Research Article

Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space

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The purpose of this paper is to present some fixed point theorem in dislocated quasimetric space for expansive type mappings.

1. Introduction and Preliminaries

It is well known that Banach Contraction mappings principle is one of the pivotal results of analysis. Generalizations of this principle have been obtained in several directions. Dass and Gupta [1] generalized Banach's Contraction principle in metric space. Also Rhoades [2] established a partial ordering for various definitions of contractive mappings. In 2005, Zeyada et al. [3] established a fixed point theorem in dislocated quasimetric spaces. In 2008, Aage and Salunke [4] proved some results on fixed point in dislocated quasimetric spaces. Recently, Isufati [5], proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric spaces. The following definitions will be needed in the sequel.

Definition 1.1 (see [3]). Let X be a nonempty set, and let $d : X \times X \rightarrow [0, \infty)$ be a function, called a distance function. One needs the following conditions:

$$(M1) \quad d(x, x) = 0,$$

$$(M2) \quad d(x, y) = d(y, x) = 0, \text{ then } x = y,$$

$$(M3) \quad d(x, y) = d(y, x),$$

$$(M4) \quad d(x, y) \leq d(x, z) + d(z, y),$$

$$(M4)' \quad d(x, y) \leq \max\{d(x, z), d(z, y)\}, \text{ for all } x, y, z \in X.$$

If d satisfies conditions (M1)–(M4), then it is called a metric on X . If d satisfies conditions (M1), (M2), and (M4), it is called a quasimetric on X . If it satisfies conditions (M2)–(M4) ((M2) and (M4)), it is called a dislocated metric (or simply d -metric) (a dislocated quasimetric (or simply dq -metric)) on X , respectively. If a metric d satisfies the strong triangle inequality (M4)', then it is called an ultrametric.

Definition 1.2 (see [3]). A sequence $\{x_n\}_{n \in \mathbb{N}}$ in dq -metric space (dislocated quasimetric space) (X, d) is called a Cauchy sequence if, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$, that is, $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$ for all $m, n \geq n_0$.

Definition 1.3 (see [3]). A sequence $\{x_n\}_{n \in \mathbb{N}}$ in dq -metric space [d -metric space] is said to be d -converge to $x \in X$ provided that

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0. \quad (1.1)$$

In this case, x is called a dq -limit [d -limit] of $\{x_n\}$ and we write $x_n \rightarrow x$.

Definition 1.4 (see [3]). A dq -metric space (X, d) is called complete if every Cauchy sequence in it is a dq -convergent.

Example 1.5. Let $X = [0, 1]$. Define $d : X \times X \rightarrow \mathbb{R}_0^+$ by $d(x, y) = \max\{x, y\}$. Then the pair (X, d) is a dislocated metric space. We define an arbitrary sequence $\{x_n\}$ in X by $x_n = 2/(2^n + 1)$, $n \in \mathbb{N} \cup \{0\}$. Let $\epsilon = \sup_{n \in \mathbb{N}} \{2/(2^n + 1)\}$, then, for $n, m \in \mathbb{N}$ and $n > m$, we have $d(x_n, x_m) = d(2/(2^n + 1), 2/(2^m + 1)) = 2/(2^m + 1) \leq \epsilon$. Thus, $\{x_n\}$ is a Cauchy sequence in X . Also as $n \rightarrow \infty$, then $x_n \rightarrow 0 \in X$. Hence, every Cauchy sequence in X is convergent with respect to d . Thus, (X, d) is a complete dislocated metric space.

2. Main Results

In this paper, we prove some fixed point theorem for continuous mapping satisfying expansion condition in complete dq -metric space.

Theorem 2.1. Let (X, d) be a complete dislocated metric space and T a continuous mapping satisfying the following condition:

$$d(Tx, Ty) + \alpha \max\{d(x, Ty), d(y, Tx)\} \geq \beta \cdot \frac{d(x, Tx)[1 + d(y, Ty)]}{1 + d(x, y)} + \gamma \cdot d(x, y) \quad (2.1)$$

for all $x, y \in X$, $x \neq y$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $\beta + \gamma > 1 + 2\alpha$, $\gamma > 1 + \alpha$. Then, T has a fixed point in X .

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows and $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \dots$. Then, using (2.1), we obtain

$$\begin{aligned}
& d(Tx_{n+1}, Tx_{n+2}) + \alpha \max\{d(x_{n+1}, Tx_{n+2}), d(x_{n+2}, Tx_{n+1})\} \\
& \geq \beta \cdot \frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_{n+2}, Tx_{n+2})]}{1 + d(x_{n+1}, x_{n+2})} + \gamma d(x_{n+1}, x_{n+2}) \\
& \Rightarrow d(x_n, x_{n+1}) + \alpha \max\{d(x_{n+1}, x_{n+1}), d(x_{n+2}, x_n)\} \\
& \geq \beta \cdot \frac{d(x_{n+1}, x_n)[1 + d(x_{n+2}, x_{n+1})]}{1 + d(x_{n+1}, x_{n+2})} + \gamma \cdot d(x_{n+1}, x_{n+2}) \\
& \Rightarrow d(x_n, x_{n+1}) + \alpha \max\{d(x_{n+1}, x_{n+1}), d(x_{n+2}, x_n)\} \\
& \geq \beta \cdot \frac{d(x_{n+1}, x_n)[1 + d(x_{n+1}, x_{n+2})]}{1 + d(x_{n+1}, x_{n+2})} + \gamma \cdot d(x_{n+1}, x_{n+2}) \\
& \Rightarrow d(x_n, x_{n+1}) + \alpha d(x_n, x_{n+2}) \geq \beta \cdot d(x_n, x_{n+1}) + \gamma \cdot d(x_{n+1}, x_{n+2}) \\
& \Rightarrow d(x_n, x_{n+1}) + \alpha d(x_n, x_{n+1}) + \alpha d(x_{n+1}, x_{n+2}) \geq \beta \cdot d(x_n, x_{n+1}) + \gamma \cdot d(x_{n+1}, x_{n+2}) \\
& \Rightarrow (1 + \alpha - \beta)d(x_n, x_{n+1}) \geq (\gamma - \alpha)d(x_{n+1}, x_{n+2}).
\end{aligned} \tag{2.2}$$

The last inequality gives

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) & \leq \frac{1 + \alpha - \beta}{\gamma - \alpha} d(x_n, x_{n+1}) \\
& = \delta d(x_n, x_{n+1}),
\end{aligned} \tag{2.3}$$

where $\delta = (1 + \alpha - \beta)/(\gamma - \alpha) < 1$. Hence, by induction, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \delta^{n+1} d(x_0, x_1). \tag{2.4}$$

Note that, for $m, n \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) & \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n) \\
& \leq [\delta^{m-1} + \delta^{m-2} + \dots + \delta^n] d(x_0, x_1) \\
& = \delta^n [1 + \delta + \dots + \delta^{m-n-1}] d(x_0, x_1) \\
& \leq \delta^n \sum_{r=0}^{\infty} \delta^r d(x_0, x_1) \\
& = \frac{\delta^n}{1 - \delta} d(x_0, x_1).
\end{aligned} \tag{2.5}$$

Since $0 \leq \delta < 1$, then as $n \rightarrow \infty$, $\delta^n(1 - \delta)^{-1} \rightarrow 0$. Hence, $d(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$. This forces that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . But X is a complete dislocated metric space;

hence, $\{x_n\}_{n \in \mathbb{N}}$ is d -converges. Call the d -limit $x^* \in X$. Then, $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By continuity of T we have,

$$Tx^* = T\left(d - \lim_{n \rightarrow \infty} x_n\right) = d - \lim_{n \rightarrow \infty} Tx_n = d - \lim_{n \rightarrow \infty} x_{n-1} = x^* \quad (2.6)$$

that is, $Tx^* = x^*$; thus, T has a fixed point in X . $\square$

Uniqueness

Let y^* be another fixed point of T in X , then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$d(Tx^*, Ty^*) + \alpha \max\{d(x^*, Ty^*), d(y^*, Tx^*)\} \geq \beta \cdot \frac{d(x^*, Tx^*)[1 + d(y^*, Ty^*)]}{1 + d(x^*, y^*)} + \gamma \cdot d(x^*, y^*). \quad (2.7)$$

This implies that

$$\begin{aligned} d(x^*, y^*) + \alpha d(x^*, y^*) &\geq \gamma \cdot d(x^*, y^*) \\ \text{that is } d(x^*, y^*) &\geq (\gamma - \alpha) d(x^*, y^*) \\ \implies d(x^*, y^*) &\leq \frac{1}{\gamma - \alpha} d(x^*, y^*). \end{aligned} \quad (2.8)$$

This is true only when $d(x^*, y^*) = 0$. Similarly, $d(y^*, x^*) = 0$. Hence, $d(x^*, y^*) = d(y^*, x^*) = 0$ and so $x^* = y^*$. Hence, T has a unique fixed point in X .

Next we prove Theorem 2.1 for surjective mapping.

Theorem 2.2. Let (X, d) be a complete dislocated metric space and T a surjective mapping satisfying the condition (2.1) for all $y \in X, y \neq y$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $\beta + \gamma > 1 + 2\alpha, \gamma > 1 + \alpha$. Then, T has a fixed point in X .

Proof. Choose $x_0 \in X$ to be arbitrary, and define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows: $Tx_n = x_{n+1}$ for $n = 0, 1, 2, 3, \dots$. Then, using (2.1), we obtain, sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . But X is a complete dislocated metric space; hence $\{x_n\}_{n \in \mathbb{N}}$ is d -converges. Call the d -limit $x^* \in X$. Then, $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Existence of Fixed Point

Since T is a Surjective map, so there exists a point y in X , such that $x = Ty$. Consider

$$\begin{aligned} d(x_n, x) &= d(Tx_{n+1}, Ty) \\ &\geq -\alpha \max\{d(x_{n+1}, Ty), d(y, Tx_{n+1})\} + \beta \cdot \frac{d(x_{n+1}, Tx_{n+1})[1 + d(y, Ty)]}{1 + d(x_{n+1}, y)} \\ &\quad + \gamma \cdot d(x_{n+1}, y). \end{aligned} \quad (2.9)$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned}
 d(x, x) &\geq -\alpha \max\{d(x, x), d(y, x)\} + \beta \cdot \frac{d(x, x)[1 + d(y, x)]}{1 + d(x, y)} + \gamma \cdot d(x, y) \\
 &\Rightarrow 0 \geq -\alpha d(x, y) + \gamma d(y, x) \\
 &\Rightarrow (\gamma - \alpha)d(x, y) \leq 0 \\
 &\Rightarrow d(x, y) = 0 \text{ as } \gamma > \alpha.
 \end{aligned} \tag{2.10}$$

Similarly, $d(y, x) = 0$. Hence, $d(x, y) = d(y, x) = 0 \Rightarrow x = y$ and so $Tx = x$, that is, x is a fixed point of T .

Uniqueness

Let y^* be another fixed point of T in X , then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$d(Tx^*, Ty^*) + \alpha \max\{d(x^*, Ty^*), d(y^*, Tx^*)\} \geq \beta \cdot \frac{d(x^*, Tx^*)[1 + d(y^*, Ty^*)]}{1 + d(x^*, y^*)} + \gamma \cdot d(x^*, y^*). \tag{2.11}$$

This implies that

$$\begin{aligned}
 d(x^*, y^*) + \alpha d(x^*, y^*) &\geq \gamma \cdot d(x^*, y^*) \\
 \text{that is, } d(x^*, y^*) &\geq (\gamma - \alpha)d(x^*, y^*) \\
 \Rightarrow d(x^*, y^*) &\leq \frac{1}{\gamma - \alpha} d(x^*, y^*).
 \end{aligned} \tag{2.12}$$

This is true only when $d(x^*, y^*) = 0$. Similarly, $d(y^*, x^*) = 0$. Hence, $d(x^*, y^*) = d(y^*, x^*) = 0$ and so $x^* = y^*$. Hence T has a unique fixed point in X . The proof is completed. $\square$

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