Research Article

Cogredient Standard Forms of Symmetric Matrices over Galois Rings of Odd Characteristic

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Let $R = \text{GR}(p^s, p^{sm})$ be a Galois ring of characteristic p^s and cardinality p^{sm} , where s and m are positive integers and p is an odd prime number. Two kinds of cogredient standard forms of symmetric matrices over R are given, and an explicit formula to count the number of all distinct cogredient classes of symmetric matrices over R is obtained.

1. Introduction and Preliminaries

Let p be a prime number, s and m be positive integers, and $R = \text{GR}(p^s, p^{sm})$ a Galois ring of characteristic p^s and cardinality p^{sm} . Then $\text{GR}(p^s, p^{sm})$ is isomorphic to the ring $\mathbb{Z}_{p^s}[x]/(h(x))$ for any basic irreducible polynomial $h(x)$ of degree m over $\mathbb{Z}_{p^s}$. It is clear that $R = \mathbb{F}_{p^m}$, that is, a finite field of p^m elements, if $s = 1$, and $R = \mathbb{Z}_{p^s}$, that is the ring of residue classes of $\mathbb{Z}$ modulo its ideal $p^s\mathbb{Z}$, if $m = 1$.

We denote by R^* the group of units of R . R is a local ring with the maximal ideal $(p) = pR$, and all ideals of R are given by $(0) = (p^s) \subset (p^{s-1}) \subset \cdots \subset (p) \subset (p^0) = R$. By [1, Theorem 14.8], there exists an element $\xi \in R^*$ of multiplicative order $p^m - 1$, which is a root of a basic primitive polynomial $h(x)$ of degree m over $\mathbb{Z}_{p^s}$ and dividing $x^{p^m-1} - 1$ in $\mathbb{Z}_{p^s}[x]$, and every element $a \in R$ can be written uniquely as

$$a = a_0 + a_1p + \cdots + a_{n-1}p^{n-1}, \quad a_0, a_1, \dots, a_{n-1} \in \mathcal{T}, \quad (1.1)$$

where $\mathcal{T} = \{0, 1, \xi, \dots, \xi^{p^m-2}\}$. Moreover, a is a unit if and only if $a_0 \neq 0$, and a is a zero divisor or 0 if and only if $a_0 = 0$. Define the p -exponent of a by $\tau(0) = s$ and $\tau(a) = i$ if $a = a_ip^i + \cdots + a_{n-1}p^{n-1}$ with $a_i \neq 0$. By [1, Corollary 14.9], $R^* \cong \langle \xi \rangle \times [1 + (p)]$, where $\langle \xi \rangle$ is the cyclic

group of order $p^m - 1$, and $1 + (p) = \{1 + x \mid x \in (p)\}$ is the one group of Galois ring R , so $|R^*| = (p^m - 1)p^{(s-1)m}$.

For a fixed positive integer n , let $M_n(R)$ and $GL_n(R)$ be the set of all $n \times n$ matrices and the multiplicative group of all $n \times n$ invertible matrices over R , and denote by $I^{(n)}$ and $0^{(n)}$ the $n \times n$ identity matrix and zero matrix, respectively. In this paper, for $l \times n$ matrix A and $q \times r$ matrix B over R , we adopt the notation $A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ which is a $(l+q) \times (n+r)$ matrix over R .

For any matrix $A \in M_n(R)$, A is said to be *symmetric* if $A^T = A$, where A^T is the transposed matrix of A . We denote the set of all $n \times n$ symmetric matrices over R by $\mathcal{S}(n, R)$. Then $(\mathcal{S}(n, R), +)$ is a group under the addition of matrices. For any matrices $S_1, S_2 \in M_n(R)$, if there exists matrix $P \in GL_n(R)$ such that $PS_1P^T = S_2$, we say that S_1 is *cogredient* to S_2 over R . It is clear that $S_1 \in \mathcal{S}(n, R)$ if and only if $S_2 \in \mathcal{S}(n, R)$. So cogredience of matrices over R is an equivalent relation on $\mathcal{S}(n, R)$. If $S_1 \in \mathcal{S}(n, R)$, we call $\{PS_1P^T \mid P \in GL_n(R)\}$ the *cogredient classes* of $\mathcal{S}(n, R)$ containing S_1 over R . Let $\mathcal{S}_0 = \{0\}$, $\mathcal{S}_1, \dots, \mathcal{S}_d$ be all distinct cogredient classes of $\mathcal{S}(n, R)$. As in [2] we define relations on $\mathcal{S}(n, R)$ by

$$\Gamma_i := \{(A, B) \mid A, B \in \mathcal{S}(n, R), A - B \in \mathcal{S}_i\}, \quad i = 0, 1, \dots, d. \quad (1.2)$$

Then the system $(\mathcal{S}(n, R), \{\Gamma_i\}_{0 \leq i \leq d})$ is an association scheme of class d on the set $\mathcal{S}(n, R)$ and denoted by $\text{Sym}(n, R)$.

Let p stand for an odd prime number in the following. When $s = 1$, we know that the class number of $\text{Sym}(n, \mathbb{F}_{p^m})$ is given by $d = 2n$ and the association scheme $\text{Sym}(n, \mathbb{F}_{p^m})$ has been investigated in [2]. When $m = 1$, two kinds of cogredient standard forms of symmetric matrices over $\mathbb{Z}_{p^s}$ are given in [3, 4]. If $n \geq 2$, $s > 1$ and $p \equiv 1 \pmod{4}$, a complex formula to count the number of all distinct cogredient classes of $\mathcal{S}(n, \mathbb{Z}_{p^s})$ is given in [3], which shows that, for example,

if m' is odd and s is odd, then

$$\begin{aligned} d+1 &= \left(\frac{m'-1}{2} + 1 \right) + \sum_{s_1 \neq 0, \text{ or } s'_i, \exists i} \left(\frac{m'-1}{2} - s_1 - \frac{s'_2 + s'_3 + s'_4 + s'_5 + \varepsilon}{2} + 1 \right) \\ &\quad \times \left[\binom{s-1}{1} + \binom{s-1}{2} \binom{s_1-1}{1} + \dots + \binom{s-1}{s_1} \right] \\ &\quad \times \binom{\frac{s-1}{2}}{s'_2} \binom{\frac{s+1}{2}}{s'_3} \binom{\frac{s-1}{2}}{s'_4} \binom{\frac{s+1}{2}}{s'_5}, \end{aligned} \quad (1.3)$$

where the meanings of $m', s_1, s'_2, s'_3, s'_4, s'_5, \varepsilon$ and formulas for other cases are referred to [3].

Then two problems arise. (1) Is there a simple and explicit formula to count the number of all distinct cogredient classes of $\mathcal{S}(n, \mathbb{Z}_{p^s})$? (2) For arbitrary Galois ring R , in order to determine precisely the class number d of the association scheme $\text{Sym}(n, R)$, we have to count the number of all distinct cogredient classes of $\mathcal{S}(n, R)$.

In Section 2 we give two kinds of cogredient standard forms for every symmetric matrix over arbitrary Galois ring R of odd characteristic. In Section 3 we obtain an explicit

formula to count the number of all distinct cogredient classes of $\mathcal{S}(n, R)$, which is simpler than that of [3] for the special case $R = \mathbb{Z}_{p^s}$.

Now, we list some properties for the Galois ring R which will be needed in the following sections. For general theory of Galois rings, one can refer to [1].

Lemma 1.1 (see [1, Theorem 14.11]). $R^* = G_1 \times G_2$ where G_1 is a cyclic group of order $p^m - 1$, and $G_2 = 1 + \langle p \rangle$ is a group of order $p^{(s-1)m}$.

Proposition 1.2. (i) R^{*2} is a subgroup of R^* with index $[R^* : R^{*2}] = 2$.

(ii) For any $z \in R^* \setminus R^{*2}$, $R^* \setminus R^{*2} = zR^{*2}$, and $|R^{*2}| = |zR^{*2}| = (1/2)|R^*|$.

(iii) For any $u \in R^*$ and $a \in \langle p \rangle$, there exists $c \in R^*$ such that $c^2(u + a) = u$.

Proof. In the notation of Lemma 1.1. Let ξ be a generator of the cyclic group G_1 . Then ξ is of order $p^m - 1$. Since p is odd and $p^m - 1$ is even, ξ^2 is of order $(1/2)(p^m - 1)$ and $G_1^2 = \langle \xi^2 \rangle$. Since $p^{(s-1)m}$ is odd and G_2 is a commutative group of order $p^{(s-1)m}$ by Lemma 1.1, for every $a \in G_2$, there exists a unique $b \in G_2$ such that $a = b^2$, so $G_2^2 = G_2$. Moreover, by Lemma 1.1 each $u \in R^*$ can be uniquely expressed as $u = gh$ where $g \in G_1$ and $h \in G_2$.

(i) For every $u = gh \in R^*$ where $g \in G_1$ and $h \in G_2$, $u \in R^{*2}$ if and only if there exist $g_1 \in G_1$ and $h_1 \in G_2$ such that $gh = (g_1h_1)^2 = g_1^2h_1^2$, which is then equivalent to that $g = g_1^2$ and $h = h_1^2$. So $u \in R^{*2}$ if and only if $u \in G_1^2 \times G_2$ by Lemma 1.1. Then $R^{*2} = G_1^2 \times G_2$ and so $|R^{*2}| = |G_1^2| \cdot |G_2| = (1/2)(p^m - 1) \cdot p^{(s-1)m} = (1/2)|R^*|$. Hence, $[R^* : R^{*2}] = 2$ by group theory.

(ii) Since $[R^* : R^{*2}] = 2$, for any $z \in R^* \setminus R^{*2}$, we have $R^* = R^{*2} \cup zR^{*2}$ and $R^{*2} \cap zR^{*2} = \emptyset$ by group theory. So $|zR^{*2}| = |R^*| - |R^{*2}| = (1/2)|R^*|$ by the proof of (i).

(iii) Let $u \in R^*$ and $a \in \langle p \rangle$. Then $u^{-1}(u + a) = 1 + u^{-1}a \in 1 + \langle p \rangle = G_2$. From this and by Lemma 1.1, there exists a unique element $b \in G_2 \subseteq R^*$ such that $u^{-1}(u + a) = b^2$. Now, let $c = b^{-1}$. Then $c \in R^*$ satisfying $c^2(u + a) = u$. $\square$

Proposition 1.3. Let $-1 \notin R^{*2}$. Then for any $z \in R^* \setminus R^{*2}$, there exist $x, y \in R^*$ such that $z = (1 + x^2)y^2$.

Proof. Let $u \in R^*$. Suppose that $1 + u^2 \notin R^*$. Then there exists $a \in R$ such that $1 + u^2 = ap$. So $u^2 = -(1 - ap)$. Since p is odd and $p^s = 0$ in R , there exists $b \in R$ such that $(u^{p^s})^2 = -(1 - ap)^{p^s} = -(1 - p^{p^s}b) = -1$. From $u^{p^s} \in R^*$ we deduce $-1 \in R^{*2}$, which is a contradiction. Hence $1 + u^2 \in R^*$. Therefore, $\sigma : w \mapsto 1 + w$ (for all $w \in R^{*2}$) is a mapping from R^{*2} to R^* . Suppose that $\sigma(R^{*2}) \subseteq R^{*2}$. Then for $1 \in R^{*2}$, there exists $w_0 \in R^{*2}$ such that $\sigma(w_0) = 1 + w_0 = 1$, which implies that $w_0 = 0$, and we get a contradiction. So there exists $x \in R^*$ such that $1 + x^2 \notin R^{*2}$, that is, $1 + x^2 \in R^* \setminus R^{*2} = zR^{*2}$ by Proposition 1.2. Then there exists $c \in R^*$ such that $1 + x^2 = zc^2$, so $(1 + x^2)y^2 = z$, where $y = c^{-1} \in R^*$. $\square$

2. Cogredient Standard Forms of Symmetric Matrices

In this section, we give two kinds of cogredient standard forms of symmetric matrices over R corresponding to that of cogredient standard forms of symmetric matrices over finite fields (see [5], or [6], Theorems 1.22 and 1.25).

Notation 1. For any nonnegative integer ν and $z \in R^* \setminus R^{*2}$, define

$$H_{2\nu} = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix}, \quad H_{2\nu+2,\Delta} = H_{2\nu} \oplus \Delta, \quad \text{where } \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}, \quad (2.1)$$

$$H_{2\nu+1,(1)} = H_{2\nu} \oplus (1), \quad H_{2\nu+1,(z)} = H_{2\nu} \oplus (z).$$

Lemma 2.1. For any $\nu \in \mathbb{Z}^+$ and $z \in R^* \setminus R^{*2}$, $zI^{(2\nu)}$ is cogredient to $I^{(2\nu)}$.

Proof. Let $-1 \in R^{*2}$. Then there exists $u \in R^*$ such that $u^2 = -1$, that is, $1 + u^2 = 0$. Since p is an odd prime number, we have $\gcd(2, p^s) = 1$ and so $2 \in R^*$. Let $P = 2^{-1} \begin{pmatrix} (1+z) & u^{-1}(1-z) \\ u(1-z) & (1+z) \end{pmatrix}$. Since R is a commutative ring, we have $\det P = (2^{-1})^2 [(1+z)(1+z) - u^{-1}(1-z)u(1-z)] = (2^{-1})^2 \cdot 2 \cdot 2z = z \in R^*$. Hence, $P \in \text{GL}_2(R)$. Then by $(u^{-1})^2 = (u^2)^{-1} = -1$ and $u(1-z^2) + u^{-1}(1-z^2) = u^{-1}(u^2 + 1)(1-z^2) = 0$, we get

$$PI^{(2)}P^T = (2^{-1})^2 \begin{pmatrix} (1+z) & u^{-1}(1-z) \\ u(1-z) & (1+z) \end{pmatrix} \begin{pmatrix} (1+z) & u(1-z) \\ u^{-1}(1-z) & (1+z) \end{pmatrix} \quad (2.2)$$

$$= (2^{-1})^2 \begin{pmatrix} 2 \cdot 2z & 0 \\ 0 & 2 \cdot 2z \end{pmatrix} = zI^{(2)},$$

so $zI^{(2)}$ is cogredient to $I^{(2)}$.

Let $-1 \notin R^{*2}$. Then by Proposition 1.3 there exist $x, y \in R^*$ such that $(1+x^2)y^2 = z$. Let $Q = \begin{pmatrix} xy & y \\ -y & xy \end{pmatrix}$. Then $\det Q = (1+x^2)y^2 = z \in R^*$ and so $Q \in \text{GL}_2(R)$. By $(1+x^2)y^2 = z$, a matrix computation shows that $QI^{(2)}Q^T = QQ^T = zI^{(2)}$. Hence, $zI^{(2)}$ is cogredient to $I^{(2)}$ as well.

$$\text{Then } zI^{(2\nu)} = \overbrace{zI^{(2)} \oplus \cdots \oplus zI^{(2)}}^{\nu \text{ times}} \text{ is cogredient to } I^{(2\nu)} = \overbrace{I^{(2)} \oplus \cdots \oplus I^{(2)}}^{\nu \text{ times}}. \quad \square$$

Lemma 2.2. Let $z \in R^* \setminus R^{*2}$ and $\nu \in \mathbb{Z}^+$.

- (i) If $-1 \in R^{*2}$, then $I^{(2\nu)}$ is cogredient to $H_{2\nu}$.
- (ii) If $-1 \notin R^{*2}$, then $I^{(\nu)} \oplus zI^{(\nu)}$ is cogredient to $H_{2\nu}$.

Proof. We select $P_1 = 2^{-1} \begin{pmatrix} I^{(\nu)} & -I^{(\nu)} \\ I^{(\nu)} & I^{(\nu)} \end{pmatrix}$ and denote that $M = 2 \begin{pmatrix} I^{(\nu)} & 0 \\ 0 & -I^{(\nu)} \end{pmatrix}$. From $P_1 \begin{pmatrix} I^{(\nu)} & I^{(\nu)} \\ 0 & I^{(\nu)} \end{pmatrix} = \begin{pmatrix} 2^{-1}I^{(\nu)} & 0 \\ 2^{-1}I^{(\nu)} & I^{(\nu)} \end{pmatrix}$ we deduce $\det P_1 = \det(2^{-1}I^{(\nu)}) = (2^{-1})^\nu \in R^*$. Hence $P_1 \in \text{GL}_{2\nu}(R)$. Then by $P_1MP_1^T = 2^{-1} \begin{pmatrix} I^{(\nu)} & -I^{(\nu)} \\ I^{(\nu)} & I^{(\nu)} \end{pmatrix} \begin{pmatrix} I^{(\nu)} & I^{(\nu)} \\ I^{(\nu)} & -I^{(\nu)} \end{pmatrix} = H_{2\nu}$, we see that M is cogredient to $H_{2\nu}$.

- (i) By $-1 \in R^{*2}$ there exists $u \in R^*$ such that $-1 = u^2$. Then M is cogredient to $2I^{(2\nu)}$. If $2 \notin R^{*2}$, $2I^{(2\nu)}$ is cogredient to $I^{(2\nu)}$ by Lemma 2.1. If $2 \in R^{*2}$, there exists $w \in R^*$ such that $2 = w^2$, so $2I^{(2\nu)}$ is cogredient to $I^{(2\nu)}$ as well. Therefore, $I^{(2\nu)}$ is cogredient to $H_{2\nu}$ in this case.
- (ii) Let $-1 \notin R^{*2}$. Then by Proposition 1.2 there exists $c \in R^*$ such that $-1 = zc^2$. Hence $I^{(\nu)} \oplus zI^{(\nu)}$ is cogredient to $\begin{pmatrix} I^{(\nu)} & 0 \\ 0 & -I^{(\nu)} \end{pmatrix}$. If $2 \in R^{*2}$, there exists $w \in R^*$ such that $2 = w^2$, so $\begin{pmatrix} I^{(\nu)} & 0 \\ 0 & -I^{(\nu)} \end{pmatrix}$ is cogredient to M . If $2 \notin R^{*2}$, then $-2 = (-1)2 \in R^{*2}$, and hence there

exists $a \in R^*$ such that $-2 = a^2$, so $(aI^{(2v)})H_{2v}\begin{pmatrix} I^{(v)} & 0 \\ 0 & -I^{(v)} \end{pmatrix}H_{2v}^T(aI^{(2v)}) = M$. Hence, $\begin{pmatrix} I^{(v)} & 0 \\ 0 & -I^{(v)} \end{pmatrix}$ is cogredient to M as well. Therefore, $I^{(v)} \oplus zI^{(v)}$ is cogredient to H_{2v} . $\square$

Lemma 2.3. *Let $z \in R^* \setminus R^{*2}$ and $D = \text{diag}(u_1, \dots, u_r)$, where $u_i \in R^*$, $i = 1, \dots, r$ and $r \in \mathbb{Z}^+$. Then, One has the following.*

- (i) *D is necessarily cogredient to either $I^{(r)}$ or $I^{(r-1)} \oplus (z)$. Moreover, these two matrices are not cogredient over R .*
- (ii) *If $r = 2v + 1$ is odd, then D is necessarily cogredient to either $H_{2v+1,(1)}$ or $H_{2v+1,(z)}$. Moreover, these two matrices are not cogredient. If $r = 2v$ is even, then D is necessarily cogredient to either H_{2v} or $H_{2(v-1)+2,\Delta}$. Moreover, these two matrices are not cogredient.*

Proof. (i) We may assume that $u_1, \dots, u_t \in R^{*2}$ and $u_{t+1}, \dots, u_r \in zR^{*2}$, where $0 \leq t \leq r$. Then D is cogredient to $I^{(t)} \oplus zI^{(r-t)}$. If $r - t$ is even, by Lemma 2.1 $zI^{(r-t)}$ is cogredient to $I^{(r-t)}$ and hence D is cogredient to $I^{(t)} \oplus I^{(r-t)} = I^{(r)}$. Now, let $r - t$ be odd. If $r - t = 1$, D is obviously cogredient to $I^{(1)} \oplus (z)$. If $r - t \geq 3$, by Lemma 2.1 $zI^{(r-t-1)}$ is cogredient to $I^{(r-t-1)}$, and hence D is cogredient to $I^{(t)} \oplus I^{(r-t-1)} \oplus (z) = I^{(r-1)} \oplus (z)$.

Suppose that $I^{(r)}$ is cogredient to $I^{(r-1)} \oplus (z)$ over R . Then there exists $Q \in \text{GL}_r(R)$ such that $QI^{(r)}Q^T = I^{(r-1)} \oplus (z)$. From this and by $\det Q \in R^*$, we obtain that $z = (\det Q)^2 \in R^{*2}$, which is a contradiction. So $I^{(r)}$ and $I^{(r-1)} \oplus (z)$ are not cogredient over R .

(ii) We have one of the following two cases.

- (ii-1) Let $r = 2v + 1$ be an odd number. Then $r - 1 = 2v$ is even and we have one of the following two cases.
 - (ii-1-1) Let $-1 \in R^{*2}$. Then $I^{(2v)}$ is cogredient to H_{2v} by Lemma 2.2(i). From this and by (i) we deduce that D is cogredient to $H_{2v+1,(1)}$ when D is cogredient to $I^{(r)}$, or D is cogredient to $H_{2v+1,(z)}$ when D is cogredient to $I^{(r-1)} \oplus (z)$.
 - (ii-1-2) Let $-1 \in zR^{*2}$. Then we have one of the following two cases.
 - (α) Let $(1/2)(r - 1) = v$ be even. Then $I^{(v)}$ is cogredient to $zI^{(v)}$ by Lemma 2.1, so $I^{(2v)}$ is cogredient to $I^{(v)} \oplus zI^{(v)}$. Since $I^{(v)} \oplus zI^{(v)}$ is cogredient to H_{2v} by Lemma 2.2(ii), by (i) we see that: D is cogredient to $H_{2v+1,(1)}$ when D is cogredient to $I^{(r)}$, or D is cogredient to $H_{2v+1,(z)}$ when D is cogredient to $I^{(r-1)} \oplus (z)$.
 - (β) Let $(1/2)(r - 1) = v$ be odd. Then $v = 2\omega + 1$ for some nonnegative integer ω and so $r - 1 = 4\omega + 2$. By Lemma 2.1 we see that $I^{(2\omega)}$ is cogredient to $zI^{(2\omega)}$, and $I^{(2)}$ is cogredient to $zI^{(2)}$. Hence $I^{(r)} = I^{(2\omega)} \oplus I^{(2\omega)} \oplus I^{(2)} \oplus (1)$ is cogredient to $I^{(2\omega)} \oplus zI^{(2\omega)} \oplus zI^{(2)} \oplus (1)$, which is then cogredient to $I^{(2\omega+1)} \oplus zI^{(2\omega+1)} \oplus (z)$. Since $I^{(2\omega+1)} \oplus zI^{(2\omega+1)}$ is cogredient to $H_{2(2\omega+1)} = H_{2v}$ by Lemma 2.2(ii), $I^{(r)}$ is cogredient to $H_{2v+1,(z)}$. Moreover, $I^{(r-1)} \oplus (z) = I^{(2\omega)} \oplus I^{(2\omega)} \oplus I^{(2)} \oplus (z)$ is cogredient to $I^{(2\omega)} \oplus zI^{(2\omega)} \oplus I^{(2)} \oplus (z)$, which is then cogredient to $I^{(2\omega+1)} \oplus zI^{(2\omega+1)} \oplus (1)$. Since $I^{(v)} \oplus zI^{(v)}$ is cogredient to H_{2v} by Lemma 2.2(ii), $I^{(r-1)} \oplus (z)$ is cogredient to $H_{2v+1,(1)}$. Therefore, D is necessarily cogredient to either $H_{2v+1,(1)}$ or $H_{2v+1,(z)}$ by (i).
- (ii-2) Let $r = 2v$ be an even number. Then $r - 2 = 2(v - 1)$ is also even and we have one of the following two cases.
 - (ii-2-1) Let $-1 \in R^{*2}$. Then $-1 = u^2$ for some $u \in R^*$ and so $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ is cogredient to $\begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} = \Delta$. By Lemma 2.2(i) D is cogredient to H_{2v} when D is cogredient to $I^{(r)}$, or D is cogredient to $H_{2(v-1)+2,\Delta}$ when D is cogredient to $I^{(r-1)} \oplus (z) = I^{(2(v-1))} \oplus \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$.

(ii-2-2) Let $-1 \in zR^{*2}$. Then $-1 = zc^2$ for some $c \in R^*$. By $1 = (-z)c^2$, we see that $I^{(2)}$ is cogredient to Δ . Now, we have one of the following two cases.

- (α) Let ν be even. Then $I^{(\nu)}$ is cogredient to $zI^{(\nu)}$ by Lemma 2.1 and so $I^{(r)} = I^{(\nu)} \oplus I^{(\nu)}$ is cogredient to $I^{(\nu)} \oplus zI^{(\nu)}$. From this and by Lemma 2.2(ii), we see that $I^{(r)}$ is cogredient to $H_{2\nu}$. Let $\nu = 2$. Since $I^{(2)}$ is cogredient to Δ and $I^{(1)} \oplus (z)$ is cogredient to H_2 by Lemma 2.2(ii), $I^{(3)} \oplus (z) = I^{(2)} \oplus I^{(1)} \oplus (z)$ is cogredient to $H_2 \oplus \Delta = H_{2+2,\Delta}$. Now, let $\nu \geq 4$. Since $\nu - 2$ is even, $I^{(\nu-2)}$ is cogredient to $zI^{(\nu-2)}$ by Lemma 2.1, so $I^{(\nu-2)} \oplus I^{(\nu-2)}$ is cogredient to $I^{(\nu-2)} \oplus zI^{(\nu-2)}$. Hence, $I^{(r-1)} \oplus (z) = I^{(\nu-2)} \oplus I^{(\nu-2)} \oplus I^{(3)} \oplus (z)$ is cogredient to $I^{(\nu-2)} \oplus zI^{(\nu-2)} \oplus I^{(3)} \oplus (z)$, which is then cogredient to $I^{(\nu-1)} \oplus zI^{(\nu-1)} \oplus I^{(2)}$. Since $I^{(2)}$ is cogredient to Δ , we see that $I^{(r-1)} \oplus (z)$ is cogredient to $H_{2(\nu-1)+2,\Delta}$ by Lemma 2.2(ii). Therefore, D is necessarily cogredient to either $H_{2\nu}$ or $H_{2(\nu-1)+2,\Delta}$ by (i).
- (β) Let ν be odd. Then there exists nonnegative integer ω such that $\nu = 2\omega + 1$ and so $r = 4\omega + 2$. Since $I^{(2\omega)}$ is cogredient to $zI^{(2\omega)}$ by Lemma 2.1, $I^{(r)} = I^{(2\omega)} \oplus I^{(2\omega)} \oplus I^{(2)}$ is cogredient to $I^{(2\omega)} \oplus zI^{(2\omega)} \oplus \Delta$, that is then cogredient to $H_{2(2\omega)+2,\Delta} = H_{2(\nu-1)+2,\Delta}$ by Lemma 2.2(ii). Now, $I^{(r-1)} \oplus (z) = I^{(2\omega)} \oplus I^{(2\omega)} \oplus (1) \oplus (z)$ is cogredient to $I^{(2\omega)} \oplus zI^{(2\omega)} \oplus (1) \oplus (z)$ by Lemma 2.1, which is then cogredient to $I^{(2\omega+1)} \oplus zI^{(2\omega+1)}$. Hence $I^{(r-1)} \oplus (z)$ is cogredient to $H_{2(2\omega+1)} = H_{2\nu}$ by Lemma 2.2(ii). Therefore, D is necessarily cogredient to either $H_{2\nu}$ or $H_{2(\nu-1)+2,\Delta}$ by (i). $\square$

Theorem 2.4. Let $z \in R^* \setminus R^{*2}$. Then every $n \times n$ symmetric matrix A over R is necessarily cogredient to one of the following matrices:

$$D_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)} := \text{diag}(p^{r_1}D_1, p^{r_2}D_2, \dots, p^{r_t}D_t, 0^{(n-k)}), \quad (2.3)$$

where $0 \leq t \leq k \leq n$, $D_i = I^{(k_i)}$ or $I^{(k_i-1)} \oplus (z)$ for all $i = 1, \dots, t$, $0 \leq r_1 < r_2 < \dots < r_t \leq s-1$, and $k_i \in \mathbb{Z}^+$ satisfy $\sum_{i=1}^t k_i = k$.

Proof. The statement holds obviously if $A = 0$ (corresponding to the case $k = 0$) or $n = 1$. Now, let $n \geq 2$ and $A = (a_{ij})_{n \times n} \neq 0$. Then, there exist $1 \leq i_0, j_0 \leq n$ such that $a_{i_0 j_0} \neq 0$ and $\tau(a_{i_0 j_0}) = \min\{\tau(a_{ij}) \mid a_{ij} \neq 0, 1 \leq i, j \leq n\}$. Let $s_1 = \nu(a_{i_0 j_0})$. Then $0 \leq s_1 \leq s-1$, and there exists $P_1 \in \text{GL}_n(R)$ such that $P_1 A P_1^T = \text{diag}(u_1 p^{s_1}, B)$ where $u_1 \in R^*$ and $B = (b_{ij})$ is a $(n-1) \times (n-1)$ symmetric matrix over R satisfying $B = 0$ or $\tau(b_{ij}) \geq s_1$ for all $b_{ij} \neq 0, 1 \leq i, j \leq n-1$. By induction there exists $X \in \text{GL}_{n-1}(R)$ such that $X B X^T = \text{diag}(u_2 p^{s_2}, \dots, u_k p^{s_k}, 0^{(n-k)})$, where $u_2, \dots, u_k \in R^*$ and $s_2 \leq \dots \leq s_k \leq s-1$. Then $P = \text{diag}(1, X) P_1 \in \text{GL}_n(R)$ satisfies $P A P^T = \text{diag}(u_1 p^{s_1}, \dots, u_k p^{s_k}, 0^{(n-k)})$.

Now, there must exist $t, k_i \in \mathbb{Z}^+$, $i = 1, \dots, t$ and $0 \leq r_1 < \dots < r_t \leq s-1$ such that $s_1 = \dots = s_{k_1} = r_1 < s_{k_1+1} = \dots = s_{k_1+k_2} = r_2 < \dots < s_{k_1+k_2+\dots+k_{t-1}+1} = \dots = s_{k_1+k_2+\dots+k_{t-1}+k_t} = r_t$. Then $\sum_{i=1}^t k_i = k$ and A is cogredient to $M = \text{diag}(p^{r_1} M_1, p^{r_2} M_2, \dots, p^{r_t} M_t, 0^{(n-k)})$, where $M_i = \text{diag}(u_{k_1+\dots+k_{i-1}+1}, \dots, u_{k_1+\dots+k_{i-1}+k_i})$ is a $k_i \times k_i$ matrix over R for all $i = 1, \dots, t$. Since M_i is cogredient to D_i for every $1 \leq i \leq t$ by Lemma 2.3(i), we deduce that A is cogredient to $\text{diag}(p^{r_1} D_1, p^{r_2} D_2, \dots, p^{r_t} D_t, 0^{(n-k)})$. $\square$

Theorem 2.5. Let $z \in R^* \setminus R^{*2}$. Then every $n \times n$ symmetric matrix A over R is necessarily cogredient to one of the following matrices:

$$H_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)} := \text{diag}(p^{r_1} H_1, p^{r_2} H_2, \dots, p^{r_t} H_t, 0^{(n-k)}), \quad (2.4)$$

where H_i is a $k_i \times k_i$ matrix over R such that H_i is equal to either $H_{2v_i+1,(1)}$ or $H_{2v_i+1,(z)}$ when $k_i = 2v_i + 1$ is odd, and H_i is equal to either H_{2v_i} or $H_{2(v_i-1)+2,\Delta}$ when $k_i = 2v_i$ is even, for all $i = 1, \dots, t$; $0 \leq t \leq k \leq n$, $0 \leq r_1 < r_2 < \dots < r_t \leq s-1$, and $k_i \in \mathbb{Z}^+$ satisfy $\sum_{i=1}^t k_i = k$.

Proof. It follows from Theorem 2.4 and the proof of Lemma 2.3(ii).

For any $n \times n$ symmetric matrix A , we call $D_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)}$ the *cogredient standard form of kind (I)* of A if A is cogredient to $D_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)}$, and call $H_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)}$ the *cogredient standard form of kind (II)* of A if A is cogredient to $H_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)}$. $\square$

3. The Number of Cogredient Classes of Symmetric Matrices

In order to count the number of all distinct cogredient classes of $n \times n$ symmetric matrices over R , we show that every $n \times n$ symmetric matrix over R has only one cogredient standard form of kind (I) first, then the number of all distinct cogredient classes of $n \times n$ symmetric matrices over R is equal to the number of all cogredient standard forms of kind (I) by Theorem 2.4.

Theorem 3.1. *The number $C_{s,n}$ of all distinct cogredient classes of $n \times n$ symmetric matrices over R is given by the following:*

$$(i) \text{ If } n \leq s, \text{ then } C_{s,n} = 1 + \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1};$$

$$(ii) \text{ If } n \geq s+1, \text{ then } C_{s,n} = 1 + \sum_{j=0}^{s-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1}.$$

Proof. Let $\widehat{D} := \text{diag}(p^{\widehat{r}_1} \widehat{D}_1, p^{\widehat{r}_2} \widehat{D}_2, \dots, p^{\widehat{r}_t} \widehat{D}_t, 0^{(n-\widehat{k})})$, where $\widehat{D}_i = I^{(\widehat{k}_i)}$ or $I^{(\widehat{k}_i-1)} \oplus (z)$ for all $i = 1, \dots, t$, $0 \leq \widehat{t} \leq \widehat{k} \leq n$, $0 \leq \widehat{r}_1 < \widehat{r}_2 < \dots < \widehat{r}_t \leq s-1$, and $\widehat{k}_i \in \mathbb{Z}^+$ satisfy $\sum_{i=1}^{\widehat{t}} \widehat{k}_i = \widehat{k}$. In the notation of Theorem 2.4, by [7, Theorem D], it follows that $D = \widehat{D}$ if $D := D_{(n,k,t;k_1,\dots,k_t;r_1,\dots,r_t)}$ is cogredient to $\widehat{D}$ over R . Hence, every $n \times n$ symmetric matrix over R has only one cogredient standard form of kind (I).

For any $1 \leq t \leq k \leq n$, denote that $S_1 = \{(k_1, \dots, k_t) \mid k_i \in \mathbb{Z}^+, \sum_{i=1}^t k_i = k\}$ and $S_2 = \{(r_1, \dots, r_t) \mid r_i \in \mathbb{Z}, 0 \leq r_1 < r_2 < \dots < r_t \leq s-1\}$. Then $|S_1| = \binom{k-1}{t-1}$, $|S_2| = \binom{s}{t}$ if $t \leq s$ and, $|S_2| = 0$ if $t \geq s$. From this and by Theorem 2.4 it follows that $C_{s,n} = 1 + \sum_{k=1}^n (\sum_{t=1}^k |S_1| \cdot |S_2| \cdot 2^t)$. Therefore, $C_{s,n} = 1 + \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1}$ if $n \leq s$ and, $C_{s,n} = 1 + \sum_{j=0}^{s-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1}$ if $n \geq s+1$.

In the notations of Section 1, we see that the class number d of the association scheme $\text{Sym}(n, R)$ is determined by $d+1 = C_{s,n}$. Then by Theorem 3.1, we have the following corollary. $\square$

Corollary 3.2. *The class number of the association scheme $\text{Sym}(n, R)$ is given by the following.*

$$(i) \text{ If } n \leq s, \text{ then } d = \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1};$$

$$(ii) \text{ If } n \geq s+1, \text{ then } d = \sum_{j=0}^{s-1} \sum_{i=j}^{n-1} \binom{i}{j} \binom{s}{j+1} 2^{j+1}.$$

Example 3.3. Let p be an odd prime number and $s = 2$. Then by Theorem 3.1 the number $C_{2,2}$ of all cogredient classes of 2×2 symmetric matrices over Galois ring $\text{GR}(p^2, p^{2m})$ is given by $C_{2,2} = 1 + \sum_{j=0}^1 \sum_{i=j}^1 \binom{i}{j} \binom{2}{j+1} 2^{j+1} = 13$. In fact, for a fixed element $z \in R^* \setminus R^{*2}$, all cogredient

standard forms of kind (I) of 2×2 symmetric matrices over $\text{GR}(p^2, p^{2m})$ are given by the following:

$$\begin{aligned} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} zp & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \\ & \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & zp \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & zp \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & zp \end{pmatrix}. \end{aligned} \quad (3.1)$$

The number $\mathcal{C}_{2,3}$ of all cogredient classes of 3×3 symmetric matrices over $\text{GR}(p^2, p^{2m})$ is given by $\mathcal{C}_{2,3} = 1 + \sum_{j=0}^1 \sum_{i=j}^2 \binom{i}{j} \binom{2}{j+1} 2^{j+1} = 25$. In fact, all cogredient standard forms of kind (I) of 3×3 symmetric matrices over $\text{GR}(p^2, p^{2m})$ are given by the following: $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$ where J is one of matrices in (3.1), and

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & zp \end{pmatrix}, \\ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & zp \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & zp \end{pmatrix}, \\ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & zp \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & zp \end{pmatrix}. \end{aligned} \quad (3.2)$$

Example 3.4. Let p be an odd prime number and $s = 5$. Then by Theorem 3.1 the number $\mathcal{C}_{5,4}$ of all cogredient classes of 4×4 symmetric matrices over Galois ring $\text{GR}(p^5, p^{5m})$ is given by $\mathcal{C}_{5,4} = 1 + \sum_{j=0}^3 \sum_{i=j}^3 \binom{i}{j} \binom{5}{j+1} 2^{j+1} = 681$; the number $\mathcal{C}_{5,7}$ of all cogredient classes of 7×7 symmetric matrices over $\text{GR}(p^5, p^{5m})$ is given by $\mathcal{C}_{5,7} = 1 + \sum_{j=0}^4 \sum_{i=j}^6 \binom{i}{j} \binom{5}{j+1} 2^{j+1} = 6943$.

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