Research Article

Dynamical Behavior in a Four-Dimensional Neural Network Model with Delay

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A four-dimensional neural network model with delay is investigated. With the help of the theory of delay differential equation and Hopf bifurcation, the conditions of the equilibrium undergoing Hopf bifurcation are worked out by choosing the delay as parameter. Applying the normal form theory and the center manifold argument, we derive the explicit formulae for determining the properties of the bifurcating periodic solutions. Numerical simulations are performed to illustrate the analytical results.

1. Introduction

The interest in the periodic orbits of a delay neural networks has increased strongly in recent years and substantial efforts have been made in neural network models, for example, Wei and Zhang [1] studied the stability and bifurcation of a class of n -dimensional neural networks with delays, Guo and Huang [2] investigated the Hopf bifurcation behavior of a ring of neurons with delays, Yan [3] discussed the stability and bifurcation of a delayed trineuron network model, Hajjhosseini et al. [4] made a discussion on the Hopf bifurcation of a delayed recurrent neural network in the frequency domain, and Liao et al. [5] did a theoretical and empirical investigation of a two-neuron system with distributed delays in the frequency domain. For more information, one can see [6–23]. In 1986 and 1987, Babcock and Westervelt [24, 25] had analyzed the stability and dynamics of the following simple neural network model of two neurons with inertial coupling:

$$\begin{aligned}\frac{dx_1}{dt} &= x_3, \\ \frac{dx_2}{dt} &= x_4, \\ \frac{dx_3}{dt} &= -2\xi x_3 - x_1 + A_2 \tanh(x_2),\end{aligned}$$

$$\frac{dx_4}{dt} = -2\xi x_4 - x_2 + A_1 \tanh(x_1), \quad (1)$$

where x_i ($i = 1, 2$) denotes the input voltage of the i th neuron, x_j ($j = 3, 4$) is the output of the j th neuron, $\xi > 0$ is the damping factor and A_i ($i = 1, 2$) is the overall gain of the neuron which determines the strength of the nonlinearity. For a more detailed interpretation of the parameters, one can see [24, 25]. In 1997, Lin and Li [26] made a detailed investigation on the bifurcation direction of periodic solution for system (1).

Considering that there exists a time delay (we assume that it is τ) in the response of the output voltages to changes in the input, then system (1) can be revised as follows:

$$\begin{aligned}\frac{dx_1}{dt} &= x_3, \\ \frac{dx_2}{dt} &= x_4, \\ \frac{dx_3}{dt} &= -2\xi x_3 - x_1 + A_2 \tanh(x_2(t - \tau)), \\ \frac{dx_4}{dt} &= -2\xi x_4 - x_2 + A_1 \tanh(x_1(t - \tau)).\end{aligned} \quad (2)$$

As is known to us that the research on the Hopf bifurcation, especially on the stability of bifurcating periodic solutions and direction of Hopf bifurcation is very critical. When delays are incorporated into the network models, stability, and Hopf bifurcation analysis become much complex. To obtain a deep and clear understanding of dynamics of neural network model with delays, we will make a investigation on system (2), that is, we study the stability, the local Hopf bifurcation for system (2).

The remainder of the paper is organized as follows. In Section 2, local stability for the equilibrium state of system (2) is discussed. We investigate the existence of the Hopf bifurcations for system (2) choosing time delay as the bifurcation parameter. In Section 3, the direction and stability of the local Hopf bifurcation are analyzed by using the normal form theory and the center manifold theorem by Hassard et al. [27]. In Section 4, numerical simulations for justifying the theoretical results are illustrated.

2. Stability of the Equilibrium and Local Hopf Bifurcations

The object of this section is to investigate the stability of the equilibrium and the existence of local Hopf bifurcations for system (2). It is easy to see that if the following condition:

$$(H1) \quad A_1 A_2 < 1$$

holds, then (2) has a unique equilibrium $E(0, 0, 0, 0)$. To investigate the local stability of the equilibrium state we linearize system (2). We expand it in a Taylor series around the origin and neglect the terms of higher order than the first order. The linearization of (2) near $E(0, 0, 0, 0)$ can be expressed as:

$$\begin{aligned} \frac{dx_1}{dt} &= x_3, \\ \frac{dx_2}{dt} &= x_4, \\ \frac{dx_3}{dt} &= -x_1 - 2\xi x_3 + A_2 x_2(t - \tau), \\ \frac{dx_4}{dt} &= -x_2 - 2\xi x_4 + A_1 x_1(t - \tau), \end{aligned} \quad (3)$$

whose characteristic equation has the form

$$\det \begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 1 & -A_2 e^{-\lambda\tau} & \lambda + 2\xi & 0 \\ -A_1 e^{-\lambda\tau} & 1 & 0 & \lambda + 2\xi \end{pmatrix} = 0, \quad (4)$$

namely,

$$\lambda^4 + 4\xi\lambda^3 + (4\xi^2 + 2)\lambda^2 + 4\xi\lambda + 1 - A_1 A_2 e^{-2\lambda\tau} = 0. \quad (5)$$

In order to investigate the distribution of roots of the transcendental equation (5), the following Lemma is necessary.

Lemma 1 (see [28]). *For the transcendental equation:*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}) \\ = \lambda^n + p_1^{(0)}\lambda^{n-1} + \dots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ + [p_1^{(1)}\lambda^{n-1} + \dots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \dots \\ + [p_1^{(m)}\lambda^{n-1} + \dots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m} = 0, \end{aligned} \quad (6)$$

as $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

For $\tau = 0$, (5) becomes

$$\lambda^4 + 4\xi\lambda^3 + (4\xi^2 + 2)\lambda^2 + 4\xi\lambda + 1 - A_1 A_2 = 0. \quad (7)$$

In view of the Routh-Hurwitz criteria, we know that all roots of (7) have a negative real part if the following condition:

$$(H2) \quad 4\xi^2 + A_1 A_2 > 0$$

is satisfied.

For $\omega > 0$, $i\omega$ is a root of (5) if and only if

$$\begin{aligned} \omega^4 - 4\xi\omega^2 i - (4\xi^2 + 2)\omega^2 + 4\xi\omega i + 1 \\ - A_1 A_2 (\cos 2\omega\tau - i \sin 2\omega\tau) = 0. \end{aligned} \quad (8)$$

Separating the real and imaginary parts gives

$$\begin{aligned} A_1 A_2 \cos 2\omega\tau &= \omega^4 - (4\xi^2 + 2)\omega^2 + 1, \\ A_1 A_2 \sin 2\omega\tau &= 4\xi\omega^2 - 4\xi\omega. \end{aligned} \quad (9)$$

It follows from (9) that

$$[\omega^4 - (4\xi^2 + 2)\omega^2 + 1]^2 + [4\xi\omega^2 - 4\xi\omega]^2 = A_1^2 A_2^2, \quad (10)$$

which is equivalent to

$$\omega^8 + m_1\omega^6 + m_2\omega^4 + m_3\omega^3 + m_4\omega^2 + m_5 = 0, \quad (11)$$

where

$$\begin{aligned} m_1 &= -4(2\xi^2 + 1), \quad m_2 = 16\xi^4 + 32\xi^2 + 6, \\ m_3 &= -32\xi^2, \quad m_4 = 8\xi^2 - 4, \quad m_5 = 1 - A_1^2 A_2^2. \end{aligned} \quad (12)$$

Without loss of generality, we assume that (11) has eight positive roots, denoted by $\omega_k (k = 1, 2, 3, \dots, 8)$. Then by (9), we derive

$$\tau_k^{(j)} = \frac{1}{2\omega_k} \left\{ \arccos \left[\frac{\omega_k^4 - (4\xi^2 + 2)\omega_k^2 + 1}{A_1 A_2} \right] + 2j\pi \right\}, \quad (13)$$

where $k = 1, 2, 3, \dots, 8$; $j = 0, 1, \dots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (5) when $\tau = \tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k \in \{1, 2, 3, \dots, 8\}} \{ \tau_k^{(0)} \}. \quad (14)$$

The above analysis leads to the Lemma as follows.

Lemma 2. *If (H1) and (H2) hold, then all roots of (5) have a negative real part when $\tau \in [0, \tau_0)$ and (5) admits a pair of purely imaginary roots $\pm \omega_k$ when $\tau = \tau_k^{(j)}$ ($k = 1, 2, 3, \dots, 8; j = 0, 1, 2, \dots$).*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (5) near $\tau = \tau_k^{(j)}$, and $\alpha(\tau_k^{(j)}) = 0$, and $\omega(\tau_k^{(j)}) = \omega_k$. Due to functional differential equation theory, for every $\tau_k^{(j)}$, $k = 1, 2, \dots, 8; j = 0, 1, 2, \dots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{(j)}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (5) and taking derivative with respect to τ , we have

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{2\lambda^3 + 6\xi\lambda^2 + 2(2\xi^2 + 1)\lambda + 2\xi}{\lambda A_1 A_2 e^{-2\lambda\tau}} - \frac{\tau}{\lambda}. \quad (15)$$

Then

$$\begin{aligned} & \left[\frac{d(\operatorname{Re} \lambda(\tau))}{d\tau} \right]^{-1}_{\tau=\tau_k^{(j)}} \\ &= \operatorname{Re} \left\{ \frac{2\lambda^3 + 6\xi\lambda^2 + 2(2\xi^2 + 1)\lambda + 2\xi}{\lambda A_1 A_2 e^{-2\lambda\tau}} \right\}_{\tau=\tau_k^{(j)}} \\ &= \operatorname{Re} \left\{ \frac{2\xi - 6\xi\omega_k^2 + (4\xi^2\omega_k + 2\omega_k - 2\omega_k^3)i}{A_1 A_2 \omega_k \sin 2\omega_k \tau_k^{(j)} + i A_1 A_2 \omega_k \cos 2\omega_k \tau_k^{(j)}} \right\} \\ &= \frac{1}{A_1^2 A_2^2 \omega_k^2} \left\{ (2\xi - 6\xi\omega_k^2) A_1 A_2 \omega_k \sin 2\omega_k \tau_k^{(j)} \right. \\ &\quad \left. - (4\xi^2\omega_k + 2\omega_k - 2\omega_k^3) A_1 A_2 \omega_k \cos 2\omega_k \tau_k^{(j)} \right\} \\ &= \frac{1}{A_1^2 A_2^2 \omega_k^2} \left\{ (2\xi - 6\xi\omega_k^2) \omega_k (4\xi\omega_k^2 - 4\xi\omega_k) \right. \\ &\quad \left. - (4\xi^2\omega_k + 2\omega_k - 2\omega_k^3) \omega_k \right. \\ &\quad \left. \times [\omega_k^4 - (4\xi^2 + 2)\omega_k^2 + 1] \right\} \\ &= \frac{2\Lambda}{A_1^2 A_2^2}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Lambda &= 4\omega_k^6 - 3(2\xi^2 + 1)\omega_k^4 - 12\xi\omega_k^3 \\ &\quad + (8\xi^4 + 8\xi^2 + 12\xi + 3)\omega_k^2 \\ &\quad + 4\xi^2\omega_k - (6\xi^2 + 1). \end{aligned} \quad (17)$$

We assume that the following condition holds:

(H3) $\Lambda \neq 0$.

According to above analysis and the results of Kuang [29] and Hale [30], we have the following theorem

Theorem 3. *If (H1) and (H2) hold, then the equilibrium $E(0, 0, 0, 0)$ of system (2) is asymptotically stable for $\tau \in [0, \tau_0)$. Under the conditions (H1) and (H2), if the condition (H3)*

holds, then system (2) undergoes a Hopf bifurcation at the equilibrium $E(0, 0, 0, 0)$ when $\tau = \tau_k^{(j)}$, $k = 1, 2, 3, \dots, 8; j = 0, 1, 2, \dots$

3. Direction and Stability of the Hopf Bifurcation

In this section, we discuss the direction, stability and the period of the bifurcating periodic solutions. The used methods are based on the normal form theory and the center manifold theorem introduced by Hassard et al. [27]. From the previous section, we know that if $\tau = \tau_k^{(j)}$, $k = 1, 2, 3, \dots, 8; j = 0, 1, 2, \dots$ any root of (5) of the form $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ satisfies $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$ and $d\alpha(\tau)/d\tau|_{\tau=\tau_k^{(j)}} \neq 0$.

For convenience, let $\bar{x}_i(t) = x_i(\tau t)$ ($i = 1, 2, 3, 4$) and $\tau = \tau_k^{(j)} + \mu$, where $\tau_k^{(j)}$ is defined by (13) and $\mu \in \mathbb{R}$, drop the bar for the simplification of notations, then system (3) can be written as an FDE in $C = C([-1, 0], \mathbb{R}^4)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \quad (18)$$

where $u(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T \in C$ and $u_t(\theta) = u(t + \theta) = (x_1(t + \theta), x_2(t + \theta), x_3(t + \theta), x_4(t + \theta))^T \in C$, and $L_\mu : C \rightarrow \mathbb{R}$, $F : \mathbb{R} \times C \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} L_\mu \phi &= (\tau_k^{(j)} + \mu) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2\xi & 0 \\ 0 & 0 & -1 & -2\xi \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{pmatrix} \\ &\quad + (\tau_k^{(j)} + \mu) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ A_1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{pmatrix}, \quad (19) \\ f(\mu, \phi) &= (\tau_k^{(j)} + \mu) \begin{pmatrix} 0 \\ 0 \\ A_2 \phi_2^3(-1) + \text{h.o.t.} \\ A_1 \phi_1^3(-1) + \text{h.o.t.} \end{pmatrix}, \end{aligned}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C$.

From the discussion in Section 2, we know that if $\mu = 0$, then system (18) undergoes a Hopf bifurcation at the equilibrium $E(0, 0, 0, 0)$ and the associated characteristic equation of system (18) has a pair of simple imaginary roots $\pm \omega_k \tau_k^{(j)}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta) \quad \text{for } \phi \in C. \quad (20)$$

(a)

(b)

(c)

(d)

(e)

(f)

FIGURE 1: Continued.

FIGURE 1: Continued.

FIGURE 1: (a)–(p) The dynamical behavior of system (37) with $\tau = 0.7 < \tau_0 \approx 0.9$ and the initial value $(0.01, 0.02, 0.02, 0.01)$. The equilibrium $E_0(0, 0, 0, 0)$ is asymptotically stable.

In fact, we can choose

$$\eta(\theta, \mu) = \begin{pmatrix} \tau_k^{(j)} + \mu \\ 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2\xi & 0 \\ 0 & -1 & 0 & -2\xi \end{pmatrix} \delta(\theta) - \begin{pmatrix} \tau_k^{(j)} + \mu \\ 0 \\ 0 \\ A_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ A_1 & 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1), \quad (21)$$

where δ is the Dirac delta function.

For $\phi \in C([-1, 0], R^4)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \quad (22)$$

$$R\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then (18) is equivalent to the abstract differential equation

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \quad (23)$$

where $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$. For $\psi \in C([0, 1], (R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases} \quad (24)$$

For $\phi \in C([-1, 0], R^4)$ and $\psi \in C([0, 1], (R^4)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi^T(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \quad (25)$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_k \tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_k \tau_k^{(j)}$ and $-i\omega_k \tau_k^{(j)}$ respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta, \gamma)^T e^{i\omega_k \tau_k^{(j)} \theta}, \quad (26)$$

$$q^*(s) = D(1, \alpha^*, \beta^*, \gamma^*) e^{i\omega_k \tau_k^{(j)} s},$$

where

$$\alpha = \frac{1 - \omega_k^2 + 2\xi\omega_k i}{A_2 e^{-i\omega_k \tau_k^{(j)}}}, \quad \beta = i\omega_k, \quad \gamma = \frac{i\omega_k (1 - \omega_k^2 + 2\xi\omega_k i)}{A_2 e^{-i\omega_k \tau_k^{(j)}}}, \quad \alpha^* = \frac{1 - \omega_k^2 - 2\xi\omega_k i}{A_1 e^{-i\omega_k \tau_k^{(j)}}},$$

$$\beta^* = \frac{1}{2\xi - i\omega_k}, \quad \gamma^* = \frac{1 - \omega_k^2 - 2\xi\omega_k i}{A_1 e^{-i\omega_k \tau_k^{(j)}} (2\xi - i\omega_k)},$$

$$D = \frac{1}{1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \bar{\gamma}\gamma^* + \tau_k^{(j)} (A_1 \gamma^* + A_2 \beta^* \bar{\alpha}) e^{i\omega_k \tau_k^{(j)}}}. \quad (27)$$

Furthermore, $\langle q^*(s), q(\theta) \rangle = 1$ and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Next, we use the same notations as those in Hassard et al. [27] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (18) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - 2 \operatorname{Re}\{z(t)q(\theta)\}, \quad (28)$$

on the center manifold C_0 , and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \quad (29)$$

FIGURE 2: Continued.

FIGURE 2: Continued.

FIGURE 2: (a)–(p) The dynamical behavior of system (37) with $\tau = 1.5 > \tau_0 \approx 0.9$ and the initial value $(0.01, 0.02, 0.02, 0.01)$. Hopf bifurcation occurs from the equilibrium $E_0(0, 0, 0, 0)$.

where

$$\begin{aligned} W(z(t), \bar{z}(t), \theta) &= W(z, \bar{z}) \\ &= W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots, \end{aligned} \quad (30)$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (18),

$$\begin{aligned} \dot{z}(t) &= i\omega_k \tau_k^{(j)} z + \bar{q}^*(\theta) f(0, W(z, \bar{z}, \theta) + 2 \operatorname{Re}\{zq(\theta)\}) \\ &\stackrel{\text{def}}{=} i\omega_k \tau_k^{(j)} z + \bar{q}^*(0) f_0. \end{aligned} \quad (31)$$

That is,

$$\dot{z}(t) = i\omega_k \tau_k^{(j)} z + g(z, \bar{z}), \quad (32)$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots \quad (33)$$

Hence we have

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f_0(z, \bar{z}) = \bar{q}^*(0) f(0, u_t) \\ &= \tau_k^{(j)} \bar{D} \begin{pmatrix} 0 \\ 0 \\ A_2 x_{2t}^3(-1) + \text{h.o.t.} \\ A_1 x_{1t}^3(-1) + \text{h.o.t.} \end{pmatrix} \\ &= \bar{D} \tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \bar{\alpha} e^{-2i\omega_k \tau_k^{(j)}} \right] z^2 \bar{z} + \text{h.o.t.} \end{aligned} \quad (34)$$

and we obtain

$$\begin{aligned} g_{20} &= g_{11} = g_{02} = 0, \\ g_{21} &= 2\bar{D} \tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \bar{\alpha} e^{-2i\omega_k \tau_k^{(j)}} \right]. \end{aligned} \quad (35)$$

Therefore, we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_k \tau_k^{(j)}} \left(g_{20} g_{11} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_k^{(j)})\}}, \\ \beta_2 &= 2 \operatorname{Re}(c_1(0)), \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_k^{(j)})\}}{\omega_k \tau_k^{(j)}}, \end{aligned} \quad (36)$$

which determine the quantities of bifurcation periodic solutions on the center manifold at the critical value $\tau_k^{(j)}$ ($k = 1, 2, 3, \dots, 8; j = 0, 1, 2, 3, \dots$). μ_2 determines the direction of the Hopf bifurcation: If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); β_2 determines the stability of the bifurcating periodic solutions: the periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period is increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical Examples

To illustrate the analytical results found, let $\xi = 0.1$, $A_1 = 0.4$, $A_2 = 0.2$, then system (2) becomes

$$\begin{aligned} \frac{dx_1}{dt} &= x_3, \\ \frac{dx_2}{dt} &= x_4, \\ \frac{dx_3}{dt} &= -0.2x_3 - x_1 + 0.2 \tanh(x_2(t - \tau)), \\ \frac{dx_4}{dt} &= -0.2x_4 - x_2 + 0.4 \tanh(x_1(t - \tau)), \end{aligned} \quad (37)$$

which has a unique equilibrium $E_0(0, 0, 0, 0)$ and satisfies the conditions indicated in Theorem 3. The equilibrium $E_0(0, 0, 0, 0)$ is asymptotically stable for $\tau = 0$. For $j = 0$,

using the software Matlab, we derive $\omega_0 \approx 3.4233$, $\tau_0 \approx 0.9$, $\lambda'(\tau_0) \approx 0.2556 - 2.3588i$, $g_{21} \approx -0.2409 - 3.7451i$. Thus by the algorithm (36) derived in Section 3, we get $c_1(0) \approx -0.1205 - 1.8726i$, $\mu_2 \approx -0.4714$, $\beta_2 \approx -0.2410$, $T_2 \approx 0.9687$. Thus the equilibrium $E_0(0, 0, 0, 0)$ is stable when $\tau < \tau_0 \approx 0.9$. Figures 1(a)–1(j) show that the equilibrium $E_0(0, 0, 0, 0)$ is asymptotically stable when $\tau = 0.7 < \tau_0 \approx 0.9$. When τ passes through the critical value $\tau_0 \approx 0.9$, the equilibrium $E_0(0, 0, 0, 0)$ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from the equilibrium $E_0(0, 0, 0, 0)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0 \approx 0.9$, and these bifurcating periodic solutions from $E_0(0, 0, 0, 0)$ at $\tau_0 \approx 0.9$ are stable. Figures 1(j)–2(d) suggest that Hopf bifurcation occurs from the equilibrium $E_0(0, 0, 0, 0)$ when $\tau = 1.5 > \tau_0 \approx 0.9$.

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