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Research Article

New Partition Theoretic Interpretations of Rogers-Ramanujan Identities

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The generating function for a restricted partition function is derived. This in conjunction with two identities of Rogers provides new partition theoretic interpretations of Rogers-Ramanujan identities.

1. Introduction, Definitions, and the Main Results

The following two "sum-product" identities are known as Rogers-Ramanujan identities

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} &= \prod_{n=1}^{\infty}\left(1-q^{5n-1}\right)^{-1}\left(1-q^{5n-4}\right)^{-1},\\ \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} &= \prod_{n=1}^{\infty}\left(1-q^{5n-2}\right)^{-1}\left(1-q^{5n-3}\right)^{-1},\end{aligned} \tag{1.1}$$

where $|q| < 1$ and $(q;q)_n$ is a rising q-factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty}\frac{(1-aq^i)}{(1-aq^{n+i})}, \tag{1.2}$$

for any constant a.

If n is a positive integer, then obviously

$$(a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), \tag{1.3}$$

$$(a;q)_\infty = (1-a)(1-aq)\left(1-aq^2\right)\cdots. \tag{1.4}$$

They were first discovered by Rogers [1] and rediscovered by Ramanujan in 1913. MacMahon [2] gave the following partition theoretic interpretations of (1.1), respectively.

Theorem 1.1. *The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 1 \pmod 5$.*

Theorem 1.2. *The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 2 \pmod 5$.*

Theorems 1.1-1.2 were generalized by Gordon [3], and Andrews [4] gave the analytic counterpart of Gordon's generalization. Partition theoretic interpretations of many more q-series identities like (1.1) have been given by several mathematicians. See, for instance, Göllnitz [5, 6], Gordon [7], Connor [8], Hirschhorn [9], Agarwal and Andrews [10], Subbarao [11], Subbarao and Agarwal [12].

Our objective in this paper is to provide new partition theoretic interpretations of identities (1.1) which will extend Theorems 1.1 and 1.2 to 3-way partition identities. In our next section, we will prove the following result.

Theorem 1.3. *For a positive integer k, let $A_k(n)$ denote the number of partitions of n such that the smallest part (or the only part) is $\equiv k \pmod 4$, and the difference between any two parts is $\equiv 2 \pmod 4$. Then*

$$\sum_{n=0}^{\infty} A_k(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}}{\left(q^4;q^4\right)_n}. \tag{1.5}$$

Theorem 1.3 in conjunction with the following two identities of Rogers [1, p.330] and [13, p.331] (see also Slater [14, Identities (20) and (16)])

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}}{\left(q^4;q^4\right)_n} &= \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \prod_{n=1}^{\infty} \left(1-q^{5n}\right)\left(1-q^{5n-2}\right)\left(1-q^{5n-3}\right), \\
\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{\left(q^4;q^4\right)_n} &= \frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \prod_{n=1}^{\infty} \left(1-q^{5n}\right)\left(1-q^{5n-1}\right)\left(1-q^{5n-4}\right),
\end{aligned} \tag{1.6}$$

extends Theorems 1.1 and 1.2 to the following 3-way partition identities, respectively.

Table 1

ν	$A_1(\nu)$	Partitions enumerated by $A_1(\nu)$	$B(\nu)$	Partitions enumerated by $B(\nu)$
0	1	empty partition	1	empty partition
1	1	1	0	—
2	0	—	1	2
3	0	—	0	—
4	1	$3+1$	1	$2+2$
5	1	5	0	—
6	0	—	2	$6,\ 2+2+2$
7	0	—	0	—
8	1	$7+1$	2	$6+2,\ 2+2+2+2$
9	2	$9,\ 5+3+1$	0	—

Theorem 1.4. *Let $B(n)$ denote the number of partitions of n into parts $\equiv 2 \pmod 4$, let $C(n)$ denote the number of partitions of n with minimal difference 2, and let $D(n)$ denote the number of partitions of n into parts $\equiv \pm 1 \pmod 5$. Then*

$$C(n) = D(n) = \sum_{r=0}^{n} A_1(r)B(n-r), \tag{1.7}$$

where $A_1(r)$ is as defined in Theorem 1.3.

Example 1.5. $C(9) = 5$, since the relevant partitions are $9, 8+1, 7+2, 6+3, 5+3+1$. $D(9) = 5$, since the relevant partitions are $9, 6+1+1+1, 4+4+1, 4+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1$.

Also,

$$\sum_{r=0}^{9} A_1(r)B(9-r) = 5. \tag{1.8}$$

Table 1 shows the relevant partitions enumerated by $A_1(\nu)$ and $B(\nu)$ for $0 \le \nu \le 9$.

Theorem 1.6. *Let $E(n)$ denote the number of partitions of n such that the parts are ≥ 2, and the minimal difference is 2. Let $F(n)$ denote the number of partitions of n into parts $\equiv \pm 2 \pmod 5$. Then*

$$E(n) = F(n) = \sum_{r=0}^{n} A_3(r)B(n-r), \quad \forall n, \tag{1.9}$$

where $A_3(r)$ is as defined in Theorem 1.3 and $B(n)$ as defined in Theorem 1.4.

Example 1.7. $E(9) = 3$, since the relevant partitions are $9, 7+2, 6+3$. $F(9) = 3$, since the relevant partitions are $7+2, 3+3+3, 3+2+2+2$.

Table 2

ν	$A_3(\nu)$	Partitions enumerated by $A_3(\nu)$
0	1	empty partition
1	0	—
2	0	—
3	1	3
4	0	—
5	0	—
6	0	—
7	1	7
8	1	5 + 3
9	0	—

Also,

$$\sum_{r=0}^{9} A_3(r)B(9-r) = 3. \tag{1.10}$$

Table 2 shows the relevant partitions enumerated by $A_3(\nu)$ for $0 \le \nu \le 9$.

2. Proof of the Theorem 1.3

Let $A_k(m,n)$ denote the number of partitions of n enumerated by $A_k(n)$ into m parts. We shall first prove that

$$A_k(m,n) = A_k(m-1, n-k-2m+2) + A_k(m, n-4m). \tag{2.1}$$

To prove the identity (2.1), we split the partitions enumerated by $A_k(m,n)$ into two classes:

(i) those that have least part equal to k,

(ii) those that have least part greater than k.

We now transform the partitions in class (i) by deleting the least part k and then subtracting 2 from all the remaining parts. This produces a partition of $n-k-2(m-1)$ into exactly $(m-1)$ parts, each of which is $\ge k$ (since originally the second smallest part was $\ge k+2$); furthermore, since this transformation does not disturb the inequalities between the parts, we see that the transformed partition is of the type enumerated by $A_k(m-1, n-k-2m+2)$.

Next, we transform the partitions in class (ii) by subtracting 4 from each part. This produces a partition of $n-4m$ into m parts, each of which is $\ge k$, as in the first case; here too, the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by $A_k(m, n-4m)$.

The above transformations establish a bijection between the partitions enumerated by $A_k(m,n)$ and those enumerated by $A_k(m-1, n-k-2m+2) + A_k(m, n-4m)$.

This proves the identity (2.1).

For $|q| < 1$ and $|z| < |q|^{-1}$, let

$$f_k(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,n) z^m q^n. \tag{2.2}$$

Substituting for $A_k(m,n)$ from (2.1) in (2.2) and then simplifying, we get

$$f_k(z,q) = zq^k f_k\left(zq^2,q\right) + f_k\left(zq^4,q\right). \tag{2.3}$$

Since $f_k(0,q) = 1$, we may easily check by coefficient comparison in (2.3) that

$$f_k(z,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)} z^n}{\left(q^4;q^4\right)_n}. \tag{2.4}$$

Now

$$\sum_{n=0}^{\infty} A_k(n) q^n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} A_k(m,n) \right) q^n = f_k(1,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}}{\left(q^4;q^4\right)_n}. \tag{2.5}$$

This completes the proof of Theorem 1.3.

3. Conclusion

In this paper MacMahon's Theorems 1.1 and 1.2 have been extended to 3-way identities. The most obvious question which arises from this work is the following: Does Gordon's generalization of Theorems 1.1 and 1.2 also admit similar extension? We must add that different partition theoretic interpretations of identities (1.6) are found in the literature (see for instance [15, 16]).

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