Research Article

A Functional for Copulas and Quasi-Copulas

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We recall and study some properties of a known functional operating on the set of n -copulas and determine conditions under such functional is well defined on the set of n -quasi-copulas. As a consequence, new families of copulas and quasi-copulas are defined, illustrating our results with several examples.

1. Introduction

The term *copula*, coined by Sklar [1], is now common in the statistical literature (for a complete survey, see [2]). The importance of copulas as a tool for statistical analysis and modeling stems largely from the observation that the joint distribution function H of a random vector $(X_1, X_2, \dots, X_n)$ —where n is a natural number such that $n \geq 2$ —with respective univariate margins $F_1, F_2, \dots, F_n$, can be expressed in the form $H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [-\infty, \infty]^n$, in terms of an n -copula C that is uniquely determined on $\times_{i=1}^n \text{Range } F_i$.

Copulas are becoming popular in development of quantitative risk management methodology within finance and insurance [3].

Alsina et al. [4] introduced the notion of (bivariate) *quasi-copula* in order to characterize operations on distribution functions that can, or cannot, be derived from operations on random variables defined on the same probability space (for the multivariate case, see [5]). Cuculescu and Theodorescu [6] have characterized an n -dimensional quasi-copula (or n -quasi-copula) as a function $Q : [0, 1]^n \rightarrow [0, 1]$ that satisfies the following:

- (Q1) $Q(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = 0$ and $Q(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $\mathbf{u} \in [0, 1]^n$ and for every $i = 1, 2, \dots, n$;
- (Q2) Q is nondecreasing in each variable;

(Q3) the 1-Lipschitz condition, that is, for each $\mathbf{u}, \mathbf{v} \in [0, 1]^n$, then $|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{i=1}^n |u_i - v_i|$.

While every n -copula is an n -quasi-copula, there exist n -quasi-copulas Q which are not n -copulas; in this case it is said that Q is a *proper* n -quasi-copula. Every n -quasi-copula Q (and hence any n -copula) satisfies the following inequalities: $W^n(\mathbf{u}) = \max(\sum_{i=1}^n u_i - n + 1, 0) \leq Q(\mathbf{u}) \leq \min(u_1, u_2, \dots, u_n) = M^n(\mathbf{u})$ for each $\mathbf{u}$ in $[0, 1]^n$. M^n is an n -copula for every $n \geq 2$; and W^2 is a 2-copula, but W^n is a proper n -quasi-copula for every $n \geq 3$.

In the last years an increasing interest has been devoted to these functions by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities, and fuzzy logics (see [7] for an overview).

Let f be a function defined on $[0, 1]^n$, and let B denote the n -box $B = \times_{i=1}^n [a_i, b_i]$ in $[0, 1]^n$ such that $a_i \leq b_i$ for all $i = 1, 2, \dots, n$. The function f is said to be n -increasing if $\sum \text{sgn}(\mathbf{c}) \cdot f(\mathbf{c}) \geq 0$, where the sum is taken over all the vertices $\mathbf{c} = (c_1, c_2, \dots, c_n)$ of B —that is, each c_k is equal to either a_k or b_k —and $\text{sgn}(\mathbf{c})$ is 1 if $c_k = a_k$ for an even number of k 's, and -1 if $c_k = a_k$ for an odd number of k 's. Thus, an n -copula is an n -increasing function satisfying condition (Q1). Some differences between copulas and quasi-copulas can be found in [8–12].

In this note, we provide new families of n -copulas using a known functional operating defined in [13]. Moreover, we also determine conditions under such functional is an n -quasi-copula. As a consequence, new families of copulas and quasi-copulas are defined.

2. The Functional

In the following, we will consider the set J defined in $[0, 1]^n$ by $J = \{\mathbf{u} \in [0, 1]^n \mid u_1 u_2 \dots u_n = u_i \text{ for some } i, 1 \leq i \leq n\}$.

Let P be a fixed n -copula. Consider the functional defined for any n -copula C and any functions $f, \alpha_1, \alpha_2, \dots, \alpha_n$ on $[0, 1]^n$ into $[0, 1]$ as follows:

$$C^*(\mathbf{u}) = \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot [f(\mathbf{u}) + C(\alpha_1(\mathbf{u}), \alpha_2(\mathbf{u}), \dots, \alpha_n(\mathbf{u}))] \quad (2.1)$$

for every $\mathbf{u}$ in $[0, 1]^n$ with $\lambda \in [0, 1]$. Conditions under this function is an n -copula for any n -copula C are given in the following result.

Theorem 2.1 ([13, Theorem 3.4]). *Let C^* be the function defined as (2.1). If*

- (i) *the function f is n -increasing;*
- (ii) *$f(\mathbf{u}) = u_1 u_2 \dots u_n - \prod_{i=1}^n \alpha_i(\mathbf{u})$ for every $\mathbf{u} \in J$;*
- (iii) *for every $\mathbf{u} \in J$, there is some $j, 1 \leq j \leq n$, such that $\prod_{i=1}^n \alpha_i(\mathbf{u}) = \alpha_j(\mathbf{u})$, and either one of the two following conditions holds;*
- (iv) *every function α_i depends solely on one variable, which is different for each α_i ; all α_i 's are monotone but the number of them that are decreasing is even;*
- (v) *there exists one variable $u_j (1 \leq j \leq n)$ such that none of the functions α_i depends on u_j .*

Then C^ is an n -copula for every n -copula C .*

2.1. Examples

In [13], there are no examples of n -copulas of type (2.1) satisfying the conditions in Theorem 2.1. In what follows, we provide examples of such n -copulas, but first we need the following result, which shows that the convex linear combination of two n -copulas is an n -copula, and whose proof is immediate.

Proposition 2.2. *Let C_1 and C_2 be two n -copulas. Then, the function $C(\mathbf{u}) = \theta \cdot C_1(\mathbf{u}) + (1-\theta) \cdot C_2(\mathbf{u})$ for all $\mathbf{u}$ in $[0, 1]^n$ with $\theta \in [0, 1]$, is an n -copula.*

Thus, taking P as any fixed n -copula, f the zero function—that is, $f(\mathbf{u}) = 0$ for all $\mathbf{u} \in [0, 1]^n$ —, and $\alpha_i(\mathbf{u}) = u_i$, for all $\mathbf{u}$ in $[0, 1]^n$ and for every $i = 1, 2, \dots, n$, in (2.1), then conditions (i)–(iv) in Theorem 2.1 are immediately satisfied, and hence we obtain the result in Proposition 2.2. In what follows we will find examples in which the function f is not identically equal to zero.

We now provide two examples.

Example 2.3. Let P be a fixed n -copula, $\alpha_i(\mathbf{u}) = u_i(1 - u_i)$, $i = 1, 2, \dots, n-1$, and $\alpha_n(\mathbf{u}) = 1$ for all $\mathbf{u}$ in $[0, 1]^n$. If f is the function defined by $f(\mathbf{u}) = u_1^2 u_2 \dots u_n$, then for any n -box $B = \times_{i=1}^n [u_i, v_i]$, we have that f is n -increasing, since $(v_2 + u_2) \prod_{i=1}^n (v_i - u_i) \geq 0$. It is easy to check that conditions (i), (ii), (iii), and (v)—but not (iv)—in Theorem 2.1 are satisfied. Thus, we have that the functions given by

$$C^*(\mathbf{u}) = \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot \left[u_1^2 u_2 \dots u_n + C(u_1(1 - u_1), \dots, u_{n-1}(1 - u_{n-1}), 1) \right], \quad \mathbf{u} \in [0, 1]^n, \quad (2.2)$$

represent a family of n -copulas for any n -copula C .

Example 2.4. Let n be a natural number such that $n \geq 3$. Let $\alpha_i(\mathbf{u}) = 1 - u_i$ for $i = 1, 2, 3$, and $\alpha_i(\mathbf{u}) = u_i$, $i \geq 4$, for every $\mathbf{u}$ in $[0, 1]^n$. If f is any n -copula, we have that conditions (i), (ii), (iii), and (iv)—but not (v)—in Theorem 2.1 are satisfied. Thus, the functions given by

$$C^*(\mathbf{u}) = \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot [f(\mathbf{u}) + C(1 - u_1, 1 - u_2, 1 - u_3, u_4, \dots, u_n)], \quad \mathbf{u} \in [0, 1]^n, \quad (2.3)$$

represent a family of n -copulas for any n -copula C .

In [13], it is noted that conditions (iv) or (v) in Theorem 2.1 could be too strong, and the authors provide a counterexample in which none of the conditions is satisfied and the function C^* given by (2.1) is not a copula. However, we stress that these two conditions are sufficient (but not necessary), as the following example shows.

Example 2.5. Let P be a fixed n -copula, and let $f(\mathbf{u}) = \prod_{i=1}^n u_i$, $\alpha_1(\mathbf{u}) = \delta \prod_{i=1}^n u_i(1 - u_i)$ with $\delta \in [-1, 1]$, and $\alpha_i(\mathbf{u}) = 1$, $i = 2, \dots, n$, for all $\mathbf{u}$ in $[0, 1]^n$. Then, it is easy to check that these

functions satisfy conditions (i), (ii), and (iii) in Theorem 2.1, but neither (iv) nor (v) hold. However, for any n -copula C , the function given by (2.1), that is,

$$\begin{aligned} C^*(\mathbf{u}) &= \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot \left[\prod_{i=1}^n u_i + C\left(\delta \prod_{i=1}^n u_i (1 - u_i), 1, \dots, 1\right) \right] \\ &= \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot \left[\prod_{i=1}^n u_i \cdot \left(1 + \delta \prod_{i=1}^n (1 - u_i) \right) \right] \end{aligned} \quad (2.4)$$

for every $\mathbf{u}$ in $[0, 1]^n$, is an n -copula, since the function $D(\mathbf{u}) = \prod_{i=1}^n u_i \cdot [1 + \delta \prod_{i=1}^n (1 - u_i)]$ is a member of the known Farlie-Gumbel-Morgenstern family of n -copulas [14, equation (44.73)], and we only need to apply Proposition 2.2. For P , we can choose any family of n -copulas different from D (see [2, 15–17] e.g.).

2.2. Association and Dependence

For statistical modelling [18], with each n -copula C we can associate, among others, a non-parametric measure of multivariate association, called the *medial correlation coefficient* (or *Blomqvist's beta*), which can be easily computed as

$$\beta_n(C) = \frac{2^{n-1} [C(1/2) + \bar{C}(1/2)] - 1}{2^{n-1} - 1} \quad (2.5)$$

(see [19, 20]), where $\bar{C}$ denotes the *survival function* of C [2].

A coefficient that summarize some statistical properties of a copula is introduced in [21]. Let $\mathbf{X}$ be a random vector with joint distribution function H and univariate margins $F_1, \dots, F_n$. Let $F_{\min} := \min(F_1(X_1), \dots, F_n(X_n))$ and $F_{\max} := \max(F_1(X_1), \dots, F_n(X_n))$. The *lower extremal dependence coefficient* of $\mathbf{X}$ is defined as $\varepsilon_L := \lim_{t \rightarrow 0^+} P[F_{\max} \leq t \mid F_{\min} \leq t]$, and the *upper extremal dependence coefficient* of $\mathbf{X}$ is defined as $\varepsilon_U := \lim_{t \rightarrow 1^-} P[F_{\min} > t \mid F_{\max} > t]$ (if the limits exist). We obtain that, for every $\mathbf{t} = (t, \dots, t) \in [0, 1]^n$, in terms of the associated copula C , these coefficients are given by

$$\varepsilon_L = \lim_{t \rightarrow 0^+} \frac{C(\mathbf{t})}{1 - \bar{C}(\mathbf{t})} \quad \varepsilon_U = \lim_{t \rightarrow 1^-} \frac{\bar{C}(\mathbf{t})}{1 - C(\mathbf{t})}, \quad (2.6)$$

respectively.

Since the general computation of these coefficients for the n -copulas given by (2.1) do not give us much information, let us take the family of n -copulas given by (2.4) with $P = M^n$. Then, it is a simple exercise to show that

$$\beta_n(C^*) = \begin{cases} \lambda + \frac{\delta(1 - \lambda)}{2^{2n-1}}, & \text{if } n \text{ is even} \\ \lambda, & \text{if } n \text{ is odd,} \end{cases} \quad (2.7)$$

and

$$\varepsilon_L = \varepsilon_U = \frac{\lambda}{\lambda + (1 - \lambda)n}. \quad (2.8)$$

2.3. Quasi-Copulas

Assume P is a fixed n -quasi-copula and consider the functional defined for any n -quasi-copula Q and any functions $f, \alpha_1, \alpha_2, \dots, \alpha_n$ on $[0, 1]^n$ into $[0, 1]$ as follows:

$$Q^*(\mathbf{u}) = \lambda \cdot P(\mathbf{u}) + (1 - \lambda) \cdot [f(\mathbf{u}) + Q(\alpha_1(\mathbf{u}), \alpha_2(\mathbf{u}), \dots, \alpha_n(\mathbf{u}))] \quad (2.9)$$

for every $\mathbf{u}$ in $[0, 1]^n$ with $\lambda \in [0, 1]$. We want to study conditions for the functions $f, \alpha_1, \alpha_2, \dots, \alpha_n$ which assure that Q^* is an n -quasi-copula for any n -quasi-copula Q —of course, Theorem 2.1 is valid for this case too. We have the following result.

Theorem 2.6. *Let Q^* be the function defined by (2.9) for which properties (ii) and (iii) in Theorem 2.1 are satisfied. Moreover, suppose*

- (vi) *the functions f and α_i , $i = 1, 2, \dots, n$, are increasing in each variable;*
- (vii) *f satisfies the r -Lipschitz condition with $r \in [0, 1]$ and α_i satisfies the $(1 - r)/n$ -Lipschitz condition for all $i = 1, 2, \dots, n$.*

Then Q^ is an n -quasi-copula for every n -quasi-copula Q .*

Proof. Condition (Q1) in the definition of n -quasi-copula is equivalent to the conditions (ii) and (iii) in Theorem 2.1 [13]. We prove now that conditions (Q2) and (Q3) are satisfied. For that, let $\mathbf{u}' = (u_1, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_n)$ and $\mathbf{u} = (u_1, \dots, u_i, \dots, u_n)$ be in $[0, 1]^n$ such that $u_i \leq u_i'$. Since

$$\begin{aligned} Q^*(\mathbf{u}') - Q^*(\mathbf{u}) &= \lambda \cdot P(\mathbf{u}') + (1 - \lambda) \cdot [f(\mathbf{u}') + Q(\alpha_1(\mathbf{u}'), \dots, \alpha_n(\mathbf{u}'))] \\ &\quad - \lambda \cdot P(\mathbf{u}) - (1 - \lambda) \cdot [f(\mathbf{u}) + Q(\alpha_1(\mathbf{u}), \dots, \alpha_n(\mathbf{u}))] \\ &= \lambda \cdot [P(\mathbf{u}') - P(\mathbf{u})] + (1 - \lambda) \\ &\quad \cdot [f(\mathbf{u}') - f(\mathbf{u}) + Q(\alpha_1(\mathbf{u}'), \dots, \alpha_n(\mathbf{u}')) - Q(\alpha_1(\mathbf{u}), \dots, \alpha_n(\mathbf{u}))], \end{aligned} \quad (2.10)$$

and P and Q are n -quasi-copulas—that is, P and Q satisfy conditions (Q2) and (Q3); using condition (vi), we obtain immediately that $Q^*(\mathbf{u}') - Q^*(\mathbf{u}) \geq 0$, that is, Q^* satisfies (Q2); and using condition (vii) we have that

$$\begin{aligned} Q^*(\mathbf{u}') - Q^*(\mathbf{u}) &\leq \lambda \cdot [P(\mathbf{u}') - P(\mathbf{u})] + (1 - \lambda) \cdot \left(f(\mathbf{u}') - f(\mathbf{u}) + \sum_{i=1}^n [\alpha_i(\mathbf{u}') - \alpha_i(\mathbf{u})] \right) \\ &\leq \lambda \cdot (u_i' - u_i) + (1 - \lambda) \cdot \left[r(u_i' - u_i) + n \cdot \frac{1 - r}{n} \cdot (u_i' - u_i) \right] \\ &= (u_i' - u_i). \end{aligned} \quad (2.11)$$

Thus, for every $\mathbf{u}, \mathbf{v}$ in $[0, 1]^n$, we have that

$$\begin{aligned}
 |Q^*(\mathbf{v}) - Q^*(\mathbf{u})| &= |Q^*(v_1, \dots, v_n) - Q^*(v_1, \dots, v_{n-1}, u_n)| \\
 &\quad + |Q^*(v_1, \dots, v_{n-1}, u_n) - Q^*(v_1, \dots, v_{n-2}, v_{n-1}, u_n)| \\
 &\quad + \dots + |Q^*(v_1, u_2, \dots, u_n) - Q^*(u_1, \dots, u_n)| \\
 &\leq \sum_{i=1}^n |v_i - u_i|,
 \end{aligned} \tag{2.12}$$

that is, Q^* satisfies (Q3), which completes the proof. $\square$

As an application of Theorem 2.6, we can generalize the result in Proposition 2.2 in the following sense: take P as any fixed n -quasi-copula, f an n -quasi-copula, and α_i the zero function for every $i = 1, 2, \dots, n$. Thus, we have a similar result to Proposition 2.2 applied to n -quasi-copulas.

Proposition 2.7. *Let Q_1 and Q_2 be two n -quasi-copulas. Then, the function $Q(\mathbf{u}) = \theta \cdot Q_1(\mathbf{u}) + (1 - \theta) \cdot Q_2(\mathbf{u})$ for all $\mathbf{u}$ in $[0, 1]^n$ with $\theta \in [0, 1]$, is an n -quasi-copula.*

Conditions (vi) and (vii) in Theorem 2.6 may be not necessary. For instance, if f is the zero function, $\alpha_i(\mathbf{u}) = u_j$ for every $i, j = 1, 2, \dots, n$, then condition (vii) in Theorem 2.6 is not satisfied; however, in this case, we obtain a family of (maybe, proper) n -quasi-copulas of type (2.9). Another example with f not identically zero is the following.

Example 2.8. Let f be the product n -copula, $\alpha_1(\mathbf{u}) = \gamma(1 - u_n)(1 - \max_{1 \leq i \leq n-1} u_i) \prod_{i=1}^n u_i$ and $\alpha_i(\mathbf{u}) = 1$, $i = 2, \dots, n$, for every $\mathbf{u} \in [0, 1]^n$ with $\gamma \in [0, 1]$. Then, conditions (ii) and (iii) in Theorem 2.1 hold, but condition (vi) in Theorem 2.6 is not satisfied. However, via Proposition 2.7, we have that

$$Q^*(\mathbf{u}) = \lambda P(\mathbf{u}) + (1 - \lambda) \cdot \prod_{i=1}^n u_i \left[1 + \gamma(1 - u_n) \left(1 - \max_{1 \leq i \leq n-1} u_i \right) \right] \tag{2.13}$$

is an n -quasi-copula, since $\prod_{i=1}^n u_i [1 + \gamma(1 - u_n)(1 - \max_{1 \leq i \leq n-1} u_i)]$ is a proper n -quasi-copula for $\gamma \in (0, 1]$ (note that the case $\gamma = 0$ corresponds to the product n -copula). If, for instance, we take $P = W^n$ in (2.13), we obtain a family of proper n -quasi-copulas.

3. Conclusion

In this note, we have recalled a known functional operating on the set of n -copulas, provided examples of n -copulas satisfying the conditions in Theorem 2.1, and studied some properties of association and dependence. Finally, we have determined conditions under such functional are well defined on the set of n -quasi-copulas.

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