Research Article

Some Almost Lacunary Double Sequence Spaces Defined by Orlicz Functions in 2-Normed Spaces

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Received 16 November 2011; Accepted 25 December 2011

Academic Editor: O. Miyagaki

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Das and Patel (1989) introduced two new sequence spaces which are called lacunary almost convergent and lacunary strongly almost convergent sequence spaces. Móricz and Rhoades (1988) defined and studied almost P -convergent double sequences spaces. Savaş and Patterson (2005) introduce the almost lacunary strong P -convergent double sequence spaces by using Orlicz functions and examined some properties of these sequences spaces. In this paper, some almost lacunary double sequences spaces are given by using 2-normed spaces.

1. Introduction

By the convergence of double sequence which is known the convergence on the Pringsheim sense, that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L denoted by $P - \lim x = L$, provided given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We write briefly as P -convergent [1–3].

Freedman et al. [4] presented a definition for lacunary refinement as follow: $p = \{\bar{k}_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{\bar{k}_r\}$ and studied many scholar [4–6]. Savaş and Patterson [7] gave some properties and theorem and also defined $S_{\theta,r,s} - P$ -convergence.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \dots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} is denoted by q_r [4].

An Orlicz Function, which was presented by Krasnoselskii and Rutisky [8], $M : [0, \infty) \rightarrow [0, \infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

An Orlicz function M can be represented in the following integral form: $M(x) = \int_0^x p(t)dt$ where p is the known kernel of M , right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Ruckle [9] and Maddox [10] described that if convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called Modulus function.

$(X, \|\cdot\|)$ be a normed space and a sequence $(x_{mn})_{(m,n \in \mathbb{N})}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{m, n \in \mathbb{N} : \|x_{mn} - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$ [11].

Let X be a real vector space of dimension d , where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$ which satisfy the following four conditions;

- (i) $\|x, y\| = 0$ if and only if x and y are linear dependent.
- (ii) $\|x, y\| = \|y, x\|$
- (iii) $\|\alpha x, y\| = |\alpha| \|x, y\|, \alpha \in \mathbb{R}$
- (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

the pair $(X, \|\cdot, \cdot\|)$ is then called a 2-normed spaces [12, 13].

The sequence $(x_k)_{k \in \mathbb{N}}$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is said to be convergent to L in X if $\lim_{k \rightarrow \infty} \|x_k - L, z\| = 0$ for every $z \in X$. In this case, we write $\lim_{k \rightarrow \infty} \|x_k, z\| = \|L, z\|$ [14].

2. Notations and Known Results

Almost P -convergent sequences have been defined by Móricz and Rhoades [15] as follow:

Definition 2.1. A double sequence $x = (x_{k,l})$ of real numbers is called almost P -convergent to a limit L if

$$P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0. \quad (2.1)$$

That is, the average $(x_{k,l})$ take over any rectangle

$$\{(k, l) : m \leq k \leq m + p - 1, n \leq l \leq n + q - 1\}, \quad (2.2)$$

tends to L as both p and q tend to ∞ , and this P -convergence is uniform in m and n . The set of sequence which satisfy this property was denoted as $[\hat{c}^2]$ by Savaş and Paterson [16].

We can define the set of almost P -convergent double sequence in $(X, \|\cdot, \cdot\|)$ similar to above definition as follow:

$$[\hat{c}^2, \|\cdot, \cdot\|] = \left\{ x = (x_{k,l}) : P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} \|x_{k,l} - L, z\| = 0 \text{ for every } z \in X \right\}. \quad (2.3)$$

Definition 2.2. Let M be an Orlicz function, $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers and $S''(2-X)$ denote all double sequence in $(X, \|\cdot, \cdot\|)$ 2-normed space we can define the following double sequence space

$$[\hat{c}^2, M, p, \|\cdot, \cdot\|] = \left\{ \begin{array}{l} x = (x_{k,l}) : P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z \right\| \right) \right]^{p_{k,l}} = 0 \\ \text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0 \text{ and } L, \text{ and every } z \in X \end{array} \right\}. \quad (2.4)$$

If we choose $M(x) = x$ and $(p_{k,l}) = 1$ for all k and l , then $[\hat{c}^2, M, p, \|\cdot, \cdot\|] = [\hat{c}^2, \|\cdot, \cdot\|]$ which was defined above.

Definition 2.3. The double sequence $\theta_{r,s} = \{k_r, l_s\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$\begin{aligned} k_0 &= 0, & h_r &= k_r - k_{r-1} \longrightarrow \infty \quad \text{as } r \longrightarrow \infty, \\ l_0 &= 0, & \overline{h}_s &= l_s - l_{s-1} \longrightarrow \infty \quad \text{as } s \longrightarrow \infty. \end{aligned} \quad (2.5)$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and $\theta_{r,s}$ is defined by

$$I_{r,s} = \{(k, l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}, \quad q_r = \frac{k_r}{k_{r-1}}, \quad \overline{q}_s = \frac{l_s}{l_{s-1}}, \quad q_{r,s} = q_r \overline{q}_s. \quad (2.6)$$

Definition 2.4. Let M be an Orlicz function, $S''(2-X)$ denote all double sequence in $(X, \|\cdot, \cdot\|)$ 2-normed space, and $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers, now we can define the following sequence spaces in $(X, \|\cdot, \cdot\|)$ 2-normed space as follows:

$$\begin{aligned} [AC_{\theta_{r,s}}, \|\cdot, \cdot\|] &= \left\{ \begin{array}{l} x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k+m,l+n} - L, z\| = 0 \\ \text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and every } z \in X \end{array} \right\} \\ [AC_{\theta_{r,s}}, \|\cdot, \cdot\|]_0 &= \left\{ \begin{array}{l} x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k+m,l+n}, z\| = 0 \\ \text{uniformly in } m \text{ and } n \text{ and every } z \in X \end{array} \right\} \\ [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|] &= \left\{ \begin{array}{l} x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z \right\| \right) \right]^{p_{k,l}} = 0 \\ \text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0 \text{ and } L, \text{ and every } z \in X \end{array} \right\} \\ [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0 &= \left\{ \begin{array}{l} x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho}, z \right\| \right) \right]^{p_{k,l}} = 0 \\ \text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0 \text{ and every } z \in X \end{array} \right\}. \end{aligned} \quad (2.7)$$

When $(p_{k,l}) = 1$ for all k and l , we shall denote $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]$ and $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0$ as $[AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]$ and $[AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]_0$. That is,

$$\begin{aligned} [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|] &= \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right] = 0 \right. \\ &\quad \left. \text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0 \text{ and } L, \text{ and every } z \in X \right\}, \\ [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]_0 &= \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho} \right\|, z \right) \right] = 0 \right. \\ &\quad \left. \text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0 \text{ and } L, \text{ and every } z \in X \right\}. \end{aligned} \quad (2.8)$$

If $x \in [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]$ we shall say that x is almost lacunary strongly P -convergent with respect to the Orlicz function M in 2-normed space. In addition if $M(x) = x$ and $(p_{k,l}) = 1$ for all k and l , then $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|] = [AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$ and $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0 = [AC_{\theta_{r,s}}, \|\cdot, \cdot\|]_0$ which are defined above. Also note that if $(p_{k,l}) = 1$ for all k and l , then $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|] = [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]$ and $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0 = [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]_0$ which are defined above.

Let us generalized almost P -convergent double sequence to Orlicz function in 2-normed spaces.

3. Main Results

Theorem 3.1. For any Orlicz function M and a bounded factorable positive double sequence $p_{k,l}$, $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]$ and $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0$ are linear spaces.

Proof. Suppose that $x, y \in [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]_0$ and $\alpha, \beta \in \mathbb{R}$. So we have

$$\begin{aligned} A_{r,s}^1 &= \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho_1} \right\|, z \right) \right]^{p_{k,l}} = 0 \right. \\ &\quad \left. \text{uniformly in } m \text{ and } n, \text{ for some } \rho_1 > 0 \text{ and every } z \in X \right\}, \\ A_{r,s}^2 &= \left\{ y = (y_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{y_{k+m,l+n}}{\rho_2} \right\|, z \right) \right]^{p_{k,l}} = 0 \right. \\ &\quad \left. \text{uniformly in } m \text{ and } n, \text{ for some } \rho_2 > 0 \text{ and every } z \in X \right\}. \end{aligned} \quad (3.1)$$

Since M is an Orlicz function we have the following inequality

$$\begin{aligned}
& \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{\alpha x_{k+m,l+n} + \beta y_{k+m,l+n}}{|\alpha|\rho_1 + |\beta|\rho_2}, z \right\| \right) \right]^{p_{k,l}} \\
& \leq D \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M \left(\left\| \frac{x_{k+m,l+n}}{\rho_1}, z \right\| \right) \right]^{p_{k,l}} \\
& \quad + D \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M \left(\left\| \frac{y_{k+m,l+n}}{\rho_2}, z \right\| \right) \right]^{p_{k,l}} \\
& \leq DF \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho_1}, z \right\| \right) \right]^{p_{k,l}} \\
& \quad + DF \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{y_{k+m,l+n}}{\rho_2}, z \right\| \right) \right]^{p_{k,l}},
\end{aligned} \tag{3.2}$$

where $F = \max[1, (|\alpha|/(|\alpha|\rho_1 + |\beta|\rho_2))^H, (|\beta|/(|\alpha|\rho_1 + |\beta|\rho_2))^H]$. When we take the limit of each side as $r, s \rightarrow \infty$

$$\left\{ x = (x_{k,l}), y = (y_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M \left(\left\| \frac{\alpha x_{k+m,l+n} + \beta y_{k+m,l+n}}{|\alpha|\rho_1 + |\beta|\rho_2}, z \right\| \right) \right]^{p_{k,l}} = o \right\}. \tag{3.3}$$

uniformly in m and n , for some $\rho_1, \rho_2 > 0$ and each $z \in X$

So this is the result. □

Definition 3.2. An Orlicz function M is said to be satisfy Δ_2 -condition for all values of u , if there exists a constant $K > 0$ such that

$$M(2u) \leq KM(u) \quad (u \geq 0). \tag{3.4}$$

Lemma 3.3. Let M be an Orlicz function which satisfies Δ_2 -condition and $0 < \delta < 1$. Then for each $x \geq \delta$ and some constant $K > 0$ we have

$$M(x) \leq K\delta^{-1}M(2). \tag{3.5}$$

Theorem 3.4. For any Orlicz function M which satisfies Δ_2 -condition, we have $[AC_{\theta_{r,s}}, \|\cdot, \cdot\|] \subseteq [AC_{\theta_{r,s}}M, \|\cdot, \cdot\|]$

Proof. Let $x \in [AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$. For each m and n

$$A_{r,s} = \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k+m,l+n} - L, z\| = 0 \right\}. \tag{3.6}$$

for some L , and every $z \in X$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for every t with $0 \leq t \leq \delta$. For every $z \in X$, we get

$$\begin{aligned}
 \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} M(\|x_{k+m,l+n} - L, z\|) &= \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L, z\| \leq \delta} M(\|x_{k+m,l+n} - L, z\|) \\
 &\quad + \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L, z\| > \delta} M(\|x_{k+m,l+n} - L, z\|) \\
 &\leq \frac{1}{h_{r,s}} (h_{r,s} \varepsilon) + \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L, z\| > \delta} M(\|x_{k+m,l+n} - L, z\|) \\
 &< \frac{1}{h_{r,s}} (h_{r,s} \varepsilon) + \frac{1}{h_{r,s}} K \delta^{-1} M(2) h_{r,s} A_{r,s}.
 \end{aligned} \tag{3.7}$$

From Lemma 3.3 as r and s goes to infinity in Pringsheim sense, for each m and n we are granted $x \in [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]$. $\square$

Theorem 3.5. Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$ then for any Orlicz function M , $[\hat{c}^2, M, p, \|\cdot, \cdot\|] \subset [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]$.

Proof. Since $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$, then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and $\bar{q}_s > 1 + \delta$. This mean $h_r/k_r \geq \delta/(1 + \delta)$, $\bar{h}_s/l_s \geq \delta/(1 + \delta)$. Then for $x \in [\hat{c}^2, M, p, \|\cdot, \cdot\|]$, we can write for each m and n , some L and every $z \in X$

$$\begin{aligned}
 B_{r,s} &= \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \\
 &= \frac{1}{h_{r,s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} - \frac{1}{h_{r,s}} \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s-1} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \\
 &\quad - \frac{1}{h_{r,s}} \sum_{k=k_r-1+1}^{k_r} \sum_{l=1}^{l_s-1} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} - \frac{1}{h_{r,s}} \sum_{k=1}^{k_r-1} \sum_{l=l_s-1+1}^{l_s} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \\
 &\quad - \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \right) \\
 &\quad - \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \right) \\
 &= \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\left\| \frac{x_{k+m,l+n} - L}{\rho}, z\right\| \right) \right]^{p_{k,l}} \right)
 \end{aligned}$$

$$\begin{aligned}
& - \frac{k_{r-1}l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1}l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}} \right) \\
& - \frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{h_s} \frac{1}{l_{s-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}} \\
& - \frac{1}{h_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}}.
\end{aligned} \tag{3.8}$$

Since $x \in [\hat{c}^2, M, p, \|\cdot, \cdot\|]$ the last two terms tends to 0 uniformly in m and n in Pringsheim sense. Thus for each m and n

$$\begin{aligned}
B_{r,s} &= \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}} \right) \\
& - \frac{k_{r-1}l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1}l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}} \right) + o(1).
\end{aligned} \tag{3.9}$$

Since $h_{r,s} = k_r l_s - k_{r-1} l_{s-1}$ we get for each m and n the following inequalities as follow:

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{1 + \delta}{\delta}, \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}. \tag{3.10}$$

Thus the terms

$$\begin{aligned}
& \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}}, \\
& \frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\left\| \frac{x_{k+m,l+n} - L}{\rho} \right\|, z \right) \right]^{p_{k,l}},
\end{aligned} \tag{3.11}$$

are both convergent to L in Pringsheim sense for all m and n , every $z \in X$ and some $\rho > 0$. Therefore $B_{r,s}$ is a convergent sequence in Pringsheim sense for each m and n . So $x \in [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]$ and this is the proof. $\square$

Theorem 3.6. Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ then for any Orlicz function M , $[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|] \subset [\hat{c}^2, M, p, \|\cdot, \cdot\|]$.

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s . Let $\varepsilon > 0$ and $x \in [AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|]$. There exists $r_0 > 0$ and $s_0 > 0$ such that for every $i \geq r_0$ and $j \geq s_0$, and all m and n , for every $z \in X$

$$A'_{i,j} = \frac{1}{h_{i,j}} \sum_{k,l \in I_{i,j}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho} \right\|, z \right) \right]^{p_{k,l}} < \varepsilon. \tag{3.12}$$

Let

$$M' = \max \left\{ A'_{l,j} : 1 \leq i \leq r_0, 1 \leq j \leq s_0 \right\} \quad (3.13)$$

and let $k_{r-1} < p \leq k_r$ and $l_{s-1} < q \leq l_s$. Hence we get

$$\begin{aligned} \frac{1}{pq} \sum_{k,l \in 1,1}^{p,q} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho}, z \right\| \right) \right]^{p_{k,l}} &\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho}, z \right\| \right) \right]^{p_{k,l}} \\ &\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} \left(\sum_{k,l \in I_{t,u}} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho}, z \right\| \right) \right]^{p_{k,l}} \right) \\ &= \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r,s} h_{t,u} A'_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} A'_{t,u} \\ &\leq \frac{M'}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} A'_{t,u} \\ &\leq \frac{M'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} A'_{t,u} \\ &\leq \frac{M'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \left(\sup_{t \geq r_0 \cup u \geq s_0} A'_{t,u} \right) \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} \\ &\leq \frac{M'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \varepsilon \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} \\ &\leq \frac{M'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \varepsilon H^2. \end{aligned} \quad (3.14)$$

Since k_r and l_s both tends to infinity as both p and q tends to infinity, uniformly in m and n , and for every $z \in X$,

$$\frac{1}{pq} \sum_{k,l \in 1,1}^{p,q} \left[M \left(\left\| \frac{x_{k+m,l+n}}{\rho}, z \right\| \right) \right]^{p_{k,l}} \longrightarrow 0. \quad (3.15)$$

Therefore $x \in [\hat{c}^2, M, p, \|\cdot, \cdot\|]$. □

The following theorem is a result of Theorems 3.4 and 3.5.

Theorem 3.7. Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{r,s} \leq \limsup_{r,s} q_{r,s} < \infty$, then for any Orlicz function $M[AC_{\theta_{r,s}}, M, p, \|\cdot, \cdot\|] = [\hat{c}^2, M, p, \|\cdot, \cdot\|]$.

Gähler [12] defined almost lacunary statistical convergence for single sequence, then Savaş and Patterson [16] defined almost lacunary statistical convergence for double sequence

by combining lacunary sequence and almost convergence. Now we can define this definition in 2-normed space as follow:

Definition 3.8. Let $\theta_{r,s}$ be a double lacunary sequence; the double number sequence x is $\bar{S}_{\theta_{r,s}} - P$ -convergent to L provided that for every $\varepsilon > 0$ and $z \in X$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \max |\{ (k, l) \in I_{r,s} : \|x_{k+m, l+n} - L, z\| \geq \varepsilon \}| = 0. \quad (3.16)$$

So we can write $\bar{S}_{\theta_{r,s}} - \lim x = L$.

Theorem 3.9. Let $\theta_{r,s}$ be a double lacunary sequence then

- (1) $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$ implies $x_{k,l} \xrightarrow{P} L[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$
- (2) $[AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$ is a proper subset of $[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$
- (3) If $x \in (l_\infty)_2$ and $x_{k,l} \xrightarrow{P} L[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$ then $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$
- (4) $[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|] \cap (l_\infty)_2 = [AC_{\theta_{r,s}}, \|\cdot, \cdot\|] \cap (l_\infty)_2$

where $(l_\infty)_2$ is the space of all bounded double sequence.

Proof. (1) Since for all m and n , and every $z \in X$

$$\begin{aligned} |\{ (k, l) \in I_{r,s} : \|x_{k+m, l+n} - L, z\| \geq \varepsilon \}| &\leq \sum_{k, l \in I_{r,s}, \|x_{k+m, l+n} - L, z\| \geq \varepsilon} \|x_{k+m, l+n} - L, z\| \\ &\leq \sum_{k, l \in I_{r,s}} \|x_{k+m, l+n} - L, z\|, \end{aligned} \quad (3.17)$$

and for all m and n

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k, l \in I_{r,s}} \|x_{k+m, l+n} - L, z\| = 0. \quad (3.18)$$

This show that for all m and n

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{ (k, l) \in I_{r,s} : \|x_{k+m, l+n} - L, z\| \geq \varepsilon \}| = 0, \quad (3.19)$$

this completes the proof of (1)

(2) Let x be defined as follows:

$$x_{k,l} = \begin{pmatrix} 1 & 2 & 3 & \cdots & \left[\sqrt[3]{h_{r,s}} \right] & 0 & \cdot & \cdot & \cdot \\ 2 & 2 & 3 & \cdots & \left[\sqrt[3]{h_{r,s}} \right] & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \left[\sqrt[3]{h_{r,s}} \right] & \left[\sqrt[3]{h_{r,s}} \right] & \cdots & \left[\sqrt[3]{h_{r,s}} \right] & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (3.20)$$

It is obvious that x is an unbounded double sequence and for $\varepsilon > 0$, for all m and n , and for every $z \in X$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k+m,l+n} - L, z\| \geq \varepsilon\}| = P - \lim_{r,s} \frac{\left[\sqrt[3]{h_{r,s}} \right]}{h_{r,s}} = 0. \quad (3.21)$$

Thus $x_{k,l} \xrightarrow{P} 0[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$. But

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k,l}, z\| = P - \lim_{r,s} \frac{\left[\sqrt[3]{h_{r,s}} \right] \left(\left[\sqrt[3]{h_{r,s}} \right] \left(\left[\sqrt[3]{h_{r,s}} \right] + 1 \right) \right)}{2h_{r,s}} = \frac{1}{2}. \quad (3.22)$$

Therefore $x_{k,l} \not\xrightarrow{P} L[AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$ which is the proof of (2).

(3) Let $x \in (l_\infty)_2$ and $x_{k,l} \xrightarrow{P} L[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$. Assuming that for all m and n , and every $z \in X$ $\|x_{k+m,l+n} - L, z\| \leq K$ for all K . And also for given $\varepsilon > 0$ and r and s large for all m and n , and every $z \in X$ we get the following inequality as follow:

$$\begin{aligned} & \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k+m,l+n} - L, z\| \\ &= \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L, z\| \geq \varepsilon} \|x_{k+m,l+n} - L, z\| + \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L, z\| < \varepsilon} \|x_{k+m,l+n} - L, z\| \quad (3.23) \\ &\leq \frac{K}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k+m,l+n} - L, z\| \geq \varepsilon\}| + \varepsilon. \end{aligned}$$

Therefore $x \in (l_\infty)_2$ and $x_{k,l} \xrightarrow{P} L[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$, this shows that $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}, \|\cdot, \cdot\|]$.

(4) from (1), (2) and (3), we get $[\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|] \cap (l_\infty)_2 = [AC_{\theta_{r,s}}, \|\cdot, \cdot\|] \cap (l_\infty)_2$.

□

Theorem 3.10. For any Orlicz function M , $[AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|] \subset [\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$

Proof. Let $x \in [AC_{\theta_{r,s}}, M, \|\cdot, \cdot\|]$ and $\varepsilon > 0$. Then for all m and n , and every $z \in X$

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} M\left(\left\|\frac{x_{k+m,l+n} - L}{\rho}, z\right\|\right) &\geq \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}, \|x_{k+m,l+n} - L\| \geq \varepsilon} M\left(\left\|\frac{x_{k+m,l+n} - L}{\rho}, z\right\|\right) \\ &> \frac{1}{h_{r,s}} M\left(\frac{\varepsilon}{\rho}\right) |\{(k,l) \in I_{r,s} : \|x_{k+m,l+n} - L, z\| \geq \varepsilon\}|. \end{aligned} \quad (3.24)$$

This shows that $x \in [\bar{S}_{\theta_{r,s}}, \|\cdot, \cdot\|]$.

□

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