Research Article

The Rate of Strong Consistency of the Nearest Neighbor Density Estimator for Negatively Dependent Random Variables

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Let $\{X_n : n \geq 1\}$ be a sequence of negatively dependent random variables. Based on $(X_1, \dots, X_n)$, in this paper we investigate the rate of pointwise consistency and strong consistency of the nonparametric density estimator proposed by Yu. We extend the correspondent results under the negatively dependent samples.

1. Introduction and Lemmas

In Yu [1] a very simple and useful nonparametric estimate of a density based on a sample on $\mathbb{R}$ has been defined. If m_n is an integer, the nonparametric estimate $f_n(x)$ of $f(x)$ is defined by dividing $2m_n$ by n times the length of the smallest interval containing x which consists of $2m_n$ of the n observations in which half of them lie on the left side of x and half on the right side. Formally,

$$f_n(x) = \begin{cases} \frac{2m_n}{n(X_{(2m_n+j)} - X_{(j+1)})} & \text{if } x \in [X_{(m_n+j)}, X_{(m_n+j+1)}], \text{ for } j = 0, 1, \dots, n - 2m_n, \\ 0 & \text{if } x < X_{(m_n)} \text{ or } x \geq X_{(n-m_n+1)}, \end{cases} \quad (1.1)$$

where $(X_{(1)}, \dots, X_{(n)})$ is the order statistic of $(X_1, \dots, X_n)$. In Yu [1] It is showed that in the case of independent and identically distributed observations $f_n(x)$ converges to $f(x)$ in probability for almost all x and converges to $f(x)$ with probability one for almost all x . Moreover, the strong uniform consistency of this estimator for f uniformly continuous was also established.

Since Bozorgnia et al. [2] studied sequences of negatively dependent random variables, many statisticians have investigated this topic with interest. A series of useful results have been established. One can refer to Amini and Bozorgnia [3] for complete convergence for negatively dependent random variables, Fakoor and Azarnoosh [4] for probability inequalities for sums of negatively dependent random variables, Liu and Zhang [5] for the consistency and asymptotic normality of nearest neighbor density estimator under β -mixing condition, Wu [6] for complete convergence for weighted sums of sequences of negatively dependent random variables, and Wu and Jiang [7] for the strong consistency of estimator in linear model for negatively dependent random samples. From (1.1), we investigate the rate of consistency of the negatively dependent samples.

The following set of definitions and Lemmas will be needed.

Definition 1.1. Random variables X and Y are said to be negatively dependent (ND) if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y), \quad (1.2)$$

for all $x, y \in \mathbb{R}$. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.2).

It is important to note that (1.2) implies

$$P(X > x, Y > y) \leq P(X > x)P(Y > y), \quad (1.3)$$

for all $x, y \in \mathbb{R}$. Moreover, it follows that (1.3) implies (1.2), and hence, (1.2) and (1.3) are equivalent. Ebrahimi and Ghosh [8] showed that (1.2) and (1.3) are not equivalent for a collection of three or more random variables. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 1.2 (see [2]). The random variables $X_1, \dots, X_n$ are said to be negatively dependent (ND) if for all real $x_1, \dots, x_n$,

$$\begin{aligned} P\left(\bigcap_{j=1}^n (X_j \leq x_j)\right) &\leq \prod_{j=1}^n P(X_j \leq x_j), \\ P\left(\bigcap_{j=1}^n (X_j > x_j)\right) &\leq \prod_{j=1}^n P(X_j > x_j). \end{aligned} \quad (1.4)$$

An infinite random sequence $\{X_n : n \geq 1\}$ is said to be ND if every finite subset $X_1, \dots, X_n$ is ND.

Lemma 1.3 (see [2]). Let $X_1, \dots, X_n$ be ND random variables and let $f_1(x), \dots, f_n(x)$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing), then $f_1(X_1), \dots, f_n(X_n)$ are still ND random variables.

Lemma 1.4 (see [7]). Let $\{X_n : n \geq 1\}$ be a sequence of negatively dependent sequence with $\mathbb{E}X_n = 0$, $|X_n| \leq b_n$, a.s. ($n = 1, 2, \dots$), $t > 0$, and $t \cdot \max_{1 \leq i \leq n} b_i \leq 1$, then

$$P\left(\left|\sum_{i=1}^n X_i\right| > \varepsilon\right) \leq 2 \exp\left\{-t\varepsilon + t^2 \sum_{i=1}^n \mathbb{E}X_i^2\right\}, \quad \text{for any } \varepsilon > 0. \quad (1.5)$$

In particular, writing $\sigma_n^2 = n^{-1} \sum_{i=1}^n \text{var}(X_i)$, $t = \varepsilon / (2\sigma_n^2 + b_n\varepsilon)$, we get

$$P\left(\frac{1}{n} \left|\sum_{i=1}^n X_i\right| > \varepsilon\right) \leq 2 \exp\left\{-\frac{n\varepsilon^2}{2(2\sigma_n^2 + b_n\varepsilon)}\right\}, \quad \text{for any } \varepsilon > 0. \quad (1.6)$$

2. Main Results

Let $\{X_n : n \geq 1\}$ be a sequence of negatively dependent random variables with common distribution function $F(x)$ and density $f(x)$ and let $(X_{(1)}, \dots, X_{(n)})$ be the order statistic of $(X_1, \dots, X_n)$.

In the case of independent observations, in order to estimate $f(x)$ from a sample $(X_1, \dots, X_n)$ Yu [1] considered (1.1) and showed both the pointwise consistency and uniform consistency for $f_n(x)$. However, in many situations we are dealing with dependent variables, so we will extend these results for the rate of consistency of nearest neighbor density estimator.

Our theorems are formulated in a more general setting.

Theorem 2.1. Let $\{X_n : n \geq 1\}$ be a negatively dependent sequence with common distribution function $F(x)$, and $0 < F(x) < 1$. There exists a sequence of positive numbers $\{\tau_n : n \geq 1\}$, $\{m_n : n \geq 1\}$ which verifies jointly with $\{m_n : n \geq 1\}$ that

$$m_n \longrightarrow \infty, \quad \tau_n \longrightarrow 0, \quad \frac{m_n}{n\tau_n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (2.1)$$

If f satisfies the local Lipschitz condition at x , and

$$\frac{\tau_n^2 m_n^2}{n} \longrightarrow \infty, \quad (2.2)$$

then

$$\lim_{n \rightarrow \infty} P(|f_n(x) - f(x)| > \varepsilon \tau_n) = 0. \quad (2.3)$$

Theorem 2.2. Under the conditions of Theorem 2.1, If (2.2) instead becomes

$$\frac{\tau_n^2 m_n^2}{n \ln n} \longrightarrow \infty, \quad (2.4)$$

then

$$|f_n(x) - f(x)| = o(\tau_n), \quad \text{a.s.} \quad (2.5)$$

Proof of Theorem 2.1. For any $\varepsilon > 0$, we have

$$\begin{aligned} P(|f_n(x) - f(x)| > \varepsilon \tau_n) &\leq P(x < X_{(m_n)}) + P(x \geq X_{(n-m_n+1)}) \\ &\quad + P(X_{(m_n)} \leq x < X_{(n-m_n+1)}, |f_n(x) - f(x)| > \varepsilon \tau_n) \\ &\triangleq J_{n1} + J_{n2} + J_{n3}. \end{aligned} \quad (2.6)$$

Let's begin with proving $\sum_{n=1}^{\infty} J_{n3} < \infty$. Denote $A_n = \{X_{(m_n)} \leq x < X_{(n-m_n+1)}\}$. When A_n is true, (1.1) implies that there exists $j(x)$, $0 \leq j(x) \leq n - m_n$, such that $x \in [X_{(m_n+j(x))}, X_{(m_n+j(x)+1)})$. Denote $X_1^{(x)} = X_{(j(x)+1)}$ and $X_2^{(x)} = X_{(2m_n+j(x))}$. Then, we get

$$\begin{aligned} J_{n3} &\leq P(A_n, f_n(x) < f(x) - \varepsilon \tau_n) + P(A_n, f_n(x) > f(x) + \varepsilon \tau_n) \\ &\leq P\left(A_n, f_n(x) < f(x) - \frac{\varepsilon \tau_n}{2}\right) + P(A_n, f_n(x) > f(x) + \varepsilon \tau_n) \\ &\triangleq P_{n1} + P_{n2}. \end{aligned} \quad (2.7)$$

If $f(x) \leq \varepsilon \tau_n$, then $(A_n, f_n(x) < f(x) - \varepsilon \tau_n)$ is the impossible event, namely, $P(A_n, f_n(x) < f(x) - \varepsilon \tau_n) = 0$. Herein below, we suppose that $f(x) > \varepsilon \tau_n$ for P_{n1} . Using the previous denotations, we obtain that

$$\begin{aligned} P_{n1} &= P\left(X_2^{(x)} - X_1^{(x)} > \frac{2m_n}{n(f(x) - \varepsilon \tau_n/2)}, A_n\right) \\ &\leq P\left(X_2^{(x)} - x > \frac{m_n}{n(f(x) - \varepsilon \tau_n/2)}, A_n\right) + P\left(x - X_1^{(x)} > \frac{m_n}{n(f(x) - \varepsilon \tau_n/2)}, A_n\right) \\ &\triangleq P'_{n1} + P''_{n1}, \\ P_{n2} &= P\left(X_2^{(x)} - X_1^{(x)} < \frac{2m_n}{n(f(x) + \varepsilon \tau_n)}, A_n\right) \\ &\leq P\left(X_2^{(x)} - x < \frac{m_n}{n(f(x) + \varepsilon \tau_n)}, A_n\right) + P\left(x - X_1^{(x)} < \frac{m_n}{n(f(x) + \varepsilon \tau_n)}, A_n\right) \\ &\triangleq P'_{n2} + P''_{n2}. \end{aligned} \quad (2.8)$$

Write

$$\begin{aligned} r_n &= \frac{m_n}{n(f(x) - \varepsilon \tau_n/2)}, \\ r'_n &= \frac{m_n}{n(f(x) + \varepsilon \tau_n)}. \end{aligned} \quad (2.9)$$

Since f satisfies the local Lipschitz condition of order one at x , there exist constants $M > 0$ and $\delta > 0$, such that $|f(t) - f(x)| \leq M|t - x|$ whenever $|t - x| < \delta$, that is,

$$f(x) - M|t - x| \leq f(t) \leq f(x) + M|t - x|. \quad (2.10)$$

From (2.1), $m_n/n\tau_n \rightarrow 0$ and $\tau_n \rightarrow 0$. Thus, when n is sufficiently large, we have

$$\begin{aligned} r_n &= \frac{m_n}{n(f(x) - \varepsilon\tau_n/2)} \leq \frac{m_n}{n(f(x) - f(x)/2)} = \frac{m_n}{(f(x)/2)n\tau_n} \tau_n < \frac{\varepsilon\tau_n}{4M} < \delta, \\ r'_n &= \frac{m_n}{n(f(x) + \varepsilon\tau_n)} \leq \frac{m_n}{nf(x)} = \frac{m_n}{n\tau_n} \frac{\tau_n}{f(x)} < \frac{\varepsilon\tau_n}{2M} < \delta. \end{aligned} \quad (2.11)$$

By (2.10), (2.11), we have

$$\begin{aligned} \mathbb{E}I_{(x \leq X_i < x+r_n)} &= \int_x^{x+r_n} f(t)dt \geq (f(x) - Mr_n)r_n = \frac{f(x) - Mr_n}{f(x) - \varepsilon\tau_n/2} \cdot \frac{m_n}{n} \\ &\geq \frac{f(x) - \varepsilon\tau_n/4}{f(x) - \varepsilon\tau_n/2} \cdot \frac{m_n}{n} = \frac{4f(x) - \varepsilon\tau_n}{4f(x) - 2\varepsilon\tau_n} \cdot \frac{m_n}{n}, \end{aligned} \quad (2.12)$$

$$\begin{aligned} \mathbb{E}I_{(x \leq X_i < x+r'_n)} &= \int_x^{x+r'_n} f(t)dt \leq (f(x) + Mr'_n)r'_n = \frac{f(x) + Mr'_n}{f(x) + \varepsilon\tau_n} \cdot \frac{m_n}{n} \\ &\leq \frac{f(x) + \varepsilon\tau_n/2}{f(x) + \varepsilon\tau_n} \cdot \frac{m_n}{n} = \frac{2f(x) + \varepsilon\tau_n}{2f(x) + 2\varepsilon\tau_n} \cdot \frac{m_n}{n}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} \mathbb{E}I_{(x-r_n \leq X_i < x)} &= \int_{x-r_n}^x f(t)dt \geq (f(x) - Mr_n)r_n \\ &\geq \frac{4f(x) - \varepsilon\tau_n}{4f(x) - 2\varepsilon\tau_n} \cdot \frac{m_n}{n}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathbb{E}I_{(x-r'_n \leq X_i < x)} &= \int_{x-r'_n}^x f(t)dt \leq (f(x) + Mr'_n)r'_n \\ &\leq \frac{2f(x) + \varepsilon\tau_n}{2f(x) + 2\varepsilon\tau_n} \cdot \frac{m_n}{n}. \end{aligned} \quad (2.15)$$

So when n is sufficiently large, we have by (2.12) and (2.13)

$$\begin{aligned} P'_{n1} &= P\left(X_2^{(x)} - x > \frac{m_n}{n(f(x) - \varepsilon\tau_n/2)}, A_n\right) \\ &\leq P\left(X_2^{(x)} - x > r_n\right) \\ &= P\left(\sum_{i=1}^n I_{(x \leq X_i < x+r_n)} \leq m_n\right) \end{aligned}$$

$$\begin{aligned}
&\leq P\left(\sum_{i=1}^n (I_{(x \leq X_i < x+r_n)} - \mathbb{E}I_{(x \leq X_i < x+r_n)}) \leq m_n - n \frac{4f(x) - \varepsilon\tau_n}{4f(x) - 2\varepsilon\tau_n} \cdot \frac{m_n}{n}\right) \\
&= P\left(\sum_{i=1}^n \xi_i \leq m_n \frac{-\varepsilon\tau_n}{4f(x) - 2\varepsilon\tau_n} < 0\right) \\
&\leq P\left(\left|\sum_{i=1}^n \xi_i\right| \geq m_n \frac{\varepsilon\tau_n}{4f(x) - 2\varepsilon\tau_n}\right) \\
&\leq P\left(\left|\sum_{i=1}^n \xi_i\right| \geq m_n \frac{\varepsilon\tau_n}{4f(x)}\right),
\end{aligned} \tag{2.16}$$

where $\xi_i = I_{(x \leq X_i < x+r_n)} - \mathbb{E}I_{(x \leq X_i < x+r_n)}$.
Consider

$$\begin{aligned}
P'_{n2} &= P\left(X_2^{(x)} - x < \frac{m_n}{n(f(x) + \varepsilon\tau_n)}, A_n\right) \\
&\leq P\left(X_2^{(x)} - x < r'_n\right) \\
&= P\left(\sum_{i=1}^n I_{(x \leq X_i < x+r'_n)} \geq m_n\right) \\
&\leq P\left(\sum_{i=1}^n (I_{(x \leq X_i < x+r'_n)} - \mathbb{E}I_{(x \leq X_i < x+r'_n)}) \geq m_n - n \frac{2f(x) + \varepsilon\tau_n}{2f(x) + 2\varepsilon\tau_n} \cdot \frac{m_n}{n}\right) \\
&= P\left(\sum_{i=1}^n \xi'_i \geq m_n \frac{\varepsilon\tau_n}{2f(x) + 2\varepsilon\tau_n}\right) \\
&\leq P\left(\sum_{i=1}^n \xi'_i \geq m_n \frac{\varepsilon\tau_n}{4f(x)}\right) \\
&\leq P\left(\left|\sum_{i=1}^n \xi'_i\right| \geq m_n \frac{\varepsilon\tau_n}{4f(x)}\right),
\end{aligned} \tag{2.17}$$

where $\xi'_i = I_{(x \leq X_i < x+r'_n)} - \mathbb{E}I_{(x \leq X_i < x+r'_n)}$. Likewise, when n is sufficiently large, we have by (2.14) and (2.15)

$$P''_{n1} \leq P\left(\left|\sum_{i=1}^n \eta_i\right| \geq m_n \frac{\varepsilon\tau_n}{4f(x)}\right), \tag{2.18}$$

$$P''_{n2} \leq P\left(\left|\sum_{i=1}^n \eta'_i\right| \geq m_n \frac{\varepsilon\tau_n}{4f(x)}\right), \tag{2.19}$$

where $\eta_i = I_{(x-r_n \leq X_i < x)} - \mathbb{E}I_{(x-r_n \leq X_i < x)}$, $\eta'_i = I_{(x-r'_n \leq X_i < x)} - \mathbb{E}I_{(x-r'_n \leq X_i < x)}$.

From (2.16) and (2.18), we get

$$P_{n1} \leq P\left(\left|\sum_{i=1}^n \xi_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) + P\left(\left|\sum_{i=1}^n \eta_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right). \quad (2.20)$$

From (2.17) and (2.19), we get

$$P_{n2} \leq P\left(\left|\sum_{i=1}^n \xi'_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) + P\left(\left|\sum_{i=1}^n \eta'_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right). \quad (2.21)$$

Write $\xi_i = I_{(-\infty < X_i < x+r_n)} - \mathbb{E}I_{(-\infty < X_i < x+r_n)}$, $\xi'_i = I_{(-\infty < X_i < x+r'_n)} - \mathbb{E}I_{(-\infty < X_i < x+r'_n)}$, $\beta_i = I_{(-\infty < X_i < x)} - \mathbb{E}I_{(-\infty < X_i < x)}$.

Then

$$\begin{aligned} P\left(\left|\sum_{i=1}^n \xi_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) &= P\left(\left|\sum_{i=1}^n \xi_i - \sum_{i=1}^n \beta_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) \\ &\leq P\left(\left|\sum_{i=1}^n \xi_i\right| \geq m_n \frac{\varepsilon \tau_n}{8f(x)}\right) + P\left(\left|\sum_{i=1}^n \beta_i\right| \geq m_n \frac{\varepsilon \tau_n}{8f(x)}\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n \xi_i\right| \geq \frac{am_n \tau_n}{n}\right) + P\left(\left|\frac{1}{n} \sum_{i=1}^n \beta_i\right| \geq \frac{am_n \tau_n}{n}\right), \end{aligned} \quad (2.22)$$

$$\begin{aligned} P\left(\left|\sum_{i=1}^n \xi'_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) &= P\left(\left|\sum_{i=1}^n \xi'_i - \sum_{i=1}^n \beta_i\right| \geq m_n \frac{\varepsilon \tau_n}{4f(x)}\right) \\ &\leq P\left(\left|\sum_{i=1}^n \xi'_i\right| \geq m_n \frac{\varepsilon \tau_n}{8f(x)}\right) + P\left(\left|\sum_{i=1}^n \beta_i\right| \geq m_n \frac{\varepsilon \tau_n}{8f(x)}\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n \xi'_i\right| \geq \frac{am_n \tau_n}{n}\right) + P\left(\left|\frac{1}{n} \sum_{i=1}^n \beta_i\right| \geq \frac{am_n \tau_n}{n}\right), \end{aligned} \quad (2.23)$$

where $a = \varepsilon/8f(x)$.

Obviously, $\{\xi_i\}$, $\{\xi'_i\}$, and $\{\beta_i\}$ are monotonic function's. According to Lemma 1.3, $\{\xi_i\}$, $\{\xi'_i\}$, and $\{\beta_i\}$ are still ND sequences. We will apply Lemma 1.4 to $\{\xi_i\}$, $\{\xi'_i\}$, and $\{\beta_i\}$. For $\{\xi_i\}$ and $\{\xi'_i\}$ we have $\mathbb{E}(\xi_i) = 0, \mathbb{E}(\xi'_i) = 0, |\xi_i| = |I_{(-\infty < X_i < x+r_n)} - \mathbb{E}I_{(-\infty < X_i < x+r_n)}| \leq |I_{(-\infty < X_i < x+r_n)}| + |\mathbb{E}I_{(-\infty < X_i < x+r_n)}| \leq 2, |\xi'_i| \leq 2, n^{-1} \sum_{i=1}^n \text{var}(\xi_i) = \text{var}(I_{(-\infty < X_i < x+r_n)} - \mathbb{E}I_{(-\infty < X_i < x+r_n)}) \leq 1, n^{-1} \sum_{i=1}^n \text{var}(\xi'_i) \leq 1$. By (2.1), when n is sufficiently large, such that $\tau_n m_n / n < 1$, then we have

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n \xi_i\right| \geq \frac{am_n \tau_n}{n}\right) \leq 2 \exp\left\{-\frac{n(am_n \tau_n / n)^2}{2(2 + 2(am_n \tau_n / n))}\right\} = 2 \exp\left\{-\frac{cm_n^2 \tau_n^2}{n}\right\}, \quad (2.24)$$

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n \xi'_i\right| \geq \frac{am_n \tau_n}{n}\right) \leq 2 \exp\left\{-\frac{cm_n^2 \tau_n^2}{n}\right\}, \quad (2.25)$$

where $c = a^2/4(1+a)$.

By (2.2) and (2.24), then we have

$$P\left(\frac{1}{n}\left|\sum_{i=1}^n \xi_i\right| \geq \frac{am_n\tau_n}{n}\right) \rightarrow 0. \quad (2.26)$$

By (2.2) and (2.25), then we have

$$P\left(\frac{1}{n}\left|\sum_{i=1}^n \xi'_i\right| \geq \frac{am_n\tau_n}{n}\right) \rightarrow 0. \quad (2.27)$$

Similarly,

$$P\left(\frac{1}{n}\left|\sum_{i=1}^n \beta_i\right| \geq \frac{am_n\tau_n}{n}\right) \rightarrow 0. \quad (2.28)$$

From (2.22), (2.26), and (2.28), we get $P(|\sum_{i=1}^n \xi_i| \geq m_n(\varepsilon\tau_n/4f(x))) \rightarrow 0$. Similarly, $P(|\sum_{i=1}^n \eta_i| \geq m_n(\varepsilon\tau_n/4f(x))) \rightarrow 0$. By (2.20), then $P_{n1} \rightarrow 0$. From (2.23), (2.27), and (2.28), we get $P(|\sum_{i=1}^n \xi'_i| \geq m_n(\varepsilon\tau_n/4f(x))) \rightarrow 0$. Similarly, $P(|\sum_{i=1}^n \eta'_i| \geq m_n(\varepsilon\tau_n/4f(x))) \rightarrow 0$. By (2.21), then $P_{n2} \rightarrow 0$. Hereby, we have proved that $J_{n3} \rightarrow 0$. It remains to show that $J_{n1} \rightarrow 0$ and $J_{n2} \rightarrow 0$. Denote

$$\beta'_i = I_{(x < X_i < \infty)} - \mathbb{E}I_{(x < X_i < \infty)}. \quad (2.29)$$

According to the conditions of Theorem 2.1, we can see $0 < \mathbb{E}I_{(-\infty < X_i < x)} = P(X_i < x) = F(x) < 1$, $m_n/n \rightarrow 0$, as $n \rightarrow \infty$. Then, there exist positive constants a_1 and a_2 such that $\mathbb{E}I_{(-\infty < X_i < x)} - m_n/n > a_1$ and $\mathbb{E}I_{(x < X_i < \infty)} - m_n/n > a_2$. When n is sufficiently large, we have

$$\begin{aligned} J_{n1} &= P(x < X_{(m_n)}) \\ &= P\left(\sum_{i=1}^n I_{(-\infty < X_i < x)} \leq m_n\right) \\ &= P\left(\sum_{i=1}^n \beta_i \leq m_n - n\mathbb{E}I_{(-\infty < X_i < x)} < -a_1n\right) \\ &\leq P\left(\left|\frac{1}{n}\sum_{i=1}^n \beta_i\right| > a_1\right), \end{aligned}$$

$$\begin{aligned}
J_{n2} &= P(x \geq X_{(n-m_n+1)}) \\
&= P\left(\sum_{i=1}^n I_{(x < X_i < \infty)} \leq m_n\right) \\
&= P\left(\sum_{i=1}^n \beta'_i \leq m_n - n\mathbb{E}I_{(x < X_i < \infty)} < -a_2 n\right) \\
&\leq P\left(\left|\frac{1}{n} \sum_{i=1}^n \beta'_i\right| > a_2\right).
\end{aligned} \tag{2.30}$$

We will apply Lemma 1.4 to $\{\beta_i\}$ and $\{\beta'_i\}$. For $\{\beta_i\}$, we have $\mathbb{E}(\beta_i) = 0$, $|\beta_i| = |I_{(-\infty < X_i < x)} - \mathbb{E}I_{(-\infty < X_i < x)}| \leq |I_{(-\infty < X_i < x)}| + |\mathbb{E}I_{(-\infty < X_i < x)}| \leq 2$, $\sigma_n^2 = n^{-1} \sum_{i=1}^n \text{var}(\beta_i) = \text{var}(\beta_i) = \text{var}(I_{(-\infty < X_i < x)} - \mathbb{E}I_{(-\infty < X_i < x)}) \leq 1$. When n is sufficiently large, then we have

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n \beta_i\right| > a_1\right) \leq 2 \exp\left\{-\frac{na_1^2}{2(2\sigma_n^2 + 2a_1)}\right\} \leq 2 \exp\left\{-\frac{na_1^2}{2(2 + 2a_1)}\right\} = 2 \exp\{-c'n\} \rightarrow 0, \tag{2.31}$$

where $c' = a_1^2/2(2 + 2a_1)$. Similarly,

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n \beta'_i\right| > a_2\right) \leq 2 \exp\left\{-\frac{na_2^2}{2(2 + 2a_2)}\right\} \rightarrow 0. \tag{2.32}$$

From (2.31) and (2.32), we can get $J_{n1} \rightarrow 0$, $J_{n2} \rightarrow 0$. This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. By (2.4), when n is sufficiently large, we get $c\tau_n^2 m_n^2/n > 2 \ln n = \ln n^2$. From (2.24), then we have

$$P\left(\frac{1}{n} \left|\sum_{i=1}^n \xi_i\right| \geq \frac{am_n \tau_n}{n}\right) \leq 2 \exp\left\{-\frac{c\tau_n^2 m_n^2}{n}\right\} \leq 2 \exp(-\ln n^2) = \frac{2}{n^2}. \tag{2.33}$$

Similarly,

$$P\left(\frac{1}{n} \left|\sum_{i=1}^n \eta_i\right| \geq \frac{am_n \tau_n}{n}\right) \leq \frac{2}{n^2}. \tag{2.34}$$

From (2.33) and (2.34), we get $\sum_{n=1}^{\infty} P_{n1} < \infty$. For P_{n2} , the proof follows in the same way. From (2.7), we can get $\sum_{n=1}^{\infty} J_{n3} < \infty$. Similarly, by (2.30), we have $\sum_{n=1}^{\infty} J_{n1} < \infty$ and $\sum_{n=1}^{\infty} J_{n2} < \infty$. From (2.6), we can get

$$\sum_{n=1}^{\infty} P(|f_n(x) - f(x)| > \varepsilon \tau_n) < \infty. \tag{2.35}$$

According to the Borel-Cantelli lemma, we have

$$|f_n(x) - f(x)| = o(\tau_n) \quad \text{a.s.} \quad (2.36)$$

□

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