Research Article

Some Results Involving the Splitting Operation on Binary Matroids

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We obtain some results concerning the planarity and graphicness of the splitting matroids. Further, we explore the effect of splitting operation on the sum of two matroids.

1. Introduction

The matroid notations and terminology used here will follow Oxley [1].

Fleischner [2] defined the splitting operation for a graph with respect to a pair of adjacent edges as follows.

Let G be a connected graph and let v be a vertex of degree at least three in G . If $x = vv_1$ and $y = vv_2$ are two edges incident at v , then splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y , and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2 . The transition from G to $G_{x,y}$ is called the splitting operation on G . Figure 1 illustrates this construction explicitly.

Fleischner [2] used the splitting operation to characterize Eulerian graphs. Fleischner [3] also developed an algorithm to find all distinct Eulerian trails in an Eulerian graph using the splitting operation. Tutte [4] characterized 3-connected graphs, and Slater [5] classified 4-connected graphs using a slight variation of this operation.

Raghunathan et al. [6] extended the splitting operation from graphs to binary matroids as follows.

Definition 1.1. Let $M = M[A]$ be a binary matroid and suppose $x, y \in E(M)$. Let $A_{x,y}$ be the matrix obtained from A by adjoining the row that is zero everywhere except for the entries of 1 in the columns labeled by x and y . The splitting matroid $M_{x,y}$ is defined to be the vector matroid of the matrix $A_{x,y}$.

Figure 1

Figure 2: Forbidden graphs.

Alternatively, the splitting operation can be defined in terms of circuits of binary matroids as follows.

Lemma 1.2 (see [6]). *Let $M = (E, \mathcal{C})$ be a binary matroid on a set E together with the set $\mathcal{C}$ of circuits and let $\{x, y\} \subseteq E$. Then $M_{x,y} = (E, \mathcal{C}')$ with $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$, where $\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C \text{ or } x \notin C, y \notin C\}$; and $\mathcal{C}_1 = \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, x \in C_1, y \in C_2, C_1 \cap C_2 = \emptyset \text{ and } C_1 \cup C_2 \text{ contains no member of } \mathcal{C}_0\}$.*

Shikare and Waphare [7] characterized graphic matroids whose splitting matroids are also graphic. In fact, they proved the following theorem.

Theorem 1.3 (see [7]). *The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the matroid has no minor isomorphic to the cycle matroid of any of the four graphs of Figure 2.*

We define a matroid to be planar if it is both graphic and cographic. Let us consider the graph (i) of Figure 2. In this graph by splitting away two non-adjacent elements x and y , we will get a matroid which is not planar. This example exhibits the fact that even if the original matroid is planar, there exist pairs of non-adjacent elements x and y such that the splitting matroid is not planar.

Now, by considering above example, we prove the following Theorem.

Theorem 1.4. *Let G be a planar graph. Then there is at least one pair of non-adjacent edges x and y such that $G_{x,y}$ is a planar graph.*

The proof of the theorem is given in Section 3.

Theorem 1.5 (see [1]). *A binary matroid is graphic if and only if it has no minor isomorphic to $F_7, F_7^*, M^*(K_5)$, or $M^*(K_{3,3})$.*

Figure 3

The following example exhibits the fact that there exists a graphic matroid M and a pair of non-adjacent edges, such that the splitting of M with respect to this pair does not yield a graphic matroid.

Example 1.6. Consider the matroid $M = M(K_5)$ where K_5 is the complete graph on 5 vertices as shown in Figure 3. The matroid has the ground set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the collection of circuits

$$\begin{aligned}
 \mathcal{C} = & \{ \{1, 2, 3\}, \{1, 4, 5\}, \{1, 9, 10\}, \{2, 4, 7\}, \{3, 6, 9\}, \{4, 8, 10\}, \{5, 8, 9\}, \{3, 5, 7\}, \\
 & \{6, 7, 8\}, \{2, 6, 10\}, \{1, 2, 6, 9\}, \{1, 4, 8, 9\}, \{1, 2, 5, 7\}, \{1, 3, 4, 7\}, \{2, 3, 4, 5\}, \\
 & \{2, 4, 6, 8\}, \{4, 6, 7, 10\}, \{3, 7, 8, 9\}, \{2, 7, 8, 10\}, \{5, 6, 7, 9\}, \{3, 5, 6, 8\}, \\
 & \{1, 5, 8, 10\}, \{2, 3, 9, 10\}, \{1, 3, 6, 10\}, \{4, 5, 9, 10\}, \{1, 2, 7, 8, 9\}, \{2, 3, 4, 8, 9\}, \\
 & \{1, 2, 5, 6, 8\}, \{2, 5, 7, 9, 10\}, \{3, 4, 5, 6, 10\}, \{1, 4, 6, 7, 9\}, \{1, 3, 7, 8, 10\}, \\
 & \{1, 5, 6, 7, 10\}, \{3, 4, 7, 9, 10\}, \{2, 3, 5, 8, 10\}, \{2, 4, 5, 6, 9\}, \{1, 3, 4, 6, 8\} \}.
 \end{aligned} \tag{1.1}$$

The matroid M is graphic and it arises as the cycle matroid of the graph G of Figure 3.

Consider the non-adjacent elements 1 and 6 of M . Then, by Lemma 1.2, $M_{1,6} = (S, \mathcal{C}_{1,6})$ is the matroid with ground set S and circuit set $\mathcal{C}_{1,6}$. By contracting 1 in $M_{1,6}$, we get a matroid $(M_{1,6})/\{1\}$, which is isomorphic to the matroid $M^*(K_{3,3})$. Thus, by Theorem 1.5, $M_{1,6}/\{1\}$ is not graphic. By similar arguments, we can check that splitting of $M(K_5)$ by any other pair of non-adjacent edges is not graphic. With this observation, we state the following theorem for the splitting matroids.

Theorem 1.7. *Let $M = M(G)$ be a graphic matroid that is not isomorphic to $M(K_5)$. Then there exists at least one pair of non-adjacent edges x and y of $E(G)$ such that $M_{x,y}$ is graphic.*

The proof of the theorem is given in Section 3.

Shikare et al. [8] introduced the concept of generalized splitting operation for a graph with respect to n adjacent edges in the following way.

Definition 1.8. Let G be a connected graph and v a vertex of G with $d(v) \geq n + 1$. Let $T = \{x_1 = vv_1, x_2 = vv_2, \dots, x_n = vv_n\}$ be a set of adjacent edges incident at v . Then splitting away the edges in T from v results in a new graph G_T obtained from G by deleting the edges $x_1, x_2, \dots, x_n$ and adding a new vertex v_T adjacent to $v_1, v_2, \dots, v_n$. We say that the graph G_T

Figure 4: Graph and its generalized splitting.

has been obtained from G by splitting away the edges $x_1, x_2, \dots, x_n$ or in short with respect to the set T .

This construction is illustrated in Figure 4 where $T = \{x_1, x_2, x_3\}$.

Shikare et al. [8] later on extended the notion of the generalized splitting operation from graphs to binary matroids in the following way.

Definition 1.9. Let $M = M[A]$ be a binary matroid on a set S and T a subset of S . Suppose that A is a matrix over $GF(2)$ that represents the matroid M . Let A_T be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to the elements of T where they take the value 1. Let M_T be the matroid represented by the matrix A_T . We say that M_T has been obtained from M by splitting the set T . The transition from M to M_T is called a generalized splitting operation.

In this paper, we explore the effect of the splitting operation on the sum of two matroids and give some application of these results.

2. The Splitting Operation on the Sum of Two Matroids

In this section, we provide some definitions and the results which are used in the proof of theorems.

Proposition 2.1 (see [1]). *The following statements are equivalent for a graph G :*

- (i) G is a planar graph;
- (ii) $M(G)$ is a planar matroid;
- (iii) $M(G)$ has no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$.

Proposition 2.2 (see [1]). *Let T_1 and T_2 be disjoint subsets of $E(M)$. Then*

- (i) $(M \setminus T_1) \setminus T_2 = M \setminus (T_1 \cup T_2) = (M \setminus T_2) \setminus T_1$;
- (ii) $(M/T_1)/T_2 = M/(T_1 \cup T_2) = (M/T_2)/T_1$;
- (iii) $(M \setminus T_1)/T_2 = (M/T_2) \setminus T_1$.

Proposition 2.3 (see [7]). *Let x and y be two elements of a binary matroid M . Then*

- (i) $M_{x,y} = M$ if and only if x and y are in series or both x and y are coloops in M ;
- (ii) $(M_{x,y})/\{x\} \setminus \{y\} \cong (M_{x,y})/\{y\} \setminus \{x\} \cong (M_{x,y}) \setminus \{x, y\} \cong M \setminus \{x, y\}$.

Definition 2.4 (see [9]). If $M_1 = (E, \mathcal{Z}_1)$ and $M_2 = (E, \mathcal{Z}_2)$ are two (not necessarily different) matroids on the same set E then let us define $\mathcal{Z}_1 \vee \mathcal{Z}_2 = \{X : X = X_1 \cup X_2, \text{ where } X_1 \in \mathcal{Z}(M_1), X_2 \in \mathcal{Z}(M_2)\}$. The matroid $M_1 \vee M_2 = (E, \mathcal{Z}_1 \vee \mathcal{Z}_2)$ is called the *sum* of two matroids.

Theorem 2.5 (see [9, 10]). Suppose $M_1, M_2, \dots, M_k$ are matroids on the same set E . If the real matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k$ coordinates the respective matroids $\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k$ and all the nonzero entries of all these matrices are algebraically independent over the field $\mathbf{Q}$ of the rationales, then the

matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}$ coordinates the matroid $\vee_{i=1}^k \mathbf{M}_i$.

Theorem 2.6 (see [6]). Suppose M is a binary matroid on a set S and $x, y \in S$. Then M is Eulerian if and only if $M_{x,y}$ is Eulerian.

Definition 2.7 (see [1]). Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2 . Let $E = E_1 \cup E_2$ and $\mathcal{Z} = \{I_1 \cup I_2 : I_1 \in \mathcal{Z}(M_1), I_2 \in \mathcal{Z}(M_2)\}$. Then $(E, \mathcal{Z})$ is a matroid. This matroid is called the *direct sum* or *1-sum* of M_1 and M_2 and is denoted $M_1 \oplus M_2$.

In the next theorem, Shikare et al. [8] explored the relation between direct sums of two matroids and the splitting operation.

Theorem 2.8 (see [8]). Let $M = (S_1, \mathcal{C}_1)$ and $N = (S_2, \mathcal{C}_2)$ be two binary matroids with $S_1 \cap S_2 = \emptyset$. Let T_1 and T_2 be subsets of S_1 and S_2 , respectively. If there is no circuit of M or N containing an odd number of elements of T_1 and T_2 , respectively, then $M_{T_1} \oplus M_{T_2} = (M \oplus N)_{T_1 \cup T_2}$.

3. Applications

In this section, we use definitions and the results of Section 2 to prove Theorems 1.4 and 1.7. We also explore the effect of the splitting operation on the sum of two matroids.

Proof of Theorem 1.4. Suppose G is a planar graph. By Proposition 2.1, $M(G)$ is a planar matroid, and therefore $M(G)$ has no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. We claim that $M(G_{x,y})$ is planar for at least one non-adjacent pair of edges x and y of G .

On the contrary, suppose that $M(G_{x,y})$ is not planar for every non-adjacent pair of edges x and y of $E(M)$. This implies that $M(G_{x,y})$ has a minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. First, suppose that $M(G_{x,y})$ has $M(K_{3,3})$ as a minor. Since no two elements in $M(K_{3,3})$ are in series, x and y cannot be elements of $M(K_{3,3})$. Let $T_1, T_2 \subseteq E(M)$ and $x, y \in T_1 \cup T_2$. We have $M(G)_{x,y} / T_1 \setminus T_2 \cong M(K_{3,3})$. This implies that $M(G)_{x,y} / (T_1 - \{x\}) / \{x\} \setminus (T_2 - \{y\}) \setminus \{y\} \cong M(K_{3,3})$. By Proposition 2.2, we have $(M(G)_{x,y} / \{x\} \setminus \{y\}) / (T_1 - \{x\}) \setminus (T_2 - \{y\}) \cong M(K_{3,3})$. Now, by Proposition 2.3 (ii), we have $M(G)_{x,y} \setminus \{x, y\} / (T_1 - \{x\}) \setminus (T_2 - \{y\}) \cong M(K_{3,3})$. So $M(G) \setminus \{x, y\} / (T_1 - \{x\}) \setminus (T_2 - \{y\}) \cong M(K_{3,3})$. We conclude that $M(G)$ has a minor isomorphic to $M(K_{3,3})$ which is a contradiction to our assumption that $M(G)$ is a planar matroid. So $M(G_{x,y})$ cannot have $M(K_{3,3})$ as a minor. Now, suppose that $M(G_{x,y})$ has $M(K_5)$ as a minor. Then, by similar arguments, we arrive to a contradiction. We conclude that $M(G_{x,y})$ cannot have $M(K_5)$ as a minor as well. Therefore, $M(G)_{x,y}$ is planar for at least one pair of non-adjacent edges x and y . Consequently, $G_{x,y}$ is planar for at least one pair of non-adjacent edges x and y . $\square$

Now, we use Theorem 1.3 to prove Theorem 1.7.

Figure 5

Proof of Theorem 1.7. Let M be a graphic matroid. Then M is isomorphic to a cycle matroid $M(G)$ of some graph G . Suppose M is not isomorphic to $M(K_5)$. By Theorem 1.3, the splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the matroid has no minor isomorphic to the cycle matroid of any of the four graphs of Figure 2. If M has no minor isomorphic to the cycle matroid of any of the four graphs of Figure 2, then there exists at least one pair of non-adjacent edges x and y of $E(G)$ such that $(M(G))_{x,y}$ is graphic. If M has minor isomorphic to the cycle matroid of any of the three graphs (i), (ii), and (iii) of Figure 5, then each of these graphs contains at least one pair of non-adjacent edges x and y of $E(G)$ such that $(M(G))_{x,y}$ is graphic.

As proved in Example 1.6 in Section 1, if $M = M(K_5)$, there is no pair of non-adjacent edges x and y such that $M_{x,y}$ is graphic and by Theorem 1.3, $M(K_5)$ is only matroid which has this property and this completes the proof. $\square$

In the next theorem, we explore the relation between the sum of two matroids and the splitting operation.

Theorem 3.1. Let $M = (S, C_1)$ and $N = (S, C_2)$ be binary matroids on the same underlying set S . If $x, y \in S$, then $M_{x,y} \vee N_{x,y} = (M \vee N)_{x,y}$.

Proof. Let A and B be the matrices that represent M and N , respectively. The matrix representation for $M_{x,y}$ and $N_{x,y}$, say A' and B' , is obtained from A and B , respectively, by adjoining extra rows to A and B , which are zero everywhere except in the columns corresponding to the elements x and y where it takes the value 1.

By Theorem 2.5, the matrix representation for $(M_{x,y} \vee N_{x,y})$ is the matrix $\begin{bmatrix} A' \\ B' \end{bmatrix}$ with two equal rows. By replacing a row by the sum of that row and the row which is equal to that, we get one zero row. By deleting the zero row, we obtain a matrix representation of the matroid $(M_{x,y} \vee N_{x,y})$.

The matrix representation of $(M \vee N)_{x,y}$, say D , can be obtained by adjoining one extra row to the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ which is zero everywhere except in the columns corresponding to the elements x and y where it takes the value 1. We observe that $D = \begin{bmatrix} A' \\ B' \end{bmatrix}$. This completes the proof. $\square$

It is well known that a graph is Eulerian if and only if its edge set can be partitioned into disjoint circuits. Generalizing this graph theoretic concepts, Welsh [11] defined Eulerian matroid. A matroid (S, C) is said to be Eulerian if the ground set S is the union of disjoint circuits of the matroid. Further, Welsh [11] proved that a binary matroid is Eulerian if and only if its dual matroid is bipartite.

The following theorem states that the splitting of the sum of two binary Eulerian matroids is the sum of the corresponding splitting matroids.

Theorem 3.2. *Let $M = (S, C_1)$ and $N = (S, C_2)$ be binary matroids with $x, y \in S$. Then M and N are Eulerian if and only if $M_{x,y} \vee N_{x,y}$ is Eulerian.*

Proof. Let A and B be the matrices that represent M and N , respectively. Suppose M and N are Eulerian matroids. By Theorem 2.6, $M_{x,y}$ and $N_{x,y}$ are Eulerian. Let A' and B' be the matrices obtained from A and B by adjoining extra rows, which are zero everywhere except in the columns corresponding to the elements x and y where they take the value 1. Then the matrix $D = \begin{bmatrix} A' \\ B' \end{bmatrix}$ is a matrix representation of the matroid $M_{x,y} \vee N_{x,y}$. Since the number of 1's in each row of the matrix D is even, the matroid $M_{x,y} \vee N_{x,y}$ is Eulerian.

The converse part of proof is straightforward. $\square$

Using Theorem 3.2, we prove that the splitting of the sum of n binary Eulerian matroids is the sum of the corresponding splitting matroids.

Theorem 3.3. *Let $M_1 = (S, C_1), M_2 = (S, C_2), \dots, M_n = (S, C_n)$ be binary matroids with $x, y \in S$. Then $M_1, M_2, \dots, M_n$ are Eulerian if and only if $(M_1)_{x,y} \vee (M_2)_{x,y} \vee \dots \vee (M_n)_{x,y}$ is Eulerian.*

Proof. Let $A_1, A_2, \dots, A_n$ be the matrices that represent $M_1, M_2, \dots, M_n$, respectively. Suppose that $M_1, M_2, \dots, M_n$ are Eulerian matroids. By Theorem 2.6, $(M_1)_{x,y}, (M_2)_{x,y}, \dots, (M_n)_{x,y}$ are Eulerian. Let $A'_1, A'_2, \dots, A'_n$ be the matrices obtained from $A_1, A_2, \dots, A_n$ by adjoining extra rows, which are zero everywhere except in the columns corresponding to the elements x and

y where they take the value 1. Then the matrix $D = \begin{bmatrix} A'_1 \\ A'_2 \\ \vdots \\ A'_n \end{bmatrix}$ is a matrix representation of the matroid $(M_1)_{x,y} \vee (M_2)_{x,y} \vee \dots \vee (M_n)_{x,y}$. Since the number of 1's in each row of the matrix D is even, the matroid $(M_1)_{x,y} \vee (M_2)_{x,y} \vee \dots \vee (M_n)_{x,y}$ is Eulerian.

The converse part of proof is straightforward. $\square$

Theorem 3.4. *Let $M = (S_1, C_1)$ and $N = (S_2, C_2)$ be two binary matroids with $S_1 \cap S_2 = \emptyset$. Let T_1 and T_2 be subsets of S_1 and S_2 , respectively. Then M and N are Eulerian if and only if $(M \oplus N)_{T_1 \cup T_2}$ is Eulerian.*

Proof. Let $M = (S_1, C_1)$ and $N = (S_2, C_2)$ be two binary matroids with $S_1 \cap S_2 = \emptyset$. Let T_1 and T_2 be subsets of S_1 and S_2 , respectively. Suppose the M and N are Eulerian. Then, S_1 and S_2 can be expressed as a disjoint unions of circuits of M and N , respectively. Suppose that $S_1 = C_1 \cup C_2 \cup \dots \cup C_m$ and $S_2 = C'_1 \cup C'_2 \cup \dots \cup C'_n$. Then $S_1 \cup S_2 = C_1 \cup C_2 \cup \dots \cup C_m \cup C'_1 \cup C'_2 \cup \dots \cup C'_n$ is a disjoint union of circuits of $M \oplus N$. So $M \oplus N$ is Eulerian. By Theorem 2.8, $(M \oplus N)_{T_1 \cup T_2}$ is Eulerian.

The converse part of the proof is straightforward. $\square$

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