$$\begin{aligned}
\frac{\partial q_{n,m}}{\partial x_i} &= c_{i,0} \left(\sum_{l=0}^{k'} \frac{1}{l!} (c_{1,n}x_1 + \cdots + c_{d,n}x_d)^l q_{n-1,m-1-la_n} \right) \\
&\quad + c_{i,n} \left(\sum_{l=1}^k \frac{1}{(l-1)!} (c_{1,n}x_1 + \cdots + c_{d,n}x_d)^{l-1} q_{n-1,m-la_n} \right) \\
&\quad + \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \cdots + c_{d,n}x_d)^l \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-la_n-a_j} \\
&= c_{i,0} q_{n,m-1} + c_{i,n} q_{n,m-a_n} + \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \cdots + c_{d,n}x_d)^l \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-la_n-a_j}.
\end{aligned} \tag{4.21}$$

It remains to show that the last row of the above equality and the right-side hand of (4.10) are equal. More precisely, for each $s_0 \leq j \leq n-1$, let k_j denote the maximum nonnegative integer satisfying $k_j a_n \leq m - a_j$, that is,

$$k_j a_n \leq m - a_j, \quad (k_j + 1) a_n > m - a_j. \tag{4.22}$$

From (4.17), we know $k_{n-1} = l_0$.

Due to (4.13), we observe that

$$q_{n,m-a_j} = \sum_{l=0}^{k_j} \frac{1}{l!} (c_{2,n}x_2 + \cdots + c_{d,n}x_d)^l q_{n-1,m-a_j-la_n}, \quad s_0 \leq j \leq n-1. \tag{4.23}$$

Furthermore,

$$q_{n,m-a_j} = \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \cdots + c_{d,n}x_d)^l q_{n-1,m-a_j-la_n}, \quad s_0 \leq j \leq n-1. \tag{4.24}$$

Here, we have used the fact that $q_{n-1,m-a_j-la_n} = 0$ for all $s_0 \leq j \leq n-1$, $k_j + 1 \leq l \leq l_0$.

Now, recall that $s_l, 0 \leq l \leq l_0$ is the minimal integer between 1 and $n-1$ such that $m - la_n - a_{s_l} \geq 0$, then

$$\sum_{j=s_0}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n} = \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n}, \quad 0 \leq l \leq l_0. \tag{4.25}$$

According to (4.24), (4.25), and the fact $s = s_0$, we find that

$$\begin{aligned} \sum_{j=s}^{n-1} c_{i,j} q_{n,m-a_j} &= \sum_{j=s_0}^{n-1} c_{i,j} \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n} x_2 + \cdots + c_{d,n} x_d)^l q_{n-1,m-a_j-la_n} \\ &= \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n} x_2 + \cdots + c_{d,n} x_d)^l \left(\sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n} \right). \end{aligned} \quad (4.26)$$

Consequently,

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0} q_{n,m-1} + \sum_{j=s}^n c_{i,j} q_{n,m-a_j}, \quad a_{n-1} \leq m \leq a_1. \quad (4.27)$$

□

Proposition 4.2. Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$, $\mathbf{c}_i = (c_{i,0}, c_{i,1}, \dots, c_{i,n})$ with $1 \leq i \leq d$, and let $q_{n,m}$ with $0 \leq m \leq a_1$ be as above, h a nonzero number in $\mathbb{F}$. Then for arbitrary $p \in \mathbb{F}[\mathbf{x}]$ and $\xi \in \mathbb{F}^d$:

$$\frac{1}{m!h^m} \sum_{0 \leq r \leq m} (-1)^{m-r} \binom{m}{r} p \left(\xi + \left(\sum_{j=0}^n c_{1,j} (rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j} (rh)^{a_j} \right) \right) = (q_{n,m}(D)p)(\xi) + O(h). \quad (4.28)$$

Proof. Applying Taylor Formulas, we obtain

$$\begin{aligned} &p \left(\xi + \left(\sum_{j=0}^n c_{1,j} (rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j} (rh)^{a_j} \right) \right) \\ &= \sum_{k=0}^{\infty} \sum_{\|\mathbf{r}_1\|_1 + \dots + \|\mathbf{r}_d\|_1 = k} (rh)^{\sum_{1 \leq i \leq d, 0 \leq j \leq n} a_j \gamma_{i,j}} \frac{\mathbf{c}_1^{\mathbf{r}_1} \cdots \mathbf{c}_d^{\mathbf{r}_d}}{\mathbf{r}_1! \cdots \mathbf{r}_d!} \frac{\partial^k p(\xi)}{\partial x_1^{\|\mathbf{r}_1\|_1} \cdots \partial x_d^{\|\mathbf{r}_d\|_1}}, \end{aligned} \quad (4.29)$$

where $\mathbf{c}_i^{\mathbf{r}_i} = \prod_{j=0}^n c_{i,j}^{\gamma_{i,j}}$, $\mathbf{r}_i = (\gamma_{i,0}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}$, $1 \leq i \leq d$.

By means of the map τ defined as in (4.3), the above equality can be rewritten as

$$\begin{aligned} &p \left(\xi + \left(\sum_{j=0}^n c_{1,j} (rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j} (rh)^{a_j} \right) \right) \\ &= \sum_{l=0}^m \sum_{\tau(\mathbf{r}_1, \dots, \mathbf{r}_d) = l} (rh)^l \frac{\mathbf{c}_1^{\mathbf{r}_1} \cdots \mathbf{c}_d^{\mathbf{r}_d}}{\mathbf{r}_1! \cdots \mathbf{r}_d!} \frac{\partial^{\|\mathbf{r}_1\|_1 + \dots + \|\mathbf{r}_d\|_1} p(\xi)}{\partial x_1^{\|\mathbf{r}_1\|_1} \cdots \partial x_d^{\|\mathbf{r}_d\|_1}} + O(h^{m+1}), \end{aligned} \quad (4.30)$$

where $O(h^{m+1})$ is a polynomial in h .

Finally, by Lemma 2.1, we can conclude that

$$\begin{aligned} & \sum_{0 \leq r \leq m} (-1)^{m-r} \binom{m}{r} p \left(\xi + \left(\sum_{j=0}^n c_{1,j} (rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j} (rh)^{a_j} \right) \right) \\ &= m! h^m \sum_{\tau(\gamma_1, \dots, \gamma_d) = m} \frac{c_1^{\gamma_1} \dots c_d^{\gamma_d}}{\gamma_1! \dots \gamma_d!} \frac{\partial^{\|\gamma_1\|_1 + \dots + \|\gamma_d\|_1} p(\xi)}{\partial x_1^{\|\gamma_1\|_1} \dots \partial x_d^{\|\gamma_d\|_1}} + O(h^{m+1}). \end{aligned} \quad (4.31)$$

□

5. Main Theorem

Let $\xi^{(1)}, \dots, \xi^{(\mu)}, \xi^{(\mu+1)}, \dots, \xi^{(\mu+\nu)} \in \mathbb{F}^d$ be distinct points. For each $1 \leq k \leq \mu$, assume that $\mathfrak{d}^{(k)} \subset \mathbb{N}^d$ is a lower set, and $\rho^{(k)} = (\rho_1^{(k)}, \dots, \rho_d^{(k)}) \in (\mathbb{F}^d)^d$ satisfies the conditions of Section 3. Likewise, for each $1 \leq l \leq \nu$, assume that $\mathbf{a}^{(l)} = (a_0^{(l)}, a_1^{(l)}, \dots, a_{n(l)}^{(l)})$, $\mathbf{c}_i^{(l)} = (c_{i,0}^{(l)}, c_{i,1}^{(l)}, \dots, c_{i,n(l)}^{(l)})$, $1 \leq i \leq d$, and $q_{n(l),m}^{(l)}$, $0 \leq m \leq a_1^{(l)}$ satisfy the conditions of Section 4.

Theorem 5.1. *With the notation above, let P be an ideal projector with*

$$\text{ran } P' = \text{span}_{\mathbb{F}} \left\{ \delta_{\xi^{(k)}} D_{\rho^{(k)}}^{\alpha} \delta_{\xi^{(\mu+l)}} q_{n(l),m}^{(l)}(D) : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \right\}, \quad (5.1)$$

and let P_h be a Lagrange projector with

$$\text{ran } P'_h = \text{span}_{\mathbb{F}} \left\{ \delta_{\xi^{(k)} + h \sum_{i=1}^d \alpha_i \rho_i^{(k)}} \delta_{\xi^{(\mu+l)} + \phi^{(l)}(mh)} : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \right\}, \quad (5.2)$$

where $h \in \mathbb{F} \setminus \{0\}$ and

$$\phi^{(l)}(mh) = \left(\sum_{j=0}^{n(l)} c_{1,j}^{(l)} (mh)^{a_j^{(l)}}, \sum_{j=0}^{n(l)} c_{2,j}^{(l)} (mh)^{a_j^{(l)}}, \dots, \sum_{j=0}^{n(l)} c_{d,j}^{(l)} (mh)^{a_j^{(l)}} \right). \quad (5.3)$$

Then, the following statements hold:

(i) *there exists a positive $\eta \in \mathbb{F}$ such that*

$$\text{ran } P_h = \text{ran } P, \quad \forall 0 < |h| < \eta; \quad (5.4)$$

(ii) *P is the pointwise limit of P_h , $0 < |h| < \eta$, as h tends to zero.*

Proof. Firstly, one can easily verify that

$$\begin{aligned}\lambda &= \left(\delta_{\xi^{(k)}} D_{\rho^{(k)}}^{\alpha}, \delta_{\xi^{(\mu+l)}} q_{n(l),m}^{(l)}(D) : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \right) \in (\mathbb{F}[\mathbf{x}])'^s, \\ \lambda_h &= \left(\delta_{\xi^{(k)} + h \sum_{i=1}^d \alpha_i \rho_i^{(k)}}, \delta_{\xi^{(\mu+l)} + \phi^{(l)}(mh)} : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \right) \in (\mathbb{F}[\mathbf{x}])'^s, \end{aligned} \quad (5.5)$$

form $\mathbb{F}$ -bases for $\text{ran } P'$ and $\text{ran } P'_h$, respectively, where

$$s = \sum_{k=1}^{\mu} \# \mathfrak{d}^{(k)} + \sum_{k=1}^{\nu} a_1^{(k)} + \nu. \quad (5.6)$$

Provided that the entries of λ and λ_h are arranged in the same order, namely, for arbitrary fixed $1 \leq k \leq \mu$, $\alpha \in \mathfrak{d}^{(k)}$, the corresponding entries of λ and λ_h are in the same position, the same as for arbitrary fixed $1 \leq l \leq \nu$, $0 \leq m \leq a_1^{(l)}$. We denote $\lambda = (\lambda_1, \dots, \lambda_s)$ and $\lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,s})$, respectively.

Let $\mathbf{q} = (q_1, q_2, \dots, q_s)$ be an $\mathbb{F}$ -basis for $\text{ran } P$. For convenience, we introduce two $s \times s$ matrices:

$$\lambda^T \mathbf{q} := (\lambda_i q_j)_{1 \leq i, j \leq s}, \quad \lambda_h^T \mathbf{q} := (\lambda_{h,i} q_j)_{1 \leq i, j \leq s} \quad (5.7)$$

and for arbitrary $f \in \mathbb{F}[\mathbf{x}]$, vectors:

$$\lambda^T f := (\lambda_i f)_{1 \leq i \leq s}, \quad \lambda_h^T f := (\lambda_{h,i} f)_{1 \leq i \leq s}. \quad (5.8)$$

For arbitrary $p \in \mathbb{F}[\mathbf{x}]$, we have the following facts according to equality (3.5) and (4.28).

- (1) For fixed $1 \leq k \leq \mu$ and $\alpha \in \mathfrak{d}^{(k)}$, $(D_{\rho^{(k)}}^{\alpha} p)(\xi^{(k)})$ can be linearly expressed by

$$\left\{ p \left(\xi^{(k)} + h \sum_{i=1}^d \beta_i \rho_i^{(k)} \right) : \beta \in \mathfrak{d}^{(k)} \right\} \cup \{O(h)\}, \quad (5.9)$$

since $\mathfrak{d}^{(k)}$ is lower, and moreover, the linear combination coefficient of each $p(\xi^{(k)} + h \sum_{i=1}^d \beta_i \rho_i^{(k)})$ is independent of $p \in \mathbb{F}[\mathbf{x}]$.

- (2) For fixed $1 \leq l \leq \nu$ and $0 \leq m \leq a_1^{(l)}$, $(q_{n(l),m}^{(l)}(D)p)(\xi^{(\mu+l)})$ can be linearly expressed by

$$\left\{ p \left(\xi^{(\mu+l)} + \phi^{(l)}(rh) \right) : 0 \leq r \leq m \right\} \cup \{O(h)\}. \quad (5.10)$$

Also, the linear combination coefficient of each $p(\xi^{(\mu+l)} + \phi^{(l)}(rh))$ is independent of $p \in \mathbb{F}[\mathbf{x}]$.

In brief, we can conclude that there exists a nonsingular matrix T such that

$$\left[\widehat{\lambda_h^T \mathbf{q}} \mid \widehat{\lambda_h^T f} \right] := T \left[\lambda_h^T \mathbf{q} \mid \lambda_h^T f \right] = \left[\lambda^T \mathbf{q} \mid \lambda^T f \right] + [E_h \mid \epsilon_h], \quad (5.11)$$

where each entry of $[E_h \mid \epsilon_h]$ has the same order as h . As a consequence, the linear systems:

$$\left(\widehat{\lambda_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\lambda_h^T f}, \quad \left(\lambda_h^T \mathbf{q} \right) \mathbf{x} = \lambda_h^T f \quad (5.12)$$

are equivalent, namely, they have the same set of solutions.

- (i) From (5.11), it follows that each entry of matrix $\widehat{\lambda_h^T \mathbf{q}}$ converges to its corresponding entry of matrix $\lambda^T \mathbf{q}$ as h tends to zero, which implies that

$$\lim_{h \rightarrow 0} \det \left(\widehat{\lambda_h^T \mathbf{q}} \right) = \det \left(\lambda^T \mathbf{q} \right). \quad (5.13)$$

Since $\det(\lambda^T \mathbf{q}) \neq 0$, there exists $\eta > 0$ such that

$$\det \left(\widehat{\lambda_h^T \mathbf{q}} \right) \neq 0, \quad 0 < |h| < \eta. \quad (5.14)$$

Notice that (5.11) directly leads to $\text{rank}(\widehat{\lambda_h^T \mathbf{q}}) = \text{rank}(\lambda_h^T \mathbf{q})$:

$$\text{ran } P_h = \text{span}_{\mathbb{F}} \mathbf{q}, \quad 0 < |h| < \eta, \quad (5.15)$$

follows, that is, $\mathbf{q}$ forms an $\mathbb{F}$ -basis for $\text{ran } P_h$. Since $\mathbf{q}$ is also an $\mathbb{F}$ -basis for $\text{ran } P$, we have

$$\text{ran } P = \text{ran } P_h, \quad 0 < |h| < \eta. \quad (5.16)$$

- (ii) Suppose that $\tilde{\mathbf{x}}$ and $\mathbf{x}_0$ be the unique solutions of nonsingular linear systems:

$$\left(\lambda_h^T \mathbf{q} \right) \mathbf{x} = \lambda_h^T f, \quad (5.17)$$

$$\left(\lambda^T \mathbf{q} \right) \mathbf{x} = \lambda^T f, \quad (5.18)$$

respectively, where $f \in \mathbb{F}[\mathbf{x}]$ and $0 < |h| < \eta$. It is easy to see that

$$P_h f = \mathbf{q} \tilde{\mathbf{x}}, \quad P f = \mathbf{q} \mathbf{x}_0. \quad (5.19)$$

Notice that, as $h \rightarrow 0$, P is the pointwise limit of P_h if and only if Pf is the coefficient-wise limit of $P_h f$ for all $f \in \mathbb{F}[\mathbf{x}]$. Therefore, it is sufficient to show that for every $f \in \mathbb{F}[\mathbf{x}]$, the solution vector of system (5.17) converges to the one of system (5.18) as h tends to zero, namely,

$$\lim_{h \rightarrow 0} \tilde{\mathbf{x}} = \mathbf{x}_0. \quad (5.20)$$

By (5.11), the linear system:

$$\left(\widehat{\lambda_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\lambda_h^T f} \quad (5.21)$$

can be rewritten as

$$\left(\lambda^T \mathbf{q} + E_h \right) \mathbf{x} = \left(\lambda^T f + \epsilon_h \right). \quad (5.22)$$

Since system (5.21) is equivalent to system (5.17), $\tilde{\mathbf{x}}$ is also the unique solution of it. Consequently, using the perturbation analysis of the sensitivity of linear systems (see, e.g., [16, page 80]), we have

$$\|\tilde{\mathbf{x}} - \mathbf{x}_0\| \leq \left\| \left(\lambda^T \mathbf{q} \right)^{-1} \right\| \|\epsilon_h - E_h \mathbf{x}_0\| + O(h^2). \quad (5.23)$$

Since each component of vector $\epsilon_h - E_h \mathbf{x}_0$ has the same order as h , it follows that $\lim_{h \rightarrow 0} \|\tilde{\mathbf{x}} - \mathbf{x}_0\| = 0$, or, equivalently, $\lim_{h \rightarrow 0} \tilde{\mathbf{x}} = \mathbf{x}_0$, which completes the proof of the theorem. $\square$

6. Conclusions

The main work of this paper is to discretize a class of ideal projectors with two special classes of D -invariant polynomial subspaces into a sequence of Lagrange projectors.

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