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Research Article

Extension of Zhou's Method to Neutral Functional-Differential Equation with Proportional Delays

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The differential transform method (DTM) is a reliable method applied by providing new theorems to develop exact and approximate solutions of neutral functional-differential equation (NFDE) with proportional delays. The results obtained with the proposed methods are in good agreement with one obtained by other methods. The advantages of this technique are illustrated. It is easy to see that the DTM is very accurate and easy to implement in finding analytical solutions of wide classes of linear and nonlinear NFDEs.

1. Introduction

The neutral functional-differential equation (NFDE) is

$$(u(t) + a(t)u(\sigma_m(t)))^{(m)} = \beta u(t) + \sum_{k=0}^{m-1} b_k(t)u^k(\sigma_k(t)) + f(t), \quad t \geq 0, \tag{1.1}$$

under the conditions

$$\sum_{k=0}^{m-1} c_{ik}u^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1, \tag{1.2}$$

where a, b_k, σ_k are analytical functions; β, c_{ik} and $\lambda_i \in \mathcal{C}$. A classical case [1]

$$\sigma_k(t) = t - \tau_k, \quad k = 0, 1, \ldots, m, \tag{1.3}$$

where τ_k is positive. Another interesting case [2] is

$$\sigma_k(t) = q_k t, \quad k = 0, 1, \ldots, m, \tag{1.4}$$

where $0 < q_k < 1$. Both cases are playing an interesting role in many applications (see [1, 2] and references therein). In recent years, there has been a growing interest in the numerical treatment of NFDE, some of which are the Adams method [3], continuous Runge-Kutta methods [4], segmented Tau approximation [5], Homotopy perturbation method [6], one-leg θ-methods [7, 8], and variational iteration method [9].

In this paper we consider the following neutral functional-differential equations with proportional delays.

Problem 1. Consider

$$\begin{gathered} u'(t) = \beta u(t) - a(t)u'(q_\ell t) + f(t, u(t), u(q_i t), u'(q_i t)), \quad i = 1, 2, \ldots, \ell - 1, \\ u(0) = u_0. \end{gathered} \tag{1.5}$$

Problem 2. Consider

$$\begin{gathered} u^{(m)}(t) = \beta u(t) - a(t)u^{(m)}(q_\ell t) + f\left(t, u(t), u(q_i t), u'(q_i t), u''(q_i t), \ldots, u^{(m-1)}(q_i t)\right), \\ \sum_{k=0}^{m-1} c_{jk} u^{(k)}(0) = \lambda_j, \quad j = 0, 1, 2, \ldots, m - 1, \end{gathered} \tag{1.6}$$

where a, f are analytical functions; β, c_{jk} and λ_j are real or complex constants; $0 < q_i < 1$, $i = 1, 2, \ldots, \ell$.

The basic motivation of this work is to extend the differential transform method (DTM) by presenting and proving new theorems to create the exact or approximate solutions to a high degree of accuracy to the Problems 1 and 2. The DTM is a numerical-analytical technique that was first proposed by Zhou (1986) [10], who solved problems in electric circuit analysis. Since then, DTM was successfully applied for a large variety problems. For example, differential-difference equations [11], Volterra integral equation with separable kernels [12], MHD boundary-layer equations [13], linear and nonlinear systems of partial differential equations [14], and nonlinear oscillators with fractional nonlinearities [15]. To the best of our knowledge differential transform method has not be used by any researcher before to solve NFDE. By this method it is possible to obtain highly accurate results when compared with existing results from variational iteration method [9] and homotopy perturbation method [6].

Table 1: Fundamental operations of differential transformation.

Original function	Transformed function
$u(t) = f(t) \pm g(t)$	$U(k) = F(k) \pm G(k)$
$u(t) = cf(t)$	$U(k) = cF(k), c \in \mathcal{R}$
$u(t) = f(t)g(t)$	$U(k) = \sum_{\ell=0}^{k} F(\ell)G(k-\ell)$
$u(t) = f_1(t)f_2(t)\cdots f_{m-1}(t)f_m(t)$	$U(k) = \sum_{\ell_{m-1}=0}^{k} \sum_{\ell_{m-2}=0}^{\ell_{m-1}} \cdots \sum_{\ell_2=0}^{\ell_3} \times \sum_{\ell_1=0}^{\ell_2} F_1(\ell_1)$ $\times F_2(\ell_2-\ell_1)\cdots F_{m-1}(\ell_{m-1}-\ell_{m-1})F(k-\ell_{m-1})$
$u(t) = (d^n/dt^n)f(t)$	$U(k) = ((k+n)!/k!)F_m(k)$
$u(t) = t^m$	$U(k) = \delta(k-m) = \begin{cases} 1, & k = m; \\ 0, & k \neq m \end{cases}$

2. Basic Idea of Differential Transform Method

The differential transform of the kth derivative of a function $u(t)$ is defined as follows:

$$U(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}u(t)\right]_{t=t_0}. \tag{2.1}$$

The original and transformed functions are denoted throughout this paper by small and capital letters, respectively. The differential inverse transform of $U(k)$ is defined as

$$u(t) = \sum_{k=0}^{N} U(k)(t-t_0)^k, \quad \text{for } N \longrightarrow \infty. \tag{2.2}$$

And from (2.1) and (2.2):

$$u(t) = \sum_{k=0}^{N} \frac{1}{k!}\left[\frac{d^k}{dt^k}u(t)\right]_{t=t_0}(t-t_0)^k, \quad \text{for } N \longrightarrow \infty, \tag{2.3}$$

which is actually the Taylor series expansion of $u(t)$ about the point t_0.

The fundamental mathematical operations performed by one-dimensional differential transform can be obtained from (1.5) and (1.6) in Table 1 (also, see [10]).

3. Main Results

In the following theorems, we find the differential transformations of given functions. These results are very useful in our approach for solving NFDEs.

Theorem 3.1. *Suppose that $U(k)$, $F(k)$, and $G(k)$ are the differential transformations of the functions $u(t)$, $f(t)$, respectively, and $g(t)$ and $0 < q$, $q_i < 1$, for $i = 1, 2, \ldots, m$:*

(I) *if $u(t) = f(qt)$, then $U(k) = q^k F(k)$,*

(II) *if $u(t) = d^n f(qt)/d(qt)^n$, then $U(k) = q^k((k+n)!/k!)F(k+n)$,*

(III) *if* $u(t) = g(t)d^n f(qt)/d(qt)^n$, *then*

$$U(k) = \sum_{\ell=0}^{k} q^{k-\ell} \frac{(k-\ell+n)}{(k-\ell)!} G(\ell) F(k-\ell+n), \tag{3.1}$$

(IV) *if* $u(t) = (d^n/d(q_1t)^n)[f(q_1t)](d^m/d(q_2t)^m)[g(q_2t)]$, *then*

$$U(k) = \sum_{\ell=0}^{k} q_1^{\ell} q_2^{k-\ell} \frac{(\ell+n)!(k-\ell+m)!}{(k-\ell)!(\ell)!} F(\ell+n) G(k-\ell+m), \tag{3.2}$$

(V) *if* $u(t) = (d^{n_1}/d(q_1t)^{n_1})[f_1(q_1t)](d^{n_2}/d(q_2t)^{n_2})[f_2(q_2t)] \cdots (d^{n_{m-1}}/d(q_{m-1}t)^{n_{m-1}})$ $[f_{m-1}(q_{m-1}t)(d^{n_m}/d(q_mt)^{n_m})[f_m(q_mt)]$, *then*

$$\begin{aligned} U(k) = {} & \sum_{\ell_{m-1}=0}^{k} \sum_{\ell_{m-2}=0}^{\ell_{m-1}} \cdots \sum_{\ell_2=0}^{\ell_3} \sum_{\ell_1=0}^{\ell_2} q_1^{\ell_1} q_2^{\ell_2-\ell_1} \cdots q_{m-1}^{\ell_{m-1}-\ell_{m-2}} q_m^{\ell_k-\ell_{m-1}} \\ & \times \frac{(\ell_1+n_1)!}{\ell_1!} \frac{(\ell_2-\ell_1+n_2)!}{(\ell_2-\ell_1)!} \cdots \frac{(\ell_{m-1}-\ell_{m-2}+n_{m-1})!}{(\ell_{m-1}-\ell_{m-2})!} \frac{(k-\ell_{m-1}+n_m)!}{(k-\ell_{m-1})!} \\ & \times F_1(\ell_1+n_1) F_2(\ell_2-\ell_1+n_2) \cdots F_{m-1}(\ell_{m-1}-\ell_{m-1}+n_{m-1}) F_m(k-\ell_{m-1}+n_m). \end{aligned} \tag{3.3}$$

Proof. (I), (II) The proof follows immediately by substituting $u(t)$ into (2.1).

(III) By using the definition of DTM (2.1), we have

$$\begin{aligned} U(k) &= \frac{1}{k!} \left[\frac{d^k}{dt^k} \left(g(t) \frac{d^n}{d(qt)^n} f(qt) \right) \right]_{t=t_0} \\ &= \frac{1}{k!} \left[\sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{\ell}}{dt^{\ell}} g(t) \frac{d^{k-\ell}}{dt^{k-\ell}} \left(\frac{d^n}{d(qt)^n} f(qt) \right) \right]_{t=t_0}, \end{aligned} \tag{3.4}$$

and from (II), we have

$$U(k-\ell) = \frac{1}{(k-\ell)!} \left[\frac{d^{k-\ell}}{dt^{k-\ell}} \frac{d^n}{d(qt)^n} f(qt) \right]_{t=t_0} = q^{k-\ell} \frac{(k-\ell+n)!}{(k-\ell)!} F(k-\ell+n). \tag{3.5}$$

By utilizing this value, we get

$$U(k) = \sum_{\ell=0}^{k} q^{k-\ell} \frac{(k-\ell+n)!}{(k-\ell)!} G(\ell) F(k-\ell+n). \tag{3.6}$$

(IV) By using the definition of DTM (2.1), we have

$$\begin{aligned} U(k) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\left[\frac{d^n}{d(q_1t)^n}\left[f(q_1t)\right]\frac{d^m}{d(q_2t)^m}\left[f(q_2t)\right]\right]\right]_{t=t_0} \\ &= \frac{1}{k!}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{d^\ell}{dt^\ell}\left[\frac{d^n}{d(q_1t)^n}f(q_1t)\right]\frac{d^{k-\ell}}{dt^{k-\ell}}\left[\frac{d^m}{d(q_{2t})^m}g(q_2t)\right]\right]_{t=t_0}, \end{aligned} \tag{3.7}$$

then from (II), we have

$$\begin{aligned} U(\ell) &= \frac{1}{\ell!}\left[\frac{d^\ell}{dt^\ell}\frac{d^n}{d(q_1t)^n}f(q_1t)\right] = q_1^\ell\frac{(\ell+n)!}{\ell!}F(\ell+n), \\ U(k-\ell) &= \frac{1}{(k-\ell)!}\left[\frac{d^{k-\ell}}{dt^{k-\ell}}\frac{d^m}{d(q_2t)^m}g(q_2t)\right] = q_2^{k-\ell}\frac{(k-\ell+m)!}{(k-\ell)!}G(k-\ell+m). \end{aligned} \tag{3.8}$$

By utilizing these values

$$U(k) = \sum_{\ell=0}^{k} q_1^\ell q_2^{k-\ell}\frac{(\ell+n)!(k-\ell+m)!}{(k-\ell)!(\ell)!}F(\ell+n)G(k-\ell+m). \tag{3.9}$$

(V) Let the differential transform of $d^{n_i}f_i/d(q_it)^{n_i}$ at $t=t_0$ for $i=1,2,\ldots,m$ be $H_i(k)$, then by using operations of differential transformation given in Table 1, we have

$$U(k) = \sum_{\ell_{m-1}=0}^{k}\sum_{\ell_{m-2}=0}^{\ell_{m-1}}\cdots\sum_{\ell_2=0}^{\ell_3}\sum_{\ell_1=0}^{\ell_2}H_1(\ell_1)H_2(\ell_2-\ell_1)\cdots H_{m-1}(\ell_{m-1}-\ell_{m-1})H(k-\ell_{m-1}), \tag{3.10}$$

and from (II), we have

$$\begin{aligned} H_1(\ell_1) &= q_1^{\ell_1}\frac{(\ell_1+n_1)!}{\ell_1!}F_1(\ell_1+n_1), \\ H_2(\ell_2-\ell_1) &= q_2^{\ell_2-\ell_1}\frac{(\ell_2-\ell_1+n_2)!}{(\ell_2-\ell_1)!}F_2(\ell_2-\ell_1+n_1), \\ &\vdots \end{aligned} \tag{3.11}$$

$$\begin{aligned} H_{m-1}(\ell_{m-1}-\ell_{m-2}) &= q_{m-1}^{\ell_{m-1}-\ell_{m-2}}\frac{(\ell_{m-1}-\ell_{m-2}+n_{m-1})!}{(\ell_{m-1}-\ell_{m-21})!}F_{m-1}(\ell_{m-1}-\ell_{m-2}+n_{m-1}), \\ H_m(k-\ell_{m-1}) &= q_m^{k-\ell_{m-1}}\frac{(k-\ell_{m-1}+n_m)!}{(k-\ell_{m-1})!}F_m(k-\ell_{m-1}+n_m). \end{aligned} \tag{3.12}$$

Substituting those values into (3.10), we obtain (3.3). □

4. Illustrative Examples

In this part, we will apply the DTM to solve NFDE with proportional delays.

The numerical solutions of Examples 4.3, 4.4, and 4.5 have been calculated by variational iteration method [9] and homotopy perturbation method [6], which did not yield the exact solutions. However, applying DTM gives the exact solutions of those examples, as we will show later.

Example 4.1 (see [6, 9]). Consider the following first-order NFDE with proportional delay:

$$u'(t) = -u(t) + \frac{1}{2}u\left(\frac{t}{2}\right) + \frac{1}{2}u'\left(\frac{t}{2}\right), \quad 0 < t < 1,$$
$$u(0) = 1. \tag{4.1}$$

Taking the differential transform of (4.1) as given in (2.1), we get

$$(k+1)U(k+1) = -U(k) + \frac{1}{2}\left(\frac{1}{2}\right)^k U(k) + \frac{1}{2}\left(\frac{1}{2}\right)^k (k+1)U(k+1), \tag{4.2}$$

which can be rewritten as follows:

$$U(k+1) = \frac{-U(k)}{(k+1)}. \tag{4.3}$$

The differential transform of the initial condition of $u(t)$ at $t_0 = 0$, is $U(0) = 1$, form (4.3) for $k = 0, 1, \ldots, 8$, we can get

$$U(1) = -1, \qquad U(2) = \frac{1}{2!}, \qquad U(3) = -\frac{1}{3!}, \qquad U(4) = \frac{1}{4!},$$
$$U(5) = -\frac{1}{5!}, \qquad U(6) = \frac{1}{6!}, \qquad U(7) = -\frac{1}{7!}, \qquad U(8) = \frac{1}{8!}, \tag{4.4}$$

substituting these values into (2.2), to get

$$u(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \frac{t^8}{8!}. \tag{4.5}$$

The closed form of the above solution, when $N \to \infty$ is $u(t) = e^{-t}$, which is the exact solution. In Table 2 the absolute errors of DTM for $N = 7, 8$, VIM [9] with eight terms and HPM [6] with eight terms (Table 3) are compared.

Example 4.2 (see [6, 9]). Consider the first-order NFDE with proportional delay

$$u'(t) = -u(t) + 0.1u(0.8t) + 0.5u'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, \quad t \geq 0,$$
$$u(0) = 0. \tag{4.6}$$

Table 2: Comparison of the absolute errors for Example 4.1.

t	VIM [9]	HPM [6]	DTM	
	$n = 8$	$n = 8$	$n = 7$	$n = 8$
0.1	$3.72E-4$	$3.36E-4$	$2.45E-13$	$2.89E-15$
0.2	$7.08E-4$	$5.80E-4$	$6.21E-11$	$1.38E-12$
0.3	$1.01E-3$	$7.50E-4$	$1.57E-9$	$5.27E-11$
0.4	$1.29E-3$	$8.64E-4$	$1.56E-8$	$6.95E-10$
0.5	$1.54E-3$	$9.33E-4$	$9.18E-8$	$5.12E-9$
0.6	$1.76E-3$	$9.68E-4$	$3.90E-7$	$2.62E-8$
0.7	$1.97E-3$	$9.78E-4$	$1.32E-6$	$1.04E-7$
0.8	$2.15E-3$	$9.68E-4$	$3.82E-6$	$3.42E-7$
0.9	$2.32E-3$	$9.44E-4$	$9.70E-6$	$9.79E-7$
1.0	$2.47E-3$	$9.10E-4$	$2.23E-5$	$2.50E-6$

Table 3: Comparison of the absolute errors for Example 4.2.

t	VIM [9]	HPM [6]	DTM	
	$n = 8$	$n = 8$	$n = 7$	$n = 8$
0.1	$1.30E-3$	$1.06E-3$	$8.20E-9$	$1.37E-10$
0.2	$2.14E-3$	$1.35E-3$	$5.16E-7$	$1.73E-8$
0.3	$2.63E-3$	$1.18E-3$	$5.78E-6$	$2.91E-7$
0.4	$2.84E-3$	$7.61E-4$	$3.20E-5$	$2.15E-6$
0.5	$2.83E-3$	$2.32E-4$	$1.20E-4$	$1.01E-5$
0.6	$2.67E-3$	$2.98E-4$	$3.53E-4$	$3.58E-5$
0.7	$2.39E-3$	$7.64E-4$	$8.77E-4$	$1.04E-4$
0.8	$2.04E-3$	$1.12E-3$	$1.92E-3$	$2.61E-4$
0.9	$1.64E-3$	$1.37E-3$	$3.84E-3$	$5.88E-4$
1.0	$1.22E-3$	$1.50E-3$	$7.12E-3$	$1.21E-3$

Let The differential transforms of $te^{-0.8t}$, $e^{-0.8t}$, and e^{-t} at $t_0 = 0$ be $\delta_1(k)$, $\delta_2(k)$, and $\delta_3(k)$, respectively:

$$\begin{aligned}
\delta_1(k) &= \begin{cases} 0, & k = 0, \\ \dfrac{(-1)^{k-1}(0.8)^{k-1}}{k-1}!, & k \neq 0, \end{cases} \\
\delta_2(k) &= \frac{(-1)^k (0.8)^k}{k!}, \\
\delta_3(k) &= \frac{(-1)^k}{k!}.
\end{aligned} \tag{4.7}$$

We can obtain the differential transform of (4.6) as

$$U(k+1) = \frac{-U(k)\left(1-0.1(0.8)^k\right) + 0.32\delta_1(k) - 0.5\delta_2(k) + \delta_3(k)}{(k+1)\left(1-0.5(0.8)^k\right)}. \tag{4.8}$$

At $t_0 = 0$, the initial condition transformed to $U(0) = 0$, so from (4.8), we have

$$\begin{gathered} U(1) = 1, \qquad U(2) = -1, \qquad U(3) = \frac{1}{2}, \qquad U(4) = -\frac{1}{6}, \\ U(5) = \frac{1}{24}, \qquad U(6) = -\frac{1}{120}, \qquad U(7) = \frac{1}{720}, \qquad U(8) = -\frac{1}{5040}. \end{gathered} \tag{4.9}$$

Substituting these values into(2.2), we get

$$u(t) = t - t^2 + \frac{1}{2}t^3 - \frac{1}{6}t^4 + \frac{1}{24}t^5 - \frac{1}{120}t^6 + \frac{1}{720}t^7 - \frac{1}{5040}t^8. \tag{4.10}$$

The closed form of the above solution, when $N \to \infty$, is $u(t) = te^{-t}$, which is the exact solution. In Table 2 we compare the absolute errors of DTM for $N = 7, 8$, VIM [9] with eight terms, and HPM [6] with eight terms.

Example 4.3 (see[6, 9]). Consider the following second-order NFDE with proportional delay:

$$\begin{gathered} u''(t) = u'\left(\frac{t}{2}\right) - \frac{1}{2}tu''\left(\frac{t}{2}\right) + 2, \quad 0 < t < 1, \\ u(0) = 1, \qquad u'(0) = 0. \end{gathered} \tag{4.11}$$

The differential transform for (4.11) is found as

$$\begin{aligned} &(k+1)(k+2)U(k+2) \\ &\quad = \frac{1}{2^k}(k+1)U(k+1) - \frac{1}{2}\sum_{\ell=0}^{k}\frac{1}{2^{k-\ell}}\frac{(k-\ell+2)!}{(k-\ell)!}\delta(\ell-1)U(k-\ell+2) + 2\delta(k), \end{aligned} \tag{4.12}$$

form the initial condition we can get $U(0) = 1$ and $U(1) = 0$. Form (4.12), we get

$$U(k) = \begin{cases} 1, & k = 2, \\ 0, & k > 2. \end{cases} \tag{4.13}$$

Then, by using (2.2), $u(t) = 1 + t^2$, which is the exact solution.

Example 4.4 (see [6, 9]). Consider the second-order NFDE with proportional delay:

$$u''(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) + u'\left(\frac{t}{2}\right) + \frac{1}{2}u''\left(\frac{t}{2}\right) - t^2 - t + 1, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0. \tag{4.14}$$

The differential transform for (4.14) at $t_0 = 0$ is given by

$$(k+1)(k+2)U(k+2) = \frac{3}{4}U(k) + \frac{1}{2^k}U(k) + \frac{1}{2^k}(k+1)U(k+1) + \frac{1}{2}\frac{1}{2^k}(k+1)$$
$$\times (k+2)U(k+2) - \delta(k-2) - \delta(k-1) + \delta(k), \tag{4.15}$$

and can be rewritten as

$$U(k+2) = \frac{(3/4 + 1/2^k)U(k) + (1/2^k)(k+1)U(k+1) - \delta(k-2) - \delta(k-1) + \delta(k)}{(k+1)(k+2)(1 - 1/2^{k+1})}. \tag{4.16}$$

Form the initial condition we can get $U(0) = U(1) = 0$, and form (4.16), we get

$$U(k) = \begin{cases} 1, & k = 2, \\ 0, & k > 2. \end{cases} \tag{4.17}$$

Then, by using (2.2), $u(t) = t^2$, which is the exact solution.

Example 4.5 (see [6, 9]). Consider the following third-order NFDE with proportional delays:

$$u'''(t) = u(t) + u'\left(\frac{t}{2}\right) + u''\left(\frac{t}{3}\right) + \frac{1}{2}u'''\left(\frac{t}{4}\right) - t^4 - \frac{t^3}{2} - \frac{4t^2}{3} + 21t,$$
$$u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = 0. \tag{4.18}$$

The differential transform of (4.18) can be written as

$$U(k) + \frac{1}{2^k}(k+1)U(k+1) + \frac{1}{3^k}(k+1)(k+2)U(k+2)$$
$$U(k+3) = \frac{-\delta(k-4) - (1/2)\delta(k-3) - (4/3)\delta(k-2) + 21\delta(k-1)}{(k+1)(k+2)(k+3)(1 - (1/2)(1/4^k))}. \tag{4.19}$$

Form the initial condition we can get $U(0) = U(1) = U(2) = 0$, so from the (4.19), we get

$$U(k) = \begin{cases} 0, & k \neq 4, \\ 1, & k = 4. \end{cases} \tag{4.20}$$

Substituting (4.20) in (2.2) gives $u(t) = t^4$, which is the exact solution.

5. Conclusion

In this study, we extended DTM to the solution of NFDE with proportional delays. New theorems are presented with their proofs. All examples results show that the DTM is more effective than VIM and HPM for solving NFDE with proportional delays. We believe that the ease of implementation and efficiency of the DTM gives it much wider applicability.

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