Research Article

Analytical Solutions for the Flow of a Fractional Second Grade Fluid due to a Rotational Constantly Accelerating Shear

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Exact analytic solutions are obtained for the flow of a generalized second grade fluid in an annular region between two infinite coaxial cylinders. The fractional calculus approach in the governing equations of a second grade fluid is used. The exact analytic solutions are constructed by means of Laplace and finite Hankel transforms. The motion is produced by the inner cylinder which is rotating about its axis due to a constantly accelerating shear. The solutions that have been obtained satisfy both the governing equations and all imposed initial and boundary conditions. Moreover, they can be easily specialized to give similar solutions for second grade and Newtonian fluids. Finally, the influence of the pertinent parameters on the fluid motion, as well as a comparison between the three models, is underlined by graphical illustrations.

1. Introduction

The study of the non-Newtonian fluids has recently achieved much importance because of well-established applications in a number of processes that occur in industry. Such applications include the extrusion of polymer fluids, cooling of the metallic plate in a bath, animal bloods, foodstuffs, exotic lubricants and colloidal and suspension solutions. For these fluids, the classical Navier-Stokes theory is inadequate. Because of their complexity, there are several models of non-Newtonian fluids in the literature. One of the most popular models for non-Newtonian fluids is the model that is called second-grade fluid [1]. Although there are some criticisms regarding the applications of this model [2], it has been shown by Walters [3] that, for many types of problems in which the flow is slow enough in the viscoelastic sense,

the results given using Oldroyd fluid will be substantially similar to those obtained for second grade fluid. Thus, if this is the manner of interpretation of the results, it is reasonable to use the second-grade fluid [4–6] to carry out the calculations. This is particularly so because of the fact that the calculations are generally simpler. This is true not only for exact analytic solutions but even for numerical solutions. The second-grade fluid is the simplest subclass of non-Newtonian fluids for which one can reasonably hope to obtain exact analytic solutions. Moreover, the exact analytic solutions are very important for several reasons. They provide a standard for checking the accuracies of many approximate solutions which can be numerical or empirical. These exact solutions can also be used as tests for verifying numerical schemes that are developed for studying more complex flow problems. Therefore, various researchers [7–9] are engaged in obtaining exact solutions.

Recently, the fractional calculus has encountered much success in the description of complex dynamics. In particular, it has been proved to be a valuable tool for handling viscoelastic properties. The starting point of the fractional derivative model of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional operator. This generalization allows one to define precisely noninteger order derivatives. The fractional calculus has been found to be quite flexible in describing viscoelastic behavior of fluids. In many different situations fractional calculus has been used to handle various rheological problems [10–21].

The aim of this note is to provide exact solutions for the flow of a generalized second grade fluid in the annular region between two infinite coaxial circular cylinders. The motion is produced by the inner cylinder that applies a time-dependent couple to the fluid. More exactly, we would like to extend the results from [7, Section 5] to a larger class of fluids and to a time-dependent couple on the boundary. The general solutions, obtained by means of the integral transforms, will be easily specialized to give the similar solutions for Newtonian and ordinary second grade fluids. Finally, the influence of the pertinent parameters on the fluid motion, as well as a comparison between models, will be underlined by graphical illustrations.

2. Governing Equations

The flows to be here considered have the velocity field of the form [22, 23]

$$\mathbf{v} = \mathbf{v}(r, t) = w(r, t)\mathbf{e}_\theta, \quad (2.1)$$

where $\mathbf{e}_\theta$ is the unit vector along the θ -direction of the cylindrical coordinate system r, θ , and z . For such flows the constraint of incompressibility is automatically satisfied. The nontrivial shear stress $\tau(r, t) = S_{r\theta}(r, t)$ corresponding to such a motion of a second grade fluid is given by [24]

$$\tau(r, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \left[\frac{\partial w(r, t)}{\partial r} - \frac{w(r, t)}{r} \right], \quad (2.2)$$

where μ is the viscosity and α_1 is a material modulus. In the absence of a pressure gradient in the flow direction and neglecting the body forces, the balance of the linear momentum leads to the relevant equation [25, 26]

$$\rho \frac{\partial w(r, t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) \tau(r, t). \quad (2.3)$$

Eliminating $\tau(r, t)$ between (2.2) and (2.3), we get the governing equation

$$\frac{\partial w(r, t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) w(r, t), \quad (2.4)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, ρ is its constant density and $\alpha = \alpha_1/\rho$.

Generally, governing equations for generalized fluids with fractional derivatives are derived from those of the ordinary fluids by replacing the inner time derivatives of an integer order with the so-called Riemann-Liouville operator [11, 27]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau, \quad 0 \leq \beta < 1, \quad (2.5)$$

where $\Gamma(\cdot)$ is the Gamma function.

Consequently, the governing equations corresponding to the motion (2.1) of a generalized second grade fluid are (cf. [22, Equations (2) and (4)])

$$\begin{aligned} \frac{\partial w(r, t)}{\partial t} &= \left(\nu + \alpha D_t^\beta \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) w(r, t); \\ \tau(r, t) &= \left(\mu + \alpha_1 D_t^\beta \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) w(r, t), \end{aligned} \quad (2.6)$$

where the new material constant α_1 (for simplicity, we are keeping the same notation) goes to the initial α_1 for $\beta \rightarrow 1$.

In this paper, we are interested into the motion of a generalized second grade fluid whose governing equations are given by (2.6). The fractional partial differential equations (2.6), with adequate initial and boundary conditions, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient, and powerful tool. The Laplace transform will be used to eliminate the time variable and the finite Hankel transform to remove the spatial variable. However, in order to avoid the lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform will be used.

3. Rotational Flow between Two Infinite Cylinders

Consider an incompressible generalized second grade fluid at rest in the annular region between two infinitely long coaxial cylinders. At time $t = 0^+$, let the inner cylinder of radius R_1 be set in rotation about its axis by a time-dependent torque per unit length $2\pi R_1 f t$ and

let the outer cylinder of radius R_2 be held stationary. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form (2.1). The governing equations are given by (2.6) and the appropriate initial and boundary conditions are (see also [7, Equations (5.2) and (5.3)])

$$w(r, 0) = 0; \quad r \in [R_1, R_2], \quad (3.1)$$

$$\tau(R_1, t) = \left(\mu + \alpha_1 D_t^\beta \right) \left(\frac{\partial w(r, t)}{\partial r} - \frac{w(r, t)}{r} \right) \Big|_{r=R_1} = ft, \quad w(R_2, t) = 0; \quad t \geq 0, \quad (3.2)$$

where f is a constant.

3.1. Calculation of the Velocity Field

Applying the Laplace transform to (2.6)₁ and (3.2), we get

$$q\bar{w}(r, q) = \left(\nu + \alpha q^\beta \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{w}(r, q), \quad (3.3)$$

$$\bar{\tau}(R_1, q) = \left(\mu + \alpha_1 q^\beta \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{w}(r, q) \Big|_{r=R_1} = \frac{f}{q^2}; \quad \bar{w}(R_2, q) = 0, \quad (3.4)$$

where $\bar{w}(r, q)$ and $\bar{\tau}(R_1, q)$ are the Laplace transforms of the functions $w(r, t)$ and $\tau(R_1, t)$, respectively.

We denote by [22, Equation (34)]

$$\bar{w}_H(r_n, q) = \int_{R_1}^{R_2} r \bar{w}(r, q) B(rr_n) dr, \quad (3.5)$$

the finite Hankel transform of the function $\bar{w}(r, q)$, where

$$B(rr_n) = J_1(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_1(rr_n), \quad (3.6)$$

where r_n are the positive roots of the equation $B(R_2r) = 0$, and $J_p(\cdot)$, $Y_p(\cdot)$ are the Bessel functions of the first and second kind of order p .

The inverse Hankel transform of $\bar{w}_H(r_n, q)$ is given by [22, Equation (35)]

$$\bar{w}(r, q) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_1^2(R_2r_n) B(rr_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)} \bar{w}_H(r_n, q). \quad (3.7)$$

By means of $(3.4)_2$ and of the identity

$$J_1(z)Y_2(z) - J_2(z)Y_1(z) = -\frac{2}{\pi z}, \quad (3.8)$$

we can easily prove that

$$\int_{R_1}^{R_2} r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{w}(r, q) B(rr_n) dr = -r_n^2 \bar{w}_H(r_n, q) + \frac{2}{\pi r_n} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{w}(r, q) |_{r=R_1}. \quad (3.9)$$

Combining (3.3), (3.4), and (3.9), we find that

$$\bar{w}_H(r_n, q) = \frac{2f}{\pi r_n} \frac{1}{q^2} \frac{1}{\rho q + \alpha_1 q^\beta r_n^2 + \mu r_n^2}. \quad (3.10)$$

Writing $\bar{w}_H(r_n, q)$ under the equivalent form

$$\bar{w}_H(r_n, q) = \frac{2f}{\mu \pi r_n^3} \left[\frac{1}{q^2} - \frac{1 + \alpha r_n^2 q^{\beta-1}}{q \{ q + (v + \alpha q^\beta) r_n^2 \}} \right] = \frac{2f}{\mu \pi r_n^3} \left[\frac{1}{q^2} - \frac{q^{-\beta-1} + \alpha r_n^2 q^{-2}}{(q^{1-\beta} + \alpha r_n^2) + v r_n^2 q^{-\beta}} \right], \quad (3.11)$$

and applying the inverse Hankel transform and using the identities

$$\frac{1}{(q^{1-\beta} + \alpha r_n^2) + v r_n^2 q^{-\beta}} = \sum_{k=0}^{\infty} \frac{(-v r_n^2)^k q^{-\beta k}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}, \quad (3.12)$$

$$\pi \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(rr_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} = \frac{1}{2} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right), \quad (3.13)$$

we find that

$$\begin{aligned} \bar{w}(r, q) &= \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) \frac{1}{q^2} - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(rr_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-v r_n^2)^k}{(q^{1-\beta} + \alpha r_n^2)^{k+1}} \left[q^{-\beta k - \beta - 1} + \alpha r_n^2 q^{-\beta k - 2} \right]. \end{aligned} \quad (3.14)$$

Now applying the inverse Laplace transform to (3.14), we find for the velocity field the expression

$$w(r, t) = \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \\ \times \sum_{k=0}^{\infty} \left(-\nu r_n^2 \right)^k \left[G_{1-\beta, -\beta k - \beta - 1, k+1} \left(-\alpha r_n^2, t \right) + \alpha r_n^2 G_{1-\beta, -\beta k - 2, k+1} \left(-\alpha r_n^2, t \right) \right], \quad (3.15)$$

where the generalized function $G_{a,b,c}(d, t)$ is defined by [28, Equations (97) and (101)]

$$G_{a,b,c}(d, t) = L^{-1} \left\{ \frac{q^b}{(q^a - d)^c} \right\} \\ = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c+j)}{\Gamma(c) \Gamma(j+1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}; \quad \text{Re}(ac-b) > 0, \left| \frac{d}{q^a} \right| < 1. \quad (3.16)$$

3.2. Calculation of the Shear Stress

Applying the Laplace transform to (2.6)₂, we find that

$$\bar{\tau}(r, q) = \left(\mu + \alpha_1 q^\beta \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{w}(r, q). \quad (3.17)$$

In order to get a suitable form for $\tau(r, t)$, we rewrite (3.10) under the equivalent form

$$\bar{w}_H(r_n, q) = \frac{2f}{\pi r_n^3} \frac{1}{q^2 (\mu + \alpha_1 q^\beta)} - \frac{2f}{\pi r_n^3} \frac{1}{q (\mu + \alpha_1 q^\beta) (q + \alpha q^\beta r_n^2 + \nu r_n^2)}. \quad (3.18)$$

Applying the inverse Hankel transform to (3.18) and using (3.7) and the identity (3.13), we find that

$$\bar{w}(r, q) = \frac{f}{2} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) \frac{1}{q^2 (\mu + \alpha_1 q^\beta)} \\ - \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \frac{1}{q (\mu + \alpha_1 q^\beta) (q + \alpha q^\beta r_n^2 + \nu r_n^2)}. \quad (3.19)$$

Figure 1: Profiles of the velocity $\omega(r, t)$ and the shear stress $\tau(r, t)$ given by (3.15) and (3.22), for $f = -1$, $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.000259$, $\mu = 0.246$, $\alpha = 0.00153$, $\beta = 0.3$, and different values of t .

Introducing (3.19) into (3.17), it results that

$$\bar{\tau}(r, q) = \left(\frac{R_1}{r}\right)^2 f \frac{1}{q^2} + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \frac{1}{q(q + \alpha q^\beta r_n^2 + \nu r_n^2)}, \quad (3.20)$$

or equivalently (see also (3.12))

$$\bar{\tau}(r, q) = \left(\frac{R_1}{r}\right)^2 f \frac{1}{q^2} + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 1}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}, \quad (3.21)$$

where $B_1(r r_n) = J_2(r r_n) Y_2(R_1 r_n) - J_2(R_1 r_n) Y_2(r r_n)$.

Now taking the inverse Laplace transform of both sides of (3.21), we get

$$\tau(r, t) = \left(\frac{R_1}{r}\right)^2 f t + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{1-\beta, -\beta k - \beta - 1, k+1}(-\alpha r_n^2, t). \quad (3.22)$$

Figure 2: Profiles of the velocity $\omega(r, t)$ and the shear stress $\tau(r, t)$ given by (3.15) and (3.22), for $f = -1$, $R_1 = 0.3$, $R_2 = 0.5$, $\alpha = 0.00153$, $\beta = 0.3$ $t = 10$ s and different values of ν .

4. The Special Case $\beta \rightarrow 1$

Making $\beta \rightarrow 1$ into (3.15) and (3.22), we obtain the similar solutions

$$\begin{aligned}
 w_{\text{SG}}(r, t) &= \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \\
 &\quad \times \left(1 + \alpha r_n^2 \right) \sum_{k=0}^{\infty} \left(-\nu r_n^2 \right)^k G_{0, -(k+2), k+1} \left(-\alpha r_n^2, t \right), \\
 \tau_{\text{SG}}(r, t) &= \left(\frac{R_1}{r} \right)^2 f t \\
 &\quad + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \sum_{k=0}^{\infty} \left(-\nu r_n^2 \right)^k G_{0, -(k+2), k+1} \left(-\alpha r_n^2, t \right),
 \end{aligned} \tag{4.1}$$

corresponding to a second grade fluid performing the same motion.

Now, in view of the identity

$$\sum_{k=0}^{\infty} \left(-\nu r_n^2 \right)^k G_{0, -(k+2), k+1} \left(-\alpha r_n^2, t \right) = \frac{1}{\nu r_n^2} \left[1 - \exp \left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right], \tag{4.2}$$

equation (4.1) can be written under the simplified forms

$$w_{SG}(r, t) = \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t - \frac{\pi f}{\mu\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n^3 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \left(1 + \alpha r_n^2 \right) \left\{ 1 - \exp \left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right\}, \quad (4.3)$$

$$\tau_{SG}(r, t) = \left(\frac{R_1}{r} \right)^2 f t + \frac{\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{r_n^2 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \left[1 - \exp \left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right]. \quad (4.4)$$

The velocity field can be also processed to give the equivalent form

$$w_{SG}(r, t) = \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) \left(t - \frac{\alpha_1}{\mu} \right) - \frac{\pi f}{\mu\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n^3 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \left\{ 1 - \left(1 + \alpha r_n^2 \right) \exp \left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right\}. \quad (4.5)$$

Making α_1 and then $\alpha \rightarrow 0$ into (4.3) and (4.4), the velocity field

$$w_N(r, t) = \frac{f}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t - \frac{\pi f}{\mu\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n^3 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \left[1 - \exp(-\nu r_n^2 t) \right] \quad (4.6)$$

and the associated shear stress

$$\tau_N(r, t) = \left(\frac{R_1}{r} \right)^2 f t + \frac{\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{r_n^2 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \left[1 - \exp(-\nu r_n^2 t) \right], \quad (4.7)$$

corresponding to a Newtonian fluid, are obtained.

5. Conclusions

The purpose of this note is to provide exact analytic solutions for the velocity field $w(r, t)$ and the shear stress $\tau(r, t)$ corresponding to the unsteady rotational flow of a generalized second grade fluid between two infinite coaxial cylinders, the inner cylinder being set in rotation about its axis by a constantly accelerating shear. The solutions that have been obtained, presented under series form in terms of usual Bessel ($J_1(\cdot)$, $J_2(\cdot)$, $Y_1(\cdot)$, and $Y_2(\cdot)$) and generalized $G_{a,b,c}(\cdot, t)$ functions, satisfy all imposed initial and boundary conditions. They can be easily specialized to give the similar solutions for ordinary second grade and Newtonian fluids. Furthermore, in view of some recent results [29, Equation (3.15)], our velocity field (4.3) is in accordance with that obtained in [7, Equation (5.17)] by a different technique.

Figure 3: Profiles of the velocity $\omega(r, t)$ and the shear stress $\tau(r, t)$ given by (3.15) and (3.22), for $f = -1$, $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.000259$, $\mu = 0.246$, $\beta = 0.3$ and different values of α .

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity $w(r, t)$ and of the shear stress $\tau(r, t)$ are depicted against r for different values of the time t and of the pertinent parameters. From Figures 1(a) and 1(b), containing the diagrams of the velocity and the shear stress at several times, it clearly results in the influence of the rigid boundary on the fluid motion. The velocity is an increasing function of t . For the same values of the parameters, the shear stress, in absolute value, is also an increasing function of t . The influence of the kinematic viscosity ν on the fluid motion is shown in Figures 2(a) and 2(b). The velocity is a decreasing function of ν . The shear stress, in absolute value, on the first part of the flow domain, near the moving cylinder, is a decreasing function of ν . It is an increasing function of ν in the neighborhood of the stationary cylinder. Figure 3 shows the influence of the parameter α on the flow motion. Both the velocity and the shear stress, in absolute value, on the first part of the flow domain, near the moving cylinder, are increasing functions of α . They are decreasing functions of α in the neighborhood of the stationary cylinder. The influence of the fractional parameter β on the fluid motion is shown in Figure 4. Its effect on the fluid motion is qualitatively opposite to that of parameter α .

Finally, for comparison, the profiles of $w(r, t)$ and $\tau(r, t)$ corresponding to the motion of the three models (Newtonian, ordinary second grade, and generalized second grade) are together depicted in Figure 5, for the same values of t and of the common material parameters. In all the cases the velocity of the fluid is a decreasing function with respect to r , and the Newtonian fluid is the swiftest, while the generalized second grade fluid is the slowest in the region near the moving cylinder. The units of the material constants are SI units within all figures, and the roots r_n have been approximated by $2(n-1)\pi/[2(R_2 - R_1)]$.

Figure 4: Profiles of the velocity $\omega(r,t)$ and the shear stress $\tau(r,t)$ given by (3.15) and (3.22), for $f = -1$, $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.000259$, $\mu = 0.246$, $\alpha = 0.00153$ and different values of β .

Figure 5: Profiles of the velocity $\omega(r,t)$ and the shear stress $\tau(r,t)$ corresponding to Newtonian, second grade and generalized second grade fluids, for $f = -1$, $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.000259$, $\mu = 0.246$, $\alpha = 0.00153$, $\beta = 0.3$ and $t = 10$ s.

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