Research Article

Two Different Classes of Wronskian Conditions to a $(3 + 1)$ -Dimensional Generalized Shallow Water Equation

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Based on the Hirota bilinear method and Wronskian technique, two different classes of sufficient conditions consisting of linear partial differential equations system are presented, which guarantee that the Wronskian determinant is a solution to the corresponding Hirota bilinear equation of a $(3 + 1)$ -dimensional generalized shallow water equation. Our results show that the nonlinear equation possesses rich and diverse exact solutions such as rational solutions, solitons, negatons, and positons.

1. Introduction

Wronskian formulations are a common feature for soliton equations, and it is a powerful tool to construct exact solutions to the soliton equations [1–4]. The resulting technique has been applied to many soliton equations such as the MKdV, NLS, derivative NLS, sine-Gordon, and other equations [5–10]. Within Wronskian formulations, soliton solutions, rational solutions, positons, and negatons are usually expressed as some kind of logarithmic derivatives of Wronskian-type determinants [11–16].

The following $(3+1)$ -dimensional generalized shallow water equation

$$u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + u_{yt} - u_{xz} = 0 \quad (1.1)$$

was investigated in different ways (see, e.g., [17, 18]). In [17], soliton-typed solutions for (1.1) were constructed by a generalized tanh algorithm with symbolic computation. In [18],

the traveling wave solutions of (1.1) expressed by hyperbolic, trigonometric, and rational functions were established by the G'/G -expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation.

Under a scale transformation $x \rightarrow -x$, (1.1) is reduced equivalently to

$$u_{xxxy} + 3u_{xx}u_y + 3u_xu_{xy} - u_{yt} - u_{xz} = 0, \quad (1.2)$$

and the Cole-Hopf transformation

$$u = 2(\ln f)_x \quad (1.3)$$

gives the corresponding Hirota bilinear equation of (1.2)

$$\begin{aligned} & (D_x^3 D_y - D_y D_t - D_x D_z) f \cdot f \\ & = (f_{xxxy} - f_{yt} - f_{xz})f - f_{xxx}f_y - 3f_{xxy}f_x + f_y f_t + f_x f_z + 3f_{xx}f_{xy} = 0, \end{aligned} \quad (1.4)$$

where D_x, D_y, D_z , and D_t are the Hirota operators [19].

As we know, in the process of employing Wronskian technique, the main difficulty lies in looking for the linear differential conditions, which the functions in the Wronskian determinant should satisfy. Moreover, the differential conditions for the Wronskian determinant solutions of many soliton equations are not unique [5, 7, 10, 12]. In this paper, we will give two different classes of linear differential conditions for the N th order Wronskian determinant solutions (simply, Wronskian conditions) of (1.4) based on the special structure of the Hirota bilinear form (1.4). Our results will show that (1.4) has diverse Wronskian determinant solutions under different linear differential conditions and further (1.2) will have diverse exact solutions such as rational solutions, solitons, negatons, and positons.

2. The First Class of Wronskian Conditions

The N th order Wronskian determinant was introduced firstly by Freeman and Nimmo [1, 20]:

$$W(\phi_1, \phi_2, \dots, \phi_N) = |\widehat{N-1}| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \dots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \dots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \dots & \phi_N^{(N-1)} \end{vmatrix}, \quad N \geq 1, \quad (2.1)$$

where

$$\phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i, \quad 0 \leq j \leq N-1, \quad 1 \leq i \leq N. \quad (2.2)$$

Solutions determined by $f = |\widehat{N-1}|$ to (1.4) are called Wronskian determinant solutions.

In this section, we present the first class of linear differential conditions for the Wronskian determinant solutions of (1.4).

Theorem 2.1. *Let a set of functions $\phi_i = \phi_i(x, y, z, t)$, $1 \leq i \leq N$ satisfy the following linear differential conditions:*

$$\phi_{i,xx} = \sum_{j=1}^N \lambda_{ij} \phi_j, \quad (2.3)$$

$$\phi_{i,y} = n \phi_{i,x},$$

$$\phi_{i,t} = -\phi_{i,(mx)}, \quad (2.4)$$

$$\phi_{i,z} = 4n \phi_{i,xxx} + n \phi_{i,(mx)},$$

where the coefficient matrix $A = (\lambda_{ij})_{1 \leq i, j \leq N}$ is an arbitrary real constant matrix (see [10, 12]), n is an arbitrary nonzero constant, m is an arbitrary positive integer, and $\phi_{i,(mx)}$ denotes the m th order derivative of ϕ_i with respect to x . Then the Wronskian determinant $f = |\widehat{N-1}|$ defined by (2.1) solves (1.4).

The proof of Theorem 2.1 needs the following two useful known Lemmas.

Lemma 2.2. *Set a_j , $j = 1, 2, \dots, N$ to be an N -dimensional column vector, and b_j , $j = 1, 2, \dots, N$ to be a real nonzero constant. Then one has*

$$\sum_{i=1}^N b_i |a_1, a_2, \dots, a_N| = \sum_{j=1}^N |a_1, a_2, \dots, b a_j, \dots, a_N|, \quad (2.5)$$

where $b a_j = (b_1 a_{1j}, b_2 a_{2j}, \dots, b_N a_{Nj})^T$.

Lemma 2.3 (see [11]). *Under the condition (2.3) and Lemma 2.2, the following equalities hold:*

$$\begin{aligned} & |\widehat{N-1}| \sum_{i=1}^N \lambda_{ii} \left(\sum_{i=1}^N \lambda_{ii} |\widehat{N-1}| \right) \\ &= \left(\sum_{i=1}^N \lambda_{ii} |\widehat{N-1}| \right)^2 \\ &= \left(-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1| \right)^2 \\ &= |\widehat{N-1}| \left(\left| \widehat{N-5}, N-3, N-2, N-1, N \right| - \left| \widehat{N-4}, N-2, N-1, N+1 \right| \right. \\ &\quad \left. + 2 \left| \widehat{N-3}, N, N+1 \right| - \left| \widehat{N-3}, N-1, N+2 \right| + \left| \widehat{N-2}, N+3 \right| \right). \end{aligned} \quad (2.6)$$

Proof of Theorem 2.1. Under the properties of the Wronskian determinant and the conditions (2.3) and (2.4), we can compute various derivatives of the Wronskian determinant $f = |\widehat{N-1}|$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_x &= |\widehat{N-2}, N|, & f_{xx} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\
f_{xxx} &= |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \\
f_y &= n|\widehat{N-2}, N|, & f_{xy} &= n(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|), \\
f_{xxy} &= n(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
f_{xxxy} &= n(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| \\
&\quad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|), \\
f_t &= -\frac{\partial^m f}{\partial x^m} = -\frac{\partial^m |\widehat{N-1}|}{\partial x^m}, & f_{yt} &= -\frac{\partial}{\partial y} \frac{\partial^m |\widehat{N-1}|}{\partial x^m} = -n \frac{\partial}{\partial x} \frac{\partial^m |\widehat{N-1}|}{\partial x^m}, \\
f_z &= 4n(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|) \\
&\quad + n \frac{\partial^m |\widehat{N-1}|}{\partial x^m}, \\
f_{xz} &= 4n(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|) \\
&\quad + n \frac{\partial}{\partial x} \frac{\partial^m |\widehat{N-1}|}{\partial x^m}.
\end{aligned} \tag{2.7}$$

Therefore, we can now compute that

$$\begin{aligned}
&(f_{xxxy} - f_{yt} - f_{xz})f \\
&= 3n(-|\widehat{N-5}, N-3, N-2, N-1, N| + |\widehat{N-4}, N-2, N-1, N+1| \\
&\quad + 2|\widehat{N-3}, N, N+1| + |\widehat{N-3}, N-1, N+2| - |\widehat{N-2}, N+3|)|\widehat{N-1}|, \\
&\quad - f_{xxx}f_y - 3f_{xxy}f_x + f_yf_t + f_xf_z \\
&= -12n|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N|,
\end{aligned}$$

$$\begin{aligned}
& 3f_{xx}f_{xy} \\
&= 3n\left(\left|\widehat{N-3}, N-1, N\right| + \left|\widehat{N-2}, N+1\right|\right)^2 \\
&= 3n\left(-\left|\widehat{N-3}, N-1, N\right| + \left|\widehat{N-2}, N+1\right|\right)^2 + 12n\left|\widehat{N-3}, N-1, N\right|\left|\widehat{N-2}, N+1\right|.
\end{aligned} \tag{2.8}$$

Using Lemma 2.3, we can further obtain that

$$\begin{aligned}
& (D_x^3 D_y - D_y D_t - D_x D_z)f \cdot f \\
&= 12n\left(\left|\widehat{N-3}, N, N+1\right|\left|\widehat{N-1}\right| - \left|\widehat{N-3}, N-1, N+1\right|\left|\widehat{N-2}, N\right| \right. \\
&\quad \left. + \left|\widehat{N-3}, N-1, N\right|\left|\widehat{N-2}, N+1\right|\right) \\
&= 0.
\end{aligned} \tag{2.9}$$

This last equality is just the Plücker relation for determinants:

$$|B, A_1, A_2||B, A_3, A_4| - |B, A_1, A_3||B, A_2, A_4| + |B, A_1, A_4||B, A_2, A_3| = 0, \tag{2.10}$$

where B denotes an $N \times (N-2)$ matrix, and A_i , $1 \leq i \leq 4$ are four N -dimensional column vectors. Therefore, we have shown that $f = \left|\widehat{N-1}\right|$ solve (1.4) under the conditions (2.3) and (2.4). Further, the corresponding solution to (1.2) is

$$u = 2(\ln f)_x, \quad f = \left|\widehat{N-1}\right|. \tag{2.11}$$

□

Remarks 1. The condition (2.4) is a generalized linear differential condition which includes many different special cases.

For example, when $m = 1$, the condition (2.4) is reduced to

$$\phi_{i,y} = n\phi_{i,x}, \quad \phi_{i,t} = -\phi_{i,x}, \quad \phi_{i,z} = 4n\phi_{i,xxx} + n\phi_{i,x}. \tag{2.12}$$

When $m = 2$, the condition (2.4) is reduced to

$$\phi_{i,y} = n\phi_{i,x}, \quad \phi_{i,t} = -\phi_{i,xx}, \quad \phi_{i,z} = 4n\phi_{i,xxx} + n\phi_{i,xx}. \tag{2.13}$$

When $m = 3$, the condition (2.4) is reduced to

$$\phi_{i,y} = n\phi_{i,x}, \quad \phi_{i,t} = -\phi_{i,xxx}, \quad \phi_{i,z} = 5n\phi_{i,xxx}. \tag{2.14}$$

Using the linear differential conditions (2.3) and (2.4) as well as the transformation (1.3), we can compute many exact solutions of (1.2) such as rational solutions, solitons, negatons, and positons.

As an example, in the special case of $m = 2$ and $n = 1$, the conditions (2.3) and (2.4) read

$$\phi_{i,xx} = \sum_{j=1}^N \lambda_{ij} \phi_j, \quad i = 1, \dots, N, \quad (2.15)$$

$$\phi_{i,y} = \phi_{i,x},$$

$$\phi_{i,t} = -\phi_{i,xx}, \quad (2.16)$$

$$\phi_{i,z} = 4\phi_{i,xxx} + \phi_{i,xx}.$$

If we let the coefficient matrix $A = (\lambda_{ij})_{1 \leq i,j \leq N}$ of condition (2.15) has the following form (see [10–12, 16] for details),

$$A = \begin{pmatrix} \lambda_{11} & & & 0 \\ & \lambda_{22} & & \\ & & \ddots & \\ 0 & & & \lambda_{NN} \end{pmatrix}_{N \times N}, \quad (\lambda_{ii} \neq \lambda_{jj}, i \neq j), \quad (2.17)$$

using the same method as that in [16], we can obtain N -soliton solutions of (1.2).

For example, when $\lambda_{11} > 0$, $\lambda_{ii} = 0$, $i = 2, \dots, N$, we can compute two exact 1-soliton solutions for (1.2),

$$\begin{aligned} u_1 &= 2\partial_x \ln \left(\cosh \left(4\lambda_{11}^{3/2} z + \lambda_{11} z + \sqrt{\lambda_{11}} y + \sqrt{\lambda_{11}} x - \lambda_{11} t + \gamma_1 \right) \right) \\ &= 2\sqrt{\lambda_{11}} \tanh \left(4\lambda_{11}^{3/2} z + \lambda_{11} z + \sqrt{\lambda_{11}} y + \sqrt{\lambda_{11}} x - \lambda_{11} t + \gamma_1 \right), \end{aligned} \quad (2.18)$$

$$\begin{aligned} u_2 &= 2\partial_x \ln \left(\sinh \left(4\lambda_{11}^{3/2} z + \lambda_{11} z + \sqrt{\lambda_{11}} y + \sqrt{\lambda_{11}} x - \lambda_{11} t + \gamma_1 \right) \right) \\ &= 2\sqrt{\lambda_{11}} \coth \left(4\lambda_{11}^{3/2} z + \lambda_{11} z + \sqrt{\lambda_{11}} y + \sqrt{\lambda_{11}} x - \lambda_{11} t + \gamma_1 \right), \end{aligned} \quad (2.19)$$

with γ_1 being a constant.

3. The Second Class of Wronskian Condition

In this section, we show another linear differential condition to the Wronskian determinant solutions of (1.4).

Theorem 3.1. Let a group of functions $\phi_i = \phi_i(x, y, z, t)$, $1 \leq i \leq N$ satisfy the following linear differential condition:

$$\begin{aligned}\phi_{i,y} &= k\phi_{i,xx}, \\ \phi_{i,t} &= -2\phi_{i,xxx}, \\ \phi_{i,z} &= 3k\phi_{i,xxx},\end{aligned}\tag{3.1}$$

where k is an arbitrary nonzero constant. Then the Wronskian determinant $f = |\widehat{N-1}|$ defined by (2.1) solves (1.4).

Proof. Under the properties of the Wronskian determinant and the condition (3.1), various derivatives of the Wronskian determinant $f = |\widehat{N-1}|$ with respect to the variables x, y, z, t are obtained as follows:

$$\begin{aligned}f_x &= |\widehat{N-2}, N|, & f_{xx} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\ f_{xxx} &= |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \\ f_y &= k(-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|), \\ f_{xy} &= k(-|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|), \\ f_{xxy} &= k(-|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N-1, N+1| \\ &\quad + |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|), \\ f_{xxxy} &= k(-|\widehat{N-6}, N-4, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N, N+1| \\ &\quad - 2|\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-3}, N, N+2| \\ &\quad + 2|\widehat{N-3}, N-1, N+3| + |\widehat{N-2}, N+4|), \\ f_t &= -2(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\ f_{yt} &= -2k(-|\widehat{N-6}, N-4, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N, N+1| \\ &\quad + |\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-3}, N, N+2| \\ &\quad - |\widehat{N-3}, N-1, N+3| + |\widehat{N-2}, N+4|), \\ f_z &= 3k(-|\widehat{N-5}, N-3, N-2, N-1, N| + |\widehat{N-4}, N-2, N-1, N+1| \\ &\quad - |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|),\end{aligned}$$

$$f_{xz} = 3k \left(-\left| \widehat{N-6}, N-4, N-3, N-2, N-1, N \right| + \left| \widehat{N-4}, N-2, N, N+1 \right| \right. \\ \left. - \left| \widehat{N-3}, N, N+2 \right| + \left| \widehat{N-2}, N+4 \right| \right). \quad (3.2)$$

Therefore, we can now compute that

$$(f_{xxx} - f_{yt} - f_{xz})f = 6k \left(-\left| \widehat{N-4}, N-2, N, N+1 \right| + \left| \widehat{N-3}, N, N+2 \right| \right) \left| \widehat{N-1} \right|, \\ -3f_{xxy}f_x + f_xf_z = 6k \left(\left| \widehat{N-4}, N-2, N-1, N+1 \right| - \left| \widehat{N-3}, N-1, N+2 \right| \right) \left| \widehat{N-2}, N \right|, \\ -f_{xxx}f_y + f_yf_t + 3f_{xx}f_{xy} = 6k \left(-\left| \widehat{N-4}, N-2, N-1, N \right| \left| \widehat{N-2}, N+1 \right| \right. \\ \left. + \left| \widehat{N-3}, N-1, N \right| \left| \widehat{N-2}, N+2 \right| \right), \quad (3.3)$$

then, substituting the above results into (1.4), we can further obtain that

$$\left(D_x^3 D_y - D_y D_t - D_x D_z \right) f \cdot f = 6k(A_1 + A_2), \quad (3.4)$$

where

$$A_1 = -\left| \widehat{N-4}, N-2, N, N+1 \right| \left| \widehat{N-1} \right| + \left| \widehat{N-4}, N-2, N-1, N+1 \right| \left| \widehat{N-2}, N \right| \\ - \left| \widehat{N-4}, N-2, N-1, N \right| \left| \widehat{N-2}, N+1 \right| \\ = \left| \widehat{N-4}, N-2, N-3, N-1 \right| \left| \widehat{N-4}, N-2, N, N+1 \right| \\ - \left| \widehat{N-4}, N-2, N-3, N \right| \left| \widehat{N-4}, N-2, N-1, N+1 \right| \\ + \left| \widehat{N-4}, N-2, N-3, N+1 \right| \left| \widehat{N-4}, N-2, N-1, N \right|, \\ A_2 = \left| \widehat{N-3}, N, N+2 \right| \left| \widehat{N-1} \right| - \left| \widehat{N-3}, N-1, N+2 \right| \left| \widehat{N-2}, N \right| \\ + \left| \widehat{N-3}, N-1, N \right| \left| \widehat{N-2}, N+2 \right| \\ = \left| \widehat{N-3}, N, N+2 \right| \left| \widehat{N-3}, N-2, N-1 \right| \\ - \left| \widehat{N-3}, N-2, N \right| \left| \widehat{N-3}, N-1, N+2 \right| \\ + \left| \widehat{N-3}, N-2, N+2 \right| \left| \widehat{N-3}, N-1, N \right|. \quad (3.5)$$

From (3.5), it is easy to see that the above expression (3.4) is nothing but zero because they both satisfy the Plücker relation (2.10). Therefore, we have shown that $f = \widehat{|N-1|}$ also solve (1.4) under the condition (3.1).

The condition (3.1) has an exponential-type function solution:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = l_{ij}x + kl_{ij}^2y + 3kl_{ij}^4z - 2l_{ij}^3t, \quad i = 1, 2, \dots, N, \quad (3.6)$$

where d_{ij} and l_{ij} are free parameters and p is an arbitrary natural number.

In particular, we can have the following Wronskian solutions of (1.2):

$$u = 2(\ln f)_x, \quad f = W(\phi_1, \phi_2, \dots, \phi_N), \quad (3.7)$$

where

$$\phi_i = e^{l_i x + kl_i^2 y + 3kl_i^4 z - 2l_i^3 t} + e^{w_i x + kw_i^2 y + 3kw_i^4 z - 2w_i^3 t}, \quad i = 1, \dots, N, \quad (3.8)$$

with l_i and w_i being free parameters. □

4. Conclusions and Remarks

In summary, we have established two different kinds of linear differential conditions for the Wronskian determinant solutions of the (3+1)-dimensional generalized shallow water equation (1.1) or equivalently (1.2). Especially, the first Wronskian conditions are generalized linear differential conditions which include many different special cases. Our results show that the nonlinear equation (1.1) carry rich and diverse Wronskian determinant solutions.

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