Research Article

On Strong Solutions of Regularized Model of a Viscoelastic Medium with Variable Boundary

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We consider the initial-value problem for systems of equations describing the evolution of a viscoelastic medium with variable boundary with memory along the trajectories of a velocity field, which generalizes the Navier-Stokes system of equations. Nonlocal existence and uniqueness theorem of strong solutions containing senior square-integrable derivatives in the planar case are established.

1. Introduction

Let $\Omega_t \subset R^n$, $n = 2, 3$, be a family of bounded domains with $\partial\Omega_t \in C^2$, depending smoothly on $0 \leq t \leq T, < +\infty$. Consider the initial-boundary problem

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{v_i \partial v}{\partial x_i} - \mu_0 \Delta v - \mu \operatorname{Div} \int_0^t \exp(\lambda(s-t)) \mathcal{E}(v)(s, z(s; t, x)) ds + \nabla p &= f(t, x), \quad (t, x) \in Q_T, \\ \operatorname{div} v(t, x) &= 0, \quad (t, x) \in Q_T, \end{aligned} \quad (1.1)$$

$$\begin{aligned} v(0, x) &= v^0(x), \quad x \in \Omega_0, \\ v(t, x) &= \psi(t, x), \quad (t, x) \in S_T, \end{aligned} \quad (1.2)$$

where $Q_T = \{(t, x) : 0 \leq t \leq T, x \in \Omega_t\}$, $S_T = \{(t, x) : t \in [0, T], x \in \partial\Omega_t\}$.

In (1.1)-(1.2), the velocity $v(t, x) = (v_1(t, x), \dots, v_n(t, x))$ and the pressure $p(t, x)$ are unknown, $f(t, x)$ is given outer forces, $v^0(x)$ is prescribed initial velocity, $\mathcal{E}(v) = \{\mathcal{E}_{ij}\}_{i,j=1}^n$ is

the rate-of-strain tensor, that is, the matrix with elements $\xi_{ij}(v) = (1/2)(\partial v_i / \partial x_j + \partial v_j / \partial x_i)$, $\mu_0 > 0$, $\mu \geq 0$, $\lambda \geq 0$. The divergence Div of a matrix is defined as a vector with elements which are the divergences of the lines. The vector function $z(\tau; t, x)$ is a solution to the Cauchy problem (in the integral form)

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s; t, x)) ds, \quad \tau \in [0, T], \quad (t, x) \in Q_T. \quad (1.3)$$

The system describes the evolution Ω_t of a viscoelastic medium Ω_0 with a memory along the trajectories of the velocity vector field.

In the case of a cylindrical domain Q_T ($\Omega_t \equiv \Omega_0$), the local existence and uniqueness theorem (e.u.t.) for strong solutions ($v \in W_q^{1,2}(Q_T)$, $q > n$) to the problem (1.1)-(1.2), (1.3) and nonlocal e.u.t. for small data were established in [1, 2]. Nonlocal e.u.t. for strong solutions ($v \in W_2^{1,2}(Q)$) to the problem (1.1)-(1.2), (1.3), where v in (1.3) is replaced by some regularized vector field $\tilde{v}$ (regularized problem) for $n = 2$, was established in [3]. There was given the motivation of the regularization of the velocity field v . In the case that $\mu = 0$ and Q_T is noncylindrical domain, the local for $n = 3$ and nonlocal for $n = 2$ (e.u.t.) of strong solutions ($v \in W_2^{1,2}(Q_T)$) to the system (1.1)-(1.2) were established in [4]. The e.u.t. of weak solutions to the regularized problem (1.1)-(1.2), (1.3) with $\mu > 0$ in noncylindrical domain was established in [5].

Our goal is to prove a nonlocal e.u.t. for solutions (v, p) , $v \in W_2^{1,2}(Q_T)$, $p \in W_2^{0,1}(Q_T)$ by $n = 2$. First, we set a priori estimates, then we prove the local solvability by means of reduction of the initial problem to a suitable operator equation, and then we obtain a nonlocal theorem using the obtained a priori estimates.

Next, referring to the solution to the problem (1.1)-(1.2), (2.1), we, as usual, keep in mind only the solenoidal part v of the pair (v, p) . The function p is found easily by the standard way (see, e.g., [6, Chapter 3, page 214]).

The constants in the estimates which do not depend on substantial parameters, we denote by M , sometimes with indices. We take the summation convention for repeated indices.

2. Notations and Statement of the Results

The norms of a function u in real spaces $L_2(\Omega_t)$, $L_2(Q_T)$ and in Sobolev spaces $W_2^k(\Omega_t)$, $W_2^{m,k}(Q_T)$ we denote by $|u|_{0,t}$, $\|u\|_0$, $|u|_{k,t}$, $\|u\|_{k,m}$, respectively. We use the same designation for scalar, vector, or matrix functions.

The index k can be negative (see [7, page 251]). We denote by $(\cdot, \cdot)_t$ the inner product in $L_2(\Omega_t)$, H_t and V_t are the closure of smooth compactly supported solenoidal functions in Ω_t in the norm of $L_2(\Omega_t)$ and $W_2^1(\Omega_t)$, respectively.

We denote by ρ_t the orthogonal projector in $L_2(\Omega_t)$ on H_t . Consider in H_t unbounded linear operator $A_t v = -\rho_t \Delta v$, with the domain $D(A_t) = W_2^2(\Omega_t) \cap H_t \cap \overset{\circ}{W}_2^1(\Omega_t)$. Operator A_t (see [8, page 54]) is a positive self-adjoint operator. We define the regularizer S_δ , $\delta > 0$ as $S_\delta v = \hat{S}_\delta(v - w) + w$, where $\hat{S}_\delta v = (I + \delta A_t)^{-1} v$. Consider the Cauchy problem

$$z(\tau; t, x) = x + \int_t^\tau \hat{v}(s, z(s; t, x)) ds, \quad \tau \in [0, T], \quad (T, x) \in Q_T, \quad (2.1)$$

where $S_\delta v \equiv \hat{v}$. If $v \in W_2^{1,2}(Q_T)$ and the evolution of Ω_t is smooth, then regularized problem (2.1) is uniquely solvable (see below). As to regularization operator S_δ , we note that it is uniformly bounded of t as an operator from $L_2(\Omega_t)$ in $C^1(\Omega_t)$. There are other constructions of such operators (see, e.g., [3, 9]). Every one of them has the property of strong convergence of S_δ to I by $\delta \rightarrow 0$.

We believe that $\Omega_t = \tilde{z}(t, 0, \Omega_0)$, where $\tilde{z}(\tau; t, x)$ is a solution to the Cauchy problem

$$\tilde{z}(\tau; t, x) = x + \int_t^\tau \tilde{w}(s, \tilde{z}(s; t, x)) ds, \quad \tau \in [0, T], \quad (t, x) \in Q_T, \quad (2.2)$$

where $\tilde{w}$ is sufficiently smooth on some neighborhood of Q_T solenoidal function. The boundary condition (1.2) is defined as $\varphi(t, x) = \tilde{w}(t, x)$, $(t, x) \in S_T$ under the natural condition $v^0(x) = \varphi(0, x)$, $x \in \partial\Omega_0$. Note that the same Q_T admits representation via evolution Ω_t by means of different functions $\tilde{w}$ with correspondent boundary values φ .

The main result reads as follows.

Theorem 2.1. *Let $n = 2$, $f \in L_2(Q_T)$, $v^0 \in W_2^1(\Omega_0)$, $\operatorname{div} v^0(x) = 0$. Then, problem (1.1)-(1.2), (2.1) has a unique solution $v \in W_2^{1,2}(Q_T)$.*

As to the form of Q_T and the boundary function $\varphi(t, x)$, we preferred to concentrate ourselves on the essence of the problem and chose a simple way to provide the solvability of (2.1) on $[0, T]$ and to avoid a complexity in the proofs. Let the assumption Q_T be a sufficiently arbitrary domain, and let $\varphi(t, x)$ be the trace of a solenoidal $\varphi(t, x) \in W_2^{1,2}(Q_T)$ and satisfy $\int_T (\varphi(t, x), n(x)) dx = 0$ ($n(x)$ is the outward unit normal) (as in [4]) leads to the effect of leakage through the boundary that in turn leads to the necessity to study the properties of solutions to (2.1) which may be defined only on a part of $[0, T]$.

We believe also $\lambda = 0$, $\mu_0 = 1$, and $\tilde{w}$ is sufficiently smooth in order to simplify the calculus (in fact, it is sufficient that $\tilde{w} \in W_q^{1,2}(Q)$, $q > n$).

We reduce the original problem to a problem with zero boundary data. Let $v = u + w$, where a solenoidal function $w \in W_2^{1,2}(Q)$ and $w(t, x) = \tilde{w}(t, x)$ on S_T . Then, u satisfies relations

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{u_i \partial u}{\partial x_i} + \frac{w_i \partial u}{\partial x_i} + \frac{u_i \partial w}{\partial x_i} - \Delta u - \mu \operatorname{Div} \int_0^t \mathcal{E}(u)(s, z(s; t, x)) ds \\ - \mu \operatorname{Div} \int_0^t \mathcal{E}(w)(s, z(s; t, x)) ds + \nabla p = F(t, x), \quad \operatorname{div} u(t, x) = 0, \quad (t, x) \in Q_T, \end{aligned} \quad (2.3)$$

$$u(0, x) = u^0(x), \quad x \in \Omega_0; \quad u(t, x) = 0, \quad (t, x) \in S_T, \quad (2.4)$$

and (2.1). Here, $u = (u_1, u_2)$, $w = (w_1, w_2)$,

$$F(t, x) = f(t, x) - \frac{\partial w}{\partial t} - \frac{w_i \partial w}{\partial x_i} + \Delta w, \quad u^0(x) = v^0(x) - w(0, x). \quad (2.5)$$

Instead of Theorem 2.1, it will be convenient for us to prove its reformulation for the problem (2.3)-(2.4), (2.1).

Theorem 2.2. *Let $F \in L_2(Q_T)$, $u^0 \in V_0$. Then, problem (2.3)-(2.4), (2.1) has a unique solution $u \in W_2^{1,2}(Q_T)$.*

The proof of Theorem 2.2 consists of several steps.

3. Properties of Solutions to Cauchy Problem

By a solution to (1.3), we mean a continuous with respect to all variables vector function $z(\tau; t, x)$, satisfying (1.3). Denote by u_x a Jacobi matrix of the vector function $u(x)$, by u_{xx} a tensor which is composed of second derivatives of the components of a vector function $u(x)$, and by $|u_x|_0^2$ and $|u_{xx}|_0^2$ the sum of squares of $L_2(\Omega)$ -norms of the first and second derivatives, respectively. The following result was established in [1] for a cylindrical domain Q_T .

Lemma 3.1 (see [2]). *Let $v, v^1, v^2 \in W_q^{1,2}(Q_T)$, $q > n$ and vanish at S_T . Then, the corresponding Cauchy problems (1.3) are uniquely solvable, and the estimates*

$$\begin{aligned} \|z^1(\tau; t, x) - z^2(\tau; t, x)\|_{W_{q_1}^1(\Omega_t)} &\leq M \left| \int_t^\tau \|v^1(s, x) - v^2(s, x)\|_{W_{q_1}(\Omega_s)} ds \right| \\ &\quad \times \exp \left(M \left| \int_t^\tau \max_{k=1,2} \|v^k(s, x)\|_{W_{q_1}^2(\Omega_s)} ds \right| \right), \\ \tau, t &\in [0, T], \quad 1 \leq q_1 \leq q, \end{aligned} \quad (3.1)$$

$$\|z_{xx}(\tau; t, x)\|_{L_q(\Omega_t)} \leq M \left(\int_0^T \|v(s, x)\|_{W_q^2(\Omega_s)} ds \right) \left| \int_t^\tau \|v(s, x)\|_{W_q^2(\Omega_s)} ds \right|$$

hold. If, moreover, $\|v(s, x)\|_{C^1(\Omega_s)} \in L_1[0, T]$, then

$$\|z_x(\tau; t, x)\|_{C(\Omega_t)} \leq M \exp \left(\left| \int_t^\tau \|v(s, x)\|_{C^1(\Omega_s)} ds \right| \right), \quad \tau, t \in [0, T]. \quad (3.2)$$

Using the smoothness of the evolution of Ω_t and the fact that the lateral side consists of the trajectories of a smooth velocity field $\tilde{w}$, we can show that the similar result holds for noncylindrical domain Q_T by $v, v^1, v^2 = \psi$ on S_T . To do this, one has as in [10, Chapter 3, Section 1.2] to continue v, v^1, v^2 on R^n with the preservation of the class by functions vanishing outside some compact domain $Q = [0, T] \times \Omega$, $\Omega \supset \Omega_t$, $0 \leq t \leq T$, and use the result for the cylindrical case. In virtue of the uniqueness theorem, the solutions to Cauchy problems of our initial problems are the restrictions of solutions of the extended Cauchy problems.

Consider the Cauchy problem (2.1). Best, compared with v , properties of $\hat{v}$ allow to strengthen the result of Lemma 3.1.

Lemma 3.2. *Let $v, v^1, v^2 \in W_2^{1,2}(Q_T)$. Then, corresponding Cauchy problems (2.1) are uniquely solvable, and the estimates*

$$\|z_x(\tau; t, x)\|_{C(\Omega_t)} \leq M \exp\left(M \left| \int_t^\tau |v(s, x)|_{1,s} ds \right| \right), \quad \tau, t \in [0, T], \quad (3.3)$$

$$\left| z_x^1(\tau; t, x) - z_x^2(\tau; t, x) \right|_{0,t} \leq M \left(\|v^1\|_{1,2}, \|v^2\|_{1,2} \right) \left| \int_t^\tau |v^1(s, x) - v^2(s, x)|_{1,s} ds \right|, \quad (3.4)$$

$$|z_{xx}(\tau; t, x)|_{0,t} \leq M (\|v\|_{1,2}) \left| \int_t^\tau |v(s, x)|_{2,s} ds \right|, \quad \tau, t \in [0, T], \quad (3.5)$$

$$\|z^1(\tau; t, x) - z^2(\tau; t, x)\|_{C(\Omega_t)} \leq M \left(\|v^1\|_{1,2}, \|v^2\|_{1,2} \right) \left| \int_t^\tau |v^1(s, x) - v^2(s, x)|_{1,s} ds \right|, \quad (3.6)$$

$\tau, t \in [0, T]$

hold.

In the proof of Lemma 3.2, Lemma 3.1 and uniform with respect to t estimate $\|\hat{v}(t, x)\|_{C^1(\Omega_t)} \leq M|v(t, x)|_{1,t}$ are used. The last estimate easily follows from the continuous imbedding $W_2^\sigma(\Omega_t) \subset C^1(\Omega_t)$, $\sigma > 2$ (see [11, page 408]), and the inclusion $D((I + \delta A_t)^{-1}) \subset W_2^2(\Omega_t)$.

4. First A Priori Estimate

Theorem 4.1. *Let u be a solution to the problem (2.3)-(2.4), (2.1). Then, inequality*

$$\sup_t |u(t, x)|_{0,t} + \|u(t, x)\|_{0,1} \leq M_1(\|w\|_{1,2}) \left(\|F\|_{0,-1} + |u^0|_{0,0} \right) \quad (4.1)$$

holds.

Recall that $\|v\|_{0,1}^2 = \int_0^T |v(t, x)|_{0,-1}^2 dt$.

Proof. Let $\Psi(t, x)$ be a sufficiently smooth scalar function. Since $\Omega_t = \tilde{z}(t, 0, \Omega_0)$, where $\tilde{z}$ is a solution to the Cauchy problem (2.2) for $\tilde{w}$ with $\text{div } \tilde{w} = 0$, then (see [12, page 8])

$$\frac{d}{dt} \int_{\Omega_t} \Psi(t, x) dx = \int_{\Omega_t} \frac{\partial}{\partial t} \Psi(t, x) dx + \int_{\partial\Omega_t} \Psi(t, x) dx. \quad (4.2)$$

It follows that

$$\frac{d}{dt} |u(t, x)|_{0,t}^2 = 2 \left(\frac{\partial}{\partial t} u(t, x), u(t, x) \right)_t, \quad (4.3)$$

where $\Psi(t, x) = u(t, x) \cdot u(t, x)$, $u \in W_2^{1,2}(Q_T)$, and vanishes on S_T .

Multiply both sides of (2.3) in $L_2(\Omega_t)$ by u . The standard arguments as in [5] and (4.2), (4.3) give

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u(t, x)|_{0,t}^2 + |u(t, x)|_{1,t}^2 \\ &= (F(t, x), u(t, x))_t + \mu \left(\int_0^t \xi(u)(s, z(s; t, x)) ds, \xi u(t, x) \right)_t + b_t(w, u, u) + b_t(u, w, u). \end{aligned} \quad (4.4)$$

Here, $b_t(v, u, w) = (v_i(t, x) \partial u(t, x) / \partial x_i, w(t, x))_t$ for $v, u, w \in W_2^1(\Omega_t)$. Since $\operatorname{div} w = 0$, then

$$B_t(w, u, u) = \int_{\Omega_t} \frac{(\partial w_i u)}{\partial x_i} \cdot u \, dx = \left(\frac{\partial(w_i u)}{\partial x_i}, u \right)_t. \quad (4.5)$$

Here, $b_t(v, u, w) = (v_i(t, x) \partial u(t, x) / \partial x_i, w(t, x))_t$ for $v, u, w \in W_2^1(\Omega_t)$. Since $\operatorname{div} w = 0$, then

$$B_t(w, u, u) = \int_{\Omega_t} \frac{(\partial w_i u)}{\partial x_i} \cdot u \, dx = \left(\frac{\partial(w_i u)}{\partial x_i}, u \right)_t. \quad (4.6)$$

Using the inequality $|(u, v)_t| \leq |u|_{-1,t} |v|_{1,t}$ (see [7, page 252]) and the boundedness of the operator $\partial / \partial x_i : W_2^1(\Omega_t) \rightarrow L_2(\Omega_t)$, we have

$$|b_t(w, u, u)| \leq M \left| \frac{\partial(w_i u)}{\partial x_i} \right|_{-1,t} |u|_{1,t} \leq M |wu|_{0,t} |u|_{1,t}. \quad (4.7)$$

With the help of Hölder's inequality and

$$\|v\|_{L_4(\Omega_t)} \leq M |v|_{1,t}^{1/2} |v|_{0,t}^{1/2}, \quad v \in \overset{\circ}{W}_2^1(\Omega_t), \quad n = 2, \quad (4.8)$$

(see [6, page 233]), we obtain

$$|wu|_{0,t} \leq M \|w\|_{L_4(\Omega_t)} \|u\|_{L_4(\Omega_t)} \leq M \|w\|_{L_4(\Omega_t)} |u|_{1,t}^{1/2} |u|_{0,t}^{1/2}. \quad (4.9)$$

Using (4.7)–(4.9) and the standard techniques, we obtain, for $\varepsilon > 0$,

$$|b_t(w, u, u)| \leq M \|w\|_{L_4(\Omega_t)} |u|_{1,t}^{3/2} |u|_{0,t}^{1/2} \leq \varepsilon |u|_{1,t}^2 + M_2(\varepsilon) \|w\|_{L_4(\Omega_t)}^4 |u|_{0,t}^2. \quad (4.10)$$

A similar estimate holds for $b_t(u, w, u)$. Integrating (4.4), using the inequality $|(F, u)_t| \leq |F|_{-1,t}|u|_{1,t}$, (4.10), and simple calculus, we have

$$\begin{aligned} |u(t, x)|_{0,t}^2 + \int_0^t |u(s, x)|_{a,s}^2 ds &\leq M_3(\varepsilon) \int_0^t |F(s, x)|_{-1,s}^2 ds \\ &+ M_4 \left(|u(0, x)|_{0,0}^2 + \int_0^t \int_0^s |u_x(\tau, z(\tau; s, x))|_{0,s} d\tau |u_x(s, x)|_{0,s} ds \right) \\ &+ \varepsilon \int_0^t |u(s, x)|_{0,s} |u(s, x)|_{1,s} ds + M_5(\varepsilon) \int_0^t \|w(s, x)\|_{L_4(\Omega_t)}^4 |u(s, x)|_{0,s}^2 ds. \end{aligned} \quad (4.11)$$

The change of variable $y = z(\tau; s, x)$ maps Ω_s into Ω_τ , by this $\det z_x(\tau; s, x) \equiv 1$ in virtue of $\operatorname{div} w = 0$. Then,

$$|u_x(\tau, z(\tau; s, x))|_{0,s}^2 = \int_{\Omega_s} |u_x(\tau, z(\tau; s, x))|^2 dx = |u_x(s, y)|_{0,\tau}. \quad (4.12)$$

Believing ε sufficiently small and using the above inequality and (4.11), one easily obtains

$$\begin{aligned} |u(t, x)|_{0,t}^2 + \int_0^t |u(s, x)|_{1,s}^2 ds &\leq M_6 \left(\|F(s, x)\|_{0,1}^2 + |u^0(x)|_{1,0}^2 + \int_0^t \int_0^s |u(\tau, x)|_{1,\tau}^2 d\tau ds \right. \\ &\left. + \int_0^t \|w(s, x)\|_{L_4(\Omega_s)}^4 |u(s, x)|_{0,s}^2 ds \right). \end{aligned} \quad (4.13)$$

Letting

$$a(s) = 1 + \|w(s, x)\|_{L_4(\Omega_s)}^4, \quad C_1 = \|F\|_{0,1}^2 + |u^0|_{1,0}^2, \quad (4.14)$$

we obtain from this that, for $b(t) = |u(t, x)|_{0,t}^2 + \int_0^t |u(s, x)|_{a,s}^2 ds$, the inequality

$$b(t) \leq M_6 \left(C_1 + \int_0^t a(s) b(s) ds \right) \quad (4.15)$$

holds. By Gronwall's lemma, this yields the inequality

$$b(t) \leq M_6 C_1 \exp \left(M_6 \int_0^T a(s) ds \right) \leq M_7 C_1 \exp \left(M_6 \int_0^T \|w(s, x)\|_{L_4(\Omega_s)}^4 ds \right). \quad (4.16)$$

Since (see [13, page 143])

$$\|v\|_{L_4(Q_T)} \leq M\|v\|_{1,2}, \quad v \in W_2^{1,2}(Q_T), \quad n = 2, \quad (4.17)$$

then

$$\int_0^T \|w(s, x)\|_{L_4(\Omega_s)}^4 ds \leq M\|w\|_{L_4(\Omega_s)}^4 \leq M\|w\|_{1,2}^4. \quad (4.18)$$

This and (4.16) yield the inequality $b(t) \leq M_8(\|w\|_{1,2})C_1$, that is,

$$|u(t, x)|_{0,t}^2 + \int_0^t |u(s, x)|_{a,s}^2 ds \leq M_8(\|w\|_{1,2}) \left(\|F(t, x)\|_{0,-1} + |u^0(x)|_{0,0} \right). \quad (4.19)$$

This gives (4.1).

Theorem 4.1 is proved. $\square$

5. Second A Priori Estimate

Theorem 5.1. *A solution to the problem (2.3)-(2.4), (2.1) satisfies the inequality*

$$\sup_t |u(t, x)|_{1,t} + \|u(t, x)\|_{1,2} \leq M_9 \left(\|w\|_{1,2}, \|F\|_0, \|u^0\|_{1,0} \right). \quad (5.1)$$

To prove Theorem 5.1, we need some estimates for solutions to the more simple linear problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{w_i \partial u}{\partial x_i} + \frac{u_i \partial w}{\partial x_i} - \Delta u + \nabla &= \Psi(t, x), \quad \operatorname{div} u(t, x) = 0, \\ u(0, x) &= u^0(x), \quad x \in \Omega_0, \\ u(t, x) &= 0, \quad (t, x) \in S_T. \end{aligned} \quad (5.2)$$

In [4], it was found that by a smooth evolution Ω_t , $\mu = 0$, $n = 2$ for $F \in L_2(Q_T)$, $u^0 \in V_0$, problem (2.3)-(2.4) has a unique solution $u \in W_2^{1,2}(Q_T)$, and the estimate

$$\|u\|_{1,2} \leq M \left(\|w\|_{1,2}, \|F\|_0, |u^0|_0 \right) \quad (5.3)$$

holds. It follows from [4] that the similar result for more simple linear problem (5.2) holds,

by this in our case, the estimate

$$\|u\|_{1,2} \leq M(\|w\|_{1,2}) \left(\|\psi\|_0 + |u^0|_0 \right) \quad (5.4)$$

holds.

It follows from [14] that for functions $\tilde{u} \in W_2^{1,2}(\tilde{Q})$ which vanish at $t = 0$ and on S_T in a cylindrical domain $\tilde{Q} = [0, T] \times \Omega_0$, the inequality

$$\sup_t |\tilde{u}(t, x)|_{1,t} \leq M \left(\|\tilde{u}\|_{W_2^{1,2}(\tilde{Q})} + |\tilde{u}(0, x)|_{W_2^1(\Omega_0)} \right) \quad (5.5)$$

is valid. The same inequality holds and for $u \in W_2^{1,2}(Q_T)$. In fact, let $\tilde{u}(t, y) = u(t, (\tilde{z}(t; 0, y)))$. Since $\tilde{z}(\tau; t, x)$ is smooth, then $\tilde{u} \in W_2^{1,2}(\tilde{Q})$, and hence inequality (5.5) for $\tilde{u}$ is valid. By means of substitution $y = \tilde{z}(0; t, x)$, we get the inequality

$$\sup_t |u(t, x)|_{1,t} \leq M(\|u\|_{1,2} + |u(0, x)|_{0,t}), \quad u \in W_2^{1,2}(Q_T), \quad (5.6)$$

for the original $u(t, x)$.

Lemma 5.2. *Let $\Psi \in L_2(Q_T)$, $u^0 \in V_0$. Then, problem (5.2) has a unique solution, and the estimates*

$$\sup_t |u(t, x)|_{1,t} + \|u\|_{1,2} \leq M_{10}(\|w\|_{1,2}) \left(\|\Psi\|_0 + |u^0|_{1,0} \right), \quad (5.7)$$

$$\sup_t |u(t, x)|_{0,t} + \|u(t, x)\|_{0,1} \leq M_{11}(\|w\|_{1,2}) \left(\|\Psi\|_{0,-1} + |u^0|_{0,0} \right) \quad (5.8)$$

hold.

The proof of (5.7) follows from [4, 14] and (5.6), and (5.8). The proof of (4.1) is similar to one of Theorem 4.1.

Proof of Theorem 5.1. Since the solution $u \in W_2^{1,2}(Q_T)$, then $u \in W_2^{1,2}(Q_t)$, $0 < t \leq T$. We remove the second, sixth, and seventh terms on the left-hand side of (2.3) to the right and apply the estimate (5.7) for Q_t :

$$\begin{aligned} & \int_0^t \left| \frac{\partial u(s, x)}{\partial s} \right|_{0,s}^2 ds + |u(t, x)|_{0,t}^2 + |u(t, x)|_{1,t}^2 + \int_0^t |u(s, x)|_{2,s}^2 ds \\ & \leq M \left(\|F\|_0^2 + |U^0|_{1,0}^2 \right) \\ & \quad + M \left(\int_0^t \left| \frac{u_i(s, x) \partial u(s, x)}{\partial x_i} \right|_{0,s}^2 ds \right) \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left| \operatorname{Div} \int_0^s \mathcal{E}(u)(\tau, z(\tau; s, x)) d\tau \right|_{0,s}^2 ds + \int_0^t \left| \operatorname{Div} \int_0^s \mathcal{E}(w)(\tau, z(\tau; s, x)) d\tau \right|_{0,s}^2 ds \\
& + \int_0^t \left| \frac{w_i(t, x) \partial u(s, x)}{\partial x_i} \right|_{0,s}^2 ds + \int_0^t \left| \frac{u_i(t, x) \partial w(s, x)}{\partial x_i} \right|_{0,s}^2 ds \Bigg) \\
& \equiv MC_0 + M \sum_{i=1}^5 Z_i, \quad C_0 = \|F\|_0^2 + \|u^0\|_{1,0}^2.
\end{aligned} \tag{5.9}$$

Let us estimate Z_i . Using Hölder's inequality and (4.8), we obtain

$$\left| \frac{u_i(s, x) \partial u(s, x)}{\partial x_i} \right|_{0,s}^2 \leq M |u(s, x)|_{0,s} |u(t, x)|_{a,s}^2 |u(s, x)|_{2,s}. \tag{5.10}$$

It follows from this and from the uniform boundedness of $|u(s, x)|_{0,s}$ (in virtue of (4.1)) that

$$\left| \frac{u_i(s, x) \partial u(s, x)}{\partial x_i} \right|_{0,s}^2 \leq \varepsilon |u(s, x)|_{2,s}^2 + M(\varepsilon, C_0) |u(t, x)|_{a,s}^4. \tag{5.11}$$

It follows that

$$Z_2^1 \leq \varepsilon \int_0^t |u(s, x)|_{2,s}^2 ds + M(\varepsilon, C_0) \int_0^t |u(s, x)|_{a,s}^4 ds. \tag{5.12}$$

Differentiating, we find that $(u = (u_1, u_2), z = (z_1, z_2))$

$$\frac{\partial}{\partial x_k} \int_0^s \frac{\partial}{\partial x_j} u_i(\tau, z(\tau; s, x)) d\tau = \int_0^s \frac{\partial^2}{\partial x_l \partial x_j} u_i(\tau, z(\tau; s, x)) \frac{\partial}{\partial x_k} z_l(\tau; s, x) d\tau. \tag{5.13}$$

It follows that

$$Z_2^2 \leq \int_0^t \left(\int_0^s |u_{xx}(\tau, z(\tau; s, x))|_{0,s} \|z_x(\tau; s, x)\|_{C(\Omega_s)} d\tau \right)^2 ds. \tag{5.14}$$

Using the change $y = z(\tau; s, x)$ in the first factor, estimates (3.3) and (4.1) in the second one, we have

$$Z_2^2 \leq M \int_0^t \left(\int_0^s |u_{xx}(\tau, x)|_{0,\tau} d\tau \right)^2 ds \leq M \int_0^t \int_0^s |u(\tau, x)|_{2,\tau}^2 d\tau ds \leq M. \tag{5.15}$$

From here and to the end of the proof, constants M depend on C_0 and $\|w\|_{1,2}$. The estimate of $Z_3 \leq M$ is established similarly. The estimates of $Z_i \leq M$, $i = 4, 5$, are established more simple.

Estimates of Z_i and (5.9) for small $\varepsilon > 0$ yield

$$\begin{aligned} & \int_0^t \left| \frac{\partial u(s, x)}{\partial s} \right|_{0,s}^2 ds + |u(t, x)|_{1,t}^2 + \int_0^t |u(s, x)|_{2,s}^2 ds \\ & \leq M \left(1 + \int_0^t |u(s, x)|_{a,s}^2 |u(s, x)|_{a,s}^2 ds + \int_0^t \int_0^s |u(\tau, x)|_{2,s}^2 d\tau ds \right). \end{aligned} \quad (5.16)$$

It follows from this that for

$$g(t) = |u(t, x)|_{1,t}^2 + \int_0^t |u(s, x)|_{2,s}^2 ds, \quad (5.17)$$

the inequality

$$g(t) \leq M_{12} + M_{13} \int_0^t |u(s, x)|_{a,s}^2 g(s) ds \quad (5.18)$$

holds. Consequently,

$$g(t) \leq M_{12} \exp \left(\int_0^t M_{13} |u(s, x)|_{a,s}^2 ds \right). \quad (5.19)$$

This and (4.1) give that

$$g(t) \leq M_{14} (\|w\|_{1,2}, C_0). \quad (5.20)$$

But then the right-hand side of (5.16) is bounded uniformly with respect to t . Hence, the left-hand side is bounded also.

Inequality (5.1) is proved.

Theorem 5.1 is proved. $\square$

6. Reduction to Operator Equation

Let $u = \mathcal{R}(\Psi)$ be a solution to (5.2) by $u^0 = 0$ and given $\Psi \in L_2(Q_T)$. Obviously, the expression $u = \mathcal{R}(\Psi)$ defines a bounded linear operator from $L_2(Q_T)$ to $W_2^{1,2}(Q_T)$. Using the operator $\mathcal{R}$, (2.5), and Lemma 5.2, we rewrite the problem (2.3)-(2.4), (2.1) by $u^0 = 0$ as

$$\begin{aligned} u &= \mathcal{R}(F) - \mathcal{R} \left(\frac{u_i \partial u}{\partial x_i} \right) + \mu \mathcal{R} \left(\text{Div} \int_0^t \mathcal{E}(u)(s, z(s; t, x)) ds \right) \\ &+ \mu \mathcal{R} \left(\text{Div} \int_0^t \mathcal{E}(w)(s, z(s; t, x)) ds \right) = \Phi + S_1(u) + \mu S_2(u) + \mu S_3(u) \equiv Q(u). \end{aligned} \quad (6.1)$$

Thus, the problem (1.1)-(1.2), (2.1) with $u^0 = 0$ is reduced to equation

$$u = Q(u). \quad (6.2)$$

We show that for sufficiently large R and sufficiently small $T > 0$, operator equation (6.2) is uniquely solvable in

$$S(R) = \left\{ u : u \in W_2^{1,2}(Q_T), u(t, x) \in V_t, \|u\|_{1,2} \leq R, u(0, x) = 0 \right\}. \quad (6.3)$$

7. Solvability of (6.2) for Small T

Lemma 7.1. *By sufficiently large R and sufficiently small T , the operator Q maps $S(R)$ into itself.*

Proof. Let $u \in S(R)$. Properties of the operator $\mathcal{R}$ and (6.1) imply $u(0, x) = 0$ and

$$\|Q(u)\|_{1,2} \leq \|\Phi\|_0 + \|S_1\|_0 + \mu\|S_2\|_0 + \mu\|S_3\|_0. \quad (7.1)$$

Let us show that

$$\|S_i\|_0 \leq M(R)T^{1/2}, \quad i = 1, 2, 3. \quad (7.2)$$

From (5.10), it follows that

$$|S_1|_{0,t} \leq M |u|_{0,t}^{1/2} |u|_{1,t} |u|_{2,t}^{1/2}. \quad (7.3)$$

Hence, from (5.6), the Hölder inequality, and $\|u\|_{1,2} \leq R$, it follows that

$$\begin{aligned} \|S_1\|_0^2 &\leq M \int_0^T |S_1(t, x)|_{0,t}^2 dt \leq M \sup_t |u(t, x)|_{0,t} \sup_t |u(t, x)|_{1,t}^2 \int_0^T |u(t, x)|_{2,t} dt \\ &\leq MR^3 \int_0^T |u(t, x)|_{2,t} dt \leq MR^3 \|u\|_{1,2} T^{1/2} \leq MR^4 T^{1/2}. \end{aligned} \quad (7.4)$$

Estimate $\|S_2\|_0$. Differentiating and applying the integral Minkowski inequality, we have

$$|S_2|_{0,t} \leq M \int_0^T |u_x(s, z(s; t, x))|_{0,t} \|z_x(s; t, x)\|_{C(\Omega_t)} ds. \quad (7.5)$$

Making the change $y = z(s; t, x)$, we get

$$|u_x(s, z(s; t, x))|_{0,t} = |u_x(s, y)|_{0,s}. \quad (7.6)$$

From (3.3), it follows that

$$\|z_x(s; t, x)\|_{C(\Omega_t)} \leq M \exp \left(M \left| \int_0^t |v(s, x)|_{a,s} ds \right| \right) \leq M \exp M(R). \quad (7.7)$$

From this by means of Hölder inequality, it follows that

$$|S_2|_{0,t} \leq M(R) \int_0^T |u_x(s, y)|_{0,s} ds \leq M(R) T^{1/2} \|u\|_{0,1} \leq M(R) T^{1/2}. \quad (7.8)$$

Integrating, we obtain (7.2) for $i = 2$. Estimate (7.2) for $i = 3$ is established easier. The statement of Lemma 7.1 follows from estimates (7.1) and (7.2).

Lemma 7.1 is proved. $\square$

Now consider the $S(R)$ as a metric space with the generated by the norm $\|\cdot\|_{0,1}$ metric. From the definitions of generalized derivatives and the weak compactness of a ball in $W_2^{1,2}(Q_T)$, it follows that the metric space $S(R)$ is complete ([10, Chapter 3, Section 1]).

Lemma 7.2. *By sufficiently small T , the operator Q on $S(R)$ is a contraction in the metric $\|\cdot\|_{0,1}$.*

Proof. Let $u^1, u^2 \in S(R)$. Then,

$$\begin{aligned} \Delta Q(u) &= Q(u^2) - Q(u^1) \\ &= \mathcal{R} \left(\frac{\partial}{\partial x_i} (u_i^1 u^1 - u_i^2 u^2) \right) \\ &\quad + \mu \mathcal{R} \left(\text{Div} \int_0^t \left(\mathcal{E}(u^2)(s, z^2(s; t, x)) - \mathcal{E}(u^1)(s, z^1(s; t, x)) \right) ds \right) \\ &\quad + \mu \mathcal{R} \left(\text{Div} \int_0^t \left(\mathcal{E}(w)(s, z^2(s; t, x)) - \mathcal{E}(w)(s, z^1(s; t, x)) \right) ds \right) \\ &= \mathcal{R} Z_1 + \mu \mathcal{R} Z_2 + \mu \mathcal{R} Z_3. \end{aligned} \quad (7.9)$$

From the inequality $|\partial v / \partial x_i|_{-1,t} \leq M |V|_{0,t}$ and (5.8), it follows that

$$\begin{aligned} \|\mathcal{R} Z_1\|_{0,1}^2 &\leq M \|Z_1\|_{0,1}^2 = M \int_0^T \left| \frac{\partial}{\partial x_i} (u_i^2 u^2 - u_i^1 u^1) \right|_{-1,t}^2 dt \\ &\leq M \int_0^T \sum_{i=1}^2 \left| u_i^2 u^2 - u_i^1 u^1 \right|_{0,t}^2 dt. \end{aligned} \quad (7.10)$$

By means of simple calculus, we obtain

$$\begin{aligned} g &= \left| u_i^2 u^2 - u_i^1 u^a \right|_{0,t}^2 \leq \int_{\Omega_t} \left| u_i^2 - u_i^1 \right|^2 |u^a|^2 dx + \int_{\Omega_t} \left| u_i^1 \right|^2 \left| u^2 - u^a \right|^2 dx \\ &\leq M \left\| u^2 - u^1 \right\|_{L_4(\Omega_t)}^2 \left(\left\| u^2 \right\|_{L_4(\Omega_t)}^2 + \left\| u^1 \right\|_{L_4(\Omega_t)}^2 \right). \end{aligned} \quad (7.11)$$

It follows from the continuous imbedding $W_2^1(\Omega_t) \subset L_4(\Omega_t)$, (4.8), and (5.6) that

$$g \leq M \left| u^2 - u^1 \right|_{1,t}^2 \max_{k=1,2} \left(\left| u^k \right|_{1,t} \left| u^k \right|_{0,t} \right) \leq MR \left| u^2 - u^1 \right|_{1,t}^2 \max_{k=1,2} \left| u^k \right|_{0,t}. \quad (7.12)$$

The inequality (4.2) for $\psi = |u^i(t, x)|^2$ and (4.3) by $u^i(t, x) = 0$ on S_T give

$$\frac{\partial |u^i(t, x)|_{0,t}^2}{dt} = 2 \left(\frac{\partial u^i(t, x)}{\partial t}, u^i(t, x) \right)_t. \quad (7.13)$$

Hence, from $u^i(0, x) = 0$ and (5.6), we obtain

$$\begin{aligned} \left| u^i(t, x) \right|_{0,t}^2 &\leq 2 \int_0^t \left| \frac{\partial}{\partial s} u^i(s, x) \right|_{0,s} \left| u^i(s, x) \right|_{0,s} ds \\ &\leq M \sup_t \left| u^i(t, x) \right|_{0,t} \int_0^t \left| \frac{\partial}{\partial s} u^i(s, x) \right|_{0,s} ds \leq MR^2 T^{1/2}. \end{aligned} \quad (7.14)$$

It follows from (7.11)–(7.14) that $G \leq M(R)T^{1/2} |u^2 - u^a|_{1,t}^2$. This and (7.10) yield

$$\|\mathcal{R}Z_1\|_{0,1}^2 \leq M(R)T^{1/2} \int_0^T \left| u^2 - u^1 \right|_{1,s}^2 ds \leq M(R)T^{1/2} \left\| u^2 - u^1 \right\|_{0,1}^2. \quad (7.15)$$

Let us estimate $\mathcal{R}Z_2$. As by the evaluation of $\mathcal{R}Z_1$, we have

$$\|\mathcal{R}Z_2\|_{0,1} \leq M \|Z_2\|_{0,-1} \leq M \left\| \int_0^t \left(\mathcal{E}(u^2)(s, z^2(s; t, x)) - \mathcal{E}(u^1)(s, z^1(s; t, x)) \right) ds \right\|_0. \quad (7.16)$$

Denoting the expression under the sign of $\| \cdot \|_0$ by G , we get

$$\begin{aligned} G &= \int_0^t \left(\mathcal{E}(u^2)(s, z^2(s; t, x)) - \mathcal{E}(u^2)(s, z^1(s; t, x)) \right) ds \\ &\quad + \int_0^t \left(\mathcal{E}(u^2)(s, z^1(s; t, x)) - \mathcal{E}(u^1)(s, z^1(s; t, x)) \right) ds = \widehat{G}_1 + \widehat{G}_2. \end{aligned} \quad (7.17)$$

Let us estimate the first term. As in [10, Chapter 3, Section 1.2], we continue the u^1 and u^2 out of Q_T on R^n with the preservation of class so that the velocity fields $v^1 = u^1 + w$ and $v^2 = u^2 + w$ of the corresponding Cauchy problems (2.1) are vanished out of a compact convex domain Ω , $\Omega \supset \Omega_t$, $0 \leq t \leq T$. By the uniqueness theorem for Cauchy problems (7.11), solutions z^1 and z^2 are the restrictions of the corresponding extended Cauchy problems. We mark the extended u^1 , u^2 , z^1 , and z^2 by the bar at the top.

We now estimate $\hat{G}_1$. Using the Newton-Leibnitz formula, we have

$$\begin{aligned} B_i &= \frac{\partial}{\partial x_i} \bar{u}^2(s, \bar{z}^2(s; t, x)) - \frac{\partial}{\partial x_i} \bar{u}^2(s, \bar{z}^1(s; t, x)) \\ &= \sum_{k=1}^2 \int_0^1 \frac{\partial^2}{\partial x_k \partial x_i} \bar{u}^2(s, \bar{y}(s, t, x, \alpha)) d\alpha (\bar{z}_k^1(s; t, x) - \bar{z}_k^2(s; t, x)), \quad \bar{z} = (\bar{z}_1, \bar{z}_2). \end{aligned} \quad (7.18)$$

Here, $Y = \bar{y}(s, t, x, \alpha) = \alpha \bar{z}^1(s; t, x) + (1 - \alpha) \bar{z}^2(s; t, x)$.

As in [10, Chapter 3, Section 1.2] in the cylindrical case $Q = [0, T] \times \Omega$, we demonstrate that, for small T and $0 \leq \alpha \leq 1$, the mapping $\bar{y}(s, t, x, \alpha)$ is diffeomorphism of Ω to Ω , and the Jacobians of the direct and inverse mappings are uniformly bounded. It follows that

$$\begin{aligned} \|B_i\|_{L_2(\Omega_t)} &\leq \|\bar{b}_i\|_{L_2(\Omega)} \leq M \sum_{k=1}^2 \int_0^1 \left\| \frac{\partial^2}{\partial x_k \partial x_i} \bar{u}^2(s, \bar{y}(s, t, x, \alpha)) \right\|_{L_2(\Omega)} d\alpha \\ &\quad \times \left\| \bar{z}^1(s; t, x) - \bar{z}^2(s; t, x) \right\|_{C(\Omega)}. \end{aligned} \quad (7.19)$$

Using the change of variable $y = \bar{y}(s, t, x, \alpha)$, we get

$$\begin{aligned} \left\| \frac{\partial^2}{\partial x_k \partial x_i} \bar{u}^2(s, \bar{y}(s, t, x, \alpha)) \right\|_{L_2(\Omega)} &\leq M \left\| \frac{\partial^2}{\partial x_k \partial x_i} \bar{u}_j^2(s, y) \right\|_{L_2(\Omega)} \\ &\leq M \left\| \frac{\partial^2}{\partial x_k \partial x_i} u_j^2(s, y) \right\|_{L_2(\Omega_s)} \\ &\leq M \left| u^2(s, y) \right|_{2,s}. \end{aligned} \quad (7.20)$$

Using (7.20), we have

$$\left\| \frac{\partial^2}{\partial x_k \partial x_i} u_j^2(s, y(s, t, x, \alpha)) \right\|_{L_2(\Omega_t)} \leq M \left\| \frac{\partial^2}{\partial x_k \partial x_i} \bar{u}^2(s, \bar{y}(s, t, x, \alpha)) \right\|_{L_2(\Omega)} \leq M \left| u^2(s, y) \right|_{2,s}. \quad (7.21)$$

From this estimate, (3.3), and (3.6), it follows that

$$\begin{aligned} \left| \widehat{G}_1 \right|_{0,t} &\leq M \int_0^t \left| u^2(s, y) \right|_{2,s} \left\| z^1(s; t, x) - z^2(s; t, x) \right\|_{C(\Omega_t)} ds \\ &\leq M(R) T^{1/2} \left\| u^2 \right\|_{0,2} \left\| u^2 - u^1 \right\|_{0,1} \leq M(R) T^{1/2} \left\| u^2 - u^1 \right\|_{0,1}. \end{aligned} \quad (7.22)$$

A similar estimate for $\widehat{G}_2$ is established easier. From the estimates (7.16), (7.21), (7.22), the inequality

$$\left\| \mathcal{R}Z_2 \right\|_{0,1} \leq M(R) T^{1/2} \left\| u^2 - u^1 \right\|_{0,1} \quad (7.23)$$

follows.

A similar estimate for $\left\| \mathcal{R}Z_3 \right\|_{0,1}$ is established similarly. From the estimates $\left\| \mathcal{R}Z_i \right\|_{0,1}$ for $i = 1, 2, 3$, it follows the inequality

$$\left\| Q(u^2) - Q(u^1) \right\|_{0,1} \leq M(R) T^{1/4} \left\| u^2 - u^1 \right\|_{0,1} \leq q \left\| u^2 - u^1 \right\|_{0,1}, \quad (7.24)$$

where $q \in (0, 1)$ by sufficiently small T .

Lemma 7.2 is proved. $\square$

Lemma 7.3. *For sufficiently large R and sufficiently small T , the operator Q in $S(R)$ has a unique fixed point.*

Proof. The proof follows from Lemmas 7.2 and 7.1 by the principle of contracting maps. Lemma 7.3 is proved. $\square$

Proof of Theorem 2.2. The theorem for a small T and $u^0 = 0$ follows from Lemma 7.3 by the equivalence of problem (2.3)-(2.4), (2.1), and (6.2). Establish the solvability of the problem (2.3)-(2.4), (2.1) at small T and $u^0 \neq 0$.

Let $\bar{u}^0(t, x)$ be a solution to the linear problem (5.2) by $u^0 \neq 0$ and $F = 0$. There exists only one solution to this problem in the force of Lemma 5.2, and the inequality

$$\sup_t \left| \bar{u}^0(t, x) \right|_{1,t} + \left\| \bar{u}^0 \right\|_{1,2} \leq M_{11} (\|w\|_{1,2}) \left| u^0 \right|_{1,0} \quad (7.25)$$

holds.

Represent the solution to (2.3)-(2.4), (2.1) as $u(t, x) = \bar{u}(t, x) + \bar{u}^0(t, x)$. It is easy to see that $\bar{u}(t, x)$ satisfies relationships

$$\begin{aligned} & \frac{\partial \bar{u}}{\partial t} + \frac{\bar{u}_i \partial \bar{u}}{\partial x_i} + \frac{\bar{w}_i \partial \bar{u}}{\partial x_i} + \frac{\bar{u}_i \partial \bar{w}}{\partial x_i} - \Delta \bar{u} - \mu \operatorname{Div} \int_0^t \mathcal{E}(\bar{u})(s, z(s; t, x)) ds \\ & - \mu \operatorname{Div} \int_0^t \mathcal{E}(\bar{u}^0)(s, z(s; t, x)) ds - \mu \operatorname{Div} \int_0^t \mathcal{E}(\bar{w})(s, z(s; t, x)) ds \\ & + \operatorname{grad} p(t, x) = \bar{F}(t, x), \quad (t, x) \in Q_T, \\ & \operatorname{div} \bar{u}(t, x) = 0, \quad (t, x) \in Q_T, \quad \bar{u}(0, x) = 0, \quad x \in \Omega_0, \quad \bar{u}(t, x) = 0, \quad (t, x) \in S_T, \end{aligned} \quad (7.26)$$

and (2.1). Here, $\bar{w} = w + \bar{u}^0$, F is defined as in (2.3)-(2.4), (2.1),

$$\bar{F}(t, x) = F(t, x) - \frac{\partial \bar{u}^0}{\partial t} - \frac{\bar{u}_i^0 \partial \bar{u}^0}{\partial x_i} - \frac{w_i \partial \bar{u}^0}{\partial x_i} - \frac{\bar{u}_i^0 \partial w}{\partial x_i} + \Delta \bar{u}^0, \quad (7.27)$$

and z is a solution to problem (2.1) by $v = u + w + \bar{u}^0$. In this case, $\bar{F}, \bar{w} \in L_2(Q_T)$. In fact, by $g, v \in L_4(Q_T)$ with the help of Hölder's inequality, we have

$$\|gv\|_0 \leq \|g\|_{L_4(Q_T)} \|v\|_{L_4(Q_T)}. \quad (7.28)$$

It follows from this and (4.17) that by $g, v \in W_2^{1,2}(Q_T)$, the inequality

$$\|gv\|_0 \leq M \|g\|_{1,2} \|v\|_{1,2} \quad (7.29)$$

holds. Setting $g = w$, $v = \bar{u}^0$, we obtain the required inclusion. Note that this and (7.25) imply the estimates

$$\|\bar{F}\|_0 \leq M_{12} \left(\|\bar{w}\|_{1,2}, \|F\|_{1,2}, \|\bar{u}^0\|_{1,2} \right), \quad \|\bar{w}\|_{1,2} \leq M_{13} \left(\|w\|_{1,2} + \|\bar{u}^0\|_{1,2} \right). \quad (7.30)$$

Thus, we have reduced the case of nonzero initial condition to the case considered of zero initial conditions. It follows from the above that for small $T = T(\|\bar{F}\|_0, \|\bar{w}\|_{1,2})$, there exists a unique solution $\bar{u}$ to problem (7.26), (2.1), and the inequality

$$\sup_t |\bar{u}(t, x)|_{1,t} + \|\bar{u}\|_{1,2} \leq M_{14} \left(\|\bar{F}\|_0, \|\bar{w}\|_{1,2} \right) \quad (7.31)$$

holds.

The inequalities (7.30)–(7.32) imply that

$$\sup_t |\bar{u}(t, x)|_{1,t} + \|\bar{u}\|_{1,2} \leq M_{15} \left(\|F\|_{1,2}, \|w\|_{1,2}, \|\bar{u}^0\|_{1,0} \right). \quad (7.32)$$

Finally, the unique solvability of (7.26), (2.1) on $[0, T]$ at small T implies the existence and uniqueness on $[0, T]$ of the solution u to problem (2.3)-(2.4), (2.1) by $u^0 \neq 0$ and the inequality

$$\sup_t |u(t, x)|_{1,t} + \|u\|_{1,2} \leq M_{16} \left(\|F\|_{1,2}, \|w\|_{1,2}, |u^0|_{1,0} \right). \quad (7.33)$$

Now let $T > 0$ be arbitrary. Let T_0 such that our problem is uniquely solvable on $[0, T_0] \subset [0, T]$. Note that the same result holds not only on $[0, T_0]$, but also on $[t_*, t_* + T_0] \subset [0, T]$ for any $t_* > 0$. Let $T_k = (kT)/N$, $k = 1, 2, \dots, N$, N be a natural number. Consider problem (2.3)-(2.4), (2.1) on the domains Q_k , where $Q_k = \{(t, x) : t_k \leq t \leq t_{k+1}, x \in \Omega_t\}$. Let us find solutions $u^k(t, x)$ to problem (2.3)-(2.4), (2.1) from $W_2^{1,2}(Q_k)$, replacing by $k > 1$ the initial condition $u(0, x) = u^0(x)$ by the initial conditions $u^k(t_k, x) = u^{k-1}(t_k, x)$, $x \in \Omega_k$. From the local solvability, which is established above, it follows that the length of the segment $[T_k, t_{k+1}]$, at which these problems are uniquely solvable are determined by $\|F\|_{L_2(Q_k)}$ and $|u^{k-1}(t_k, x)|_{1,t_k}$. If these values are uniformly bounded with respect to t , then there exists a sufficiently large N , that all these problems are uniquely solvable at $[t_k, t_{k+1}]$. We solve these problems sequentially, starting with $k = 0$. By a priori estimate (5.1), $|u^{k-1}(t_k, x)|_{1,t_k}$ is uniformly bounded on t . In addition, it is obvious that $\|F\|_{L_2(Q_k)} \leq \|F\|_{L_2(Q_T)}$. Thus, we obtain the solvability to problems for all k . The function $u(t, x)$, whose restriction to $[t_k, t_{k+1}]$ is $u^k(t, x)$, is obviously the unique solution to (2.3)-(2.4), (2.1) at Q_T . Theorem 2.2 is proved completely. $\square$

Proof of Theorem 2.1. The conditions of Theorem 2.1 and (2.5) imply that the following conditions of Theorem 2.2 are fulfilled. Obviously, the function $v = u + w$, where u is a solution to (2.3)-(2.4), (2.1), is the unique solution to (1.1)-(1.2), (2.1). Theorem 2.1 is proved. $\square$

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