Research Article

Convergence of a General Composite Iterative Method for a Countable Family of Nonexpansive Mappings

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We propose a general composite iterative method for computing common fixed points of a countable family of nonexpansive mappings in the framework of Hilbert spaces. Our results improve and complement the corresponding ones announced by many others.

1. Introduction and Main Result

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A mapping $T : H \rightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points, $\text{Fix}(T) = \{x \in H : Tx = x\}$, of a nonexpansive mapping is always a closed and convex subset of H .

In addition to nonexpansive mappings, we are going to use contractions and k -Lipschitzian and η -strongly monotone operators. A self-mapping $f : H \rightarrow H$ is a contraction on H , if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in H$. We use Π_H to denote the collection of mappings f verifying the above inequality. That is, $\Pi_H = \{f : H \rightarrow H \mid f \text{ is a contraction with constant } \alpha\}$. A mapping $F : H \rightarrow H$ is called k -Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \leq k\|x - y\|, \quad \forall x, y \in H. \quad (1.1)$$

F is said to be η -strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H. \quad (1.2)$$

Recently, Jung [1] introduced the following composite iterative scheme for the solution of a specific minimization problem, which involves a closed convex subset $C \subseteq H$, a nonexpansive mapping $T : C \rightarrow C$, and a contraction $f : C \rightarrow C$,

$$\begin{aligned} x_1 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_nTy_n, \quad n \geq 1. \end{aligned} \tag{1.3}$$

Therein the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy certain conditions. He proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to $\tilde{x} = Q(f) \in \text{Fix}(T)$, which is the unique solution of the variational inequality $\langle \tilde{x} - f(\tilde{x}), \tilde{x} - z \rangle \leq 0, z \in \text{Fix}(T)$. The results of Jung [1] are even stronger than stated here. In fact, they hold for a finite family of nonexpansive mappings and in the setting of Banach spaces.

Very recently, Tian [2] considered the following iterative method: for nonexpansive mapping $T : H \rightarrow H$ with $\text{Fix}(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F)Tx_n, \quad n \geq 1, \tag{1.4}$$

where F is a k -Lipschitzian and η -strongly monotone operator. He obtained that the sequence $\{x_n\}$ generated by (1.4) converges to a point q in $\text{Fix}(T)$, which is the unique solution of the variational inequality $\langle (\gamma f - \mu F)q, p - q \rangle \leq 0, p \in \text{Fix}(T)$.

In this connection, notice that Aoyama et al. [3] proposed a Halpern approximation method for finding a common fixed point of a countable family of nonexpansive mappings. As their main result, they established the following strong convergence theorem.

Theorem 1.1 (see [3, Theorem 3.4]). *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, and let C be a nonempty closed convex subset of E . Let $\{T_n\}$ be a sequence of nonexpansive mappings $T_n : C \rightarrow C$ such that*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty \tag{1.5}$$

for each bounded subset B of C . Suppose in addition that

$$\text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset, \tag{1.6}$$

where $T : C \rightarrow C$ is the nonexpansive mapping defined by $Tz = \lim_{n \rightarrow \infty} T_nz$. Let $\alpha_n \in (0, 1]$ be a sequence satisfying the conditions (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$. Given $x \in C$, define $\{x_n\}_{n=1}^{\infty}$ by

$$x_1 = x, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)T_nx_n. \tag{1.7}$$

Then $\{x_n\}$ converges strongly to Qx , where Q is a sunny nonexpansive retraction of E onto $\text{Fix}(T)$.

Inspired by Jung [1], Tian [2] and Aoyama et al. [3], the goal of this paper is to combine these three results into a single method. In this connection, observe that the iteration methods (1.3) and (1.4) just can compute a fixed point of one nonexpansive mapping (or, perhaps, finitely many), while the iteration methods (1.3) and (1.7) do not contain the k -Lipschitzian and η -strongly monotone operator F . On the other hand, in contrast to the result of Jung [1] and Aoyama et al. [3], our result will be restricted to the setting of Hilbert spaces, which on the other hand is natural when dealing with a k -Lipschitzian and η -strongly monotone operator F .

First of all, let us remark that condition (1.5) implies that sequence of mappings $\{T_n\}$ are uniformly Cauchy on each bounded subset $B \subset C$. Hence the limiting map T is well defined and is in fact the uniform limit (on B) of the maps T_n . In other words,

$$\lim_{n \rightarrow \infty} \sup \{ \|Tz - T_n z\| : z \in B \} = 0. \quad (1.8)$$

Notice that condition (1.5) is quite strong so that, in general, we cannot apply the result directly to an arbitrary countable family of nonexpansive mappings. Furthermore, we have to assume the nontrivial condition (1.6) that the fixed point set of T coincides with the set of common fixed points of the family $\{T_n\}$.

Fortunately, as pointed out in [3, Section 4], given an arbitrary countable family of nonexpansive mappings $S_n : C \rightarrow C$, which have at least one common fixed point, one can find nonnegative numbers β_n^k ($1 \leq k \leq n$, $n \in \mathbb{N}$) such that the mappings

$$T_n x = \sum_{k=1}^n \beta_n^k S_k x \quad (1.9)$$

are nonexpansive self-maps of C satisfying (1.5) and (1.6). More specifically, the T_n are constructed as certain convex combinations of the $S_1, \dots, S_n$. (For details, see the reference.) Another possibility of a construction will be mentioned below. Furthermore, we establish the following strong convergence theorem.

Theorem 1.2. *Let H be a real Hilbert space. Let F be a k -Lipschitzian and η -strongly monotone operator on H with $0 < \mu < 2\eta/k^2$ and $f \in \Pi_H$ with $0 < \gamma < \mu(\eta - \mu k^2/2)/\alpha = \tau/\alpha$ and $\tau < 1$. Assume that $\{T_n\}$ is a sequence of nonexpansive mappings from H into itself such that the condition (1.5). Suppose that $T : H \rightarrow H$ is defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ such that the condition (1.6). Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions:*

- (B1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (B2) $b := \limsup_{n \rightarrow \infty} \beta_n < 1$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
- (B3) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$ with $\sigma_n \geq 0$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Then, for arbitrary $x_1 = x \in H$, the sequence $\{x_n\}_{n=1}^{\infty}$, defined by

$$\begin{aligned} y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_n x_n, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 1, \end{aligned} \quad (1.10)$$

converges strongly to some $q \in \text{Fix}(T)$, which satisfies the variational inequality $\langle (\mu F - \gamma f)q, q - z \rangle \leq 0$, for all $z \in \text{Fix}(T)$.

The iterative scheme (1.10) is a direct generalization of the three iteration methods considered before. Thus Theorem 1.2 complements or improves Theorem 3.2 of Jung [1], Theorem 3.2 of Tian [2], Theorem 3.4 of Aoyama et al. [3], and Theorem 3.1 of Jung [4].

An important special case is obtained for $F = I$, the identity mapping. Then $\eta = k = 1$ and we can choose $\gamma = \mu = 1$ (compare with iteration scheme (1.3) above).

Besides the basic conditions (B1) and (B2) on the sequences α_n and β_n , we have the “control condition” (B3). It can obviously be replaced by one of the following ones:

$$(B3-1) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(B3-2) \alpha_n \in (0, 1] \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \rightarrow \infty} (\alpha_{n+1} / \alpha_n) = 1.$$

Indeed, (B3-1) implies (B3) by choosing $\sigma_n = |\alpha_{n+1} - \alpha_n|$, and (B3-2) implies (B3) by choosing $\sigma_n = 0$. In this sense, (B3) is a weaker condition than the previous condition (C3).

As has already noticed in Theorem 1.1, the assumptions (1.5) and (1.6) do not apply to arbitrary families of nonexpansive mappings. Besides the construction above, we mention another construction, which has appeared in the literature. See [5–8], the references therein, and also Remark 3.1 of Peng and Yao [9]. Proofs are given there.

Let $\{S_i : H \rightarrow H\}$ be a countable family of nonexpansive mappings, and let $\{\xi_i\}$ be a sequence of real numbers such that $0 \leq \xi_i \leq c < 1$, for all $i \geq 1$. For any $n \geq 1$, define a mapping $T_n : H \rightarrow H$ as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n S_n U_{n,n+1} + (1 - \xi_n) I, \\ U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \xi_k S_k U_{n,k+1} + (1 - \xi_k) I, \\ &\vdots \\ U_{n,2} &= \xi_2 S_2 U_{n,3} + (1 - \xi_2) I, \\ U_{n,1} &= \xi_1 S_1 U_{n,2} + (1 - \xi_1) I, \\ T_n &= U_{n,1}. \end{aligned} \tag{1.11}$$

Proposition 1.3. *Let H be a real Hilbert space, $\{S_i : H \rightarrow H\}$ a sequence nonexpansive mappings with $\bigcap_{i=1}^{\infty} \text{Fix}(S_i) \neq \emptyset$, and $\{\xi_i\}$ a real sequence such that $0 \leq \xi_i \leq c < 1$, for all $i \geq 1$. Define $\{T_n\}$ as above. Then*

- (1) T_n is a nonexpansive and $\text{Fix}(T_n) = \bigcap_{i=1}^n \text{Fix}(S_i)$ for each $n \geq 1$;
- (2) the mappings $\{T_n\}$ satisfy the condition (1.5);
- (3) defining $T : H \rightarrow H$ by $Tx = \lim_{n \rightarrow \infty} T_n x$, we have

$$\text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n). \tag{1.12}$$

2. Proof of the Main Results

Ahead of the proof, we start with recalling some known auxiliary results.

Lemma 2.1 (see [10]). *Let F be a k -Lipschitzian and η -strongly monotone operator on a Hilbert space H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$, and $0 < t < 1$. Then $S = (I - t\mu F) : H \rightarrow H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = (1/2)\mu(2\eta - \mu k^2)$.*

Lemma 2.2 (see [2]). *Let H be a Hilbert space. Assume that $\{x_t\}$ is defined by*

$$x_t = t\gamma f(x_t) + (I - t\mu F)Tx_t, \quad (2.1)$$

where F is a k -Lipschitzian and η -strongly monotone operator on a Hilbert space H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$ and $0 < t < 1$, $f \in \Pi_H$, and $T : H \rightarrow H$ is a nonexpansive mapping. Then x_t converges strongly as $t \rightarrow 0$ to a fixed point q of T , which solves the variational inequality $\langle (\mu F - \gamma f)q, q - p \rangle \leq 0$, $p \in F(T)$.

Some more auxiliary results are given next.

Lemma 2.3 (see [11]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C which converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y , then $(I - T)x = y$.*

Lemma 2.4 (see [3, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\delta_n\}$ a sequence of numbers in $[0, 1]$ such that $\sum_{n=1}^{\infty} \delta_n = \infty$, $\{\mu_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \mu_n < \infty$, and $\{\theta_n\}$ a sequence of real numbers with $\limsup_{n \rightarrow \infty} \theta_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \delta_n)s_n + \delta_n\theta_n + \mu_n \quad (2.2)$$

for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Now we are prepared to prove the main result.

Proof of Theorem 1.2. We divide the proof into seven steps.

Step 1. We claim that $\{x_n\}$ is bounded. Taking any point $q \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ and using Lemma 2.1, we obtain

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \beta_n)(y_n - q) + \beta_n(T_n y_n - q)\| \\ &\leq \|y_n - q\| \\ &= \|\alpha_n(\gamma f(x_n) - \mu F q) + (I - \alpha_n \mu F)T_n x_n - (I - \alpha_n \mu F)q\| \\ &\leq (1 - \alpha_n \tau)\|x_n - q\| + \alpha_n \|\gamma f(x_n) - \mu F q\| \\ &\leq (1 - \alpha_n \tau)\|x_n - q\| + \alpha_n \|\gamma f(x_n) - \gamma f(q)\| + \alpha_n \|\gamma f(q) - \mu F q\| \\ &\leq (1 - \alpha_n \tau)\|x_n - q\| + \alpha_n \gamma \alpha \|x_n - q\| + \alpha_n \|\gamma f(q) - \mu F q\| \end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n(\tau - \gamma\alpha))\|x_n - q\| + \alpha_n(\tau - \gamma\alpha) \frac{\|\gamma f(q) - \mu Fq\|}{\tau - \gamma\alpha} \\
&\leq \max \left\{ \|x_n - q\|, \frac{\|\gamma f(q) - \mu Fq\|}{\tau - \gamma\alpha} \right\}.
\end{aligned} \tag{2.3}$$

By induction, it follows

$$\|x_n - q\| \leq \max \left\{ \|x_1 - q\|, \frac{\|\gamma f(q) - \mu Fq\|}{\tau - \gamma\alpha} \right\}, \quad n \geq 1, \tag{2.4}$$

and hence $\{x_n\}$ is bounded. From this, we also obtain that $\{y_n\}$, $\{T_n y_n\}$, $\{FT_n x_n\}$, and $\{f(x_n)\}$ are all bounded. In what follows, let B stand for some bounded set of H , which contains all of $\{x_n\}$, $\{y_n\}$, $\{T_n y_n\}$, $\{FT_n x_n\}$, $\{f(x_n)\}$.

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Let $M = \sup \{\|T_n y_n - y_n\|, \|f(x_n)\|, \|FT_n x_n\| : n \in \mathbb{N}\} < \infty$. From the definition of $\{x_n\}$, we obtain

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|(1 - \beta_{n+1})y_{n+1} + \beta_{n+1}T_{n+1}y_{n+1} - (1 - \beta_n)y_n - \beta_n T_n y_n\| \\
&= \|(1 - \beta_{n+1})y_{n+1} + \beta_{n+1}T_{n+1}y_{n+1} - (1 - \beta_{n+1})y_n - (\beta_{n+1} - \beta_n)y_n \\
&\quad - \beta_{n+1}T_n y_n + (\beta_{n+1} - \beta_n)T_n y_n\| \\
&= \|(1 - \beta_{n+1})(y_{n+1} - y_n) + \beta_{n+1}(T_{n+1}y_{n+1} - T_n y_n) + (\beta_{n+1} - \beta_n)(T_n y_n - y_n)\| \\
&\leq (1 - \beta_{n+1})\|y_{n+1} - y_n\| + \beta_{n+1}\|T_{n+1}y_{n+1} - T_n y_n\| + |\beta_{n+1} - \beta_n|M \\
&\leq (1 - \beta_{n+1})\|y_{n+1} - y_n\| + \beta_{n+1}\|T_{n+1}y_{n+1} - T_n y_{n+1}\| + \beta_{n+1}\|y_{n+1} - y_n\| \\
&\quad + |\beta_{n+1} - \beta_n|M \\
&\leq \|y_{n+1} - y_n\| + \|T_{n+1}y_{n+1} - T_n y_{n+1}\| + |\beta_{n+1} - \beta_n|M
\end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$. From (1.10), we have

$$\begin{aligned}
y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_n x_n, \\
y_{n+1} &= \alpha_{n+1} \gamma f(x_{n+1}) + (I - \alpha_{n+1} \mu F)T_{n+1} x_{n+1}.
\end{aligned} \tag{2.6}$$

With $M_1 = (\mu + \gamma)M$, we thus have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}\mu F)T_{n+1}x_{n+1} - \alpha_n\gamma f(x_n) - (I - \alpha_n\mu F)T_nx_n\| \\
&= \|(I - \alpha_{n+1}\mu F)T_{n+1}x_{n+1} - (I - \alpha_{n+1}\mu F)T_nx_n - (\alpha_{n+1} - \alpha_n)\mu FT_nx_n \\
&\quad + \alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n)\| \\
&\leq (1 - \alpha_{n+1}\tau)\|T_{n+1}x_{n+1} - T_nx_n\| + |\alpha_{n+1} - \alpha_n|\mu\|FT_nx_n\| + \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| \\
&\quad + \gamma|\alpha_{n+1} - \alpha_n|\|f(x_n)\| \\
&\leq (1 - \alpha_{n+1}\tau)\|T_{n+1}x_{n+1} - T_nx_n\| + \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|M_1 \\
&\leq (1 - \alpha_{n+1}\tau)(\|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_nx_n\|) + \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| \\
&\quad + M_1|\alpha_{n+1} - \alpha_n| \\
&\leq [1 - \alpha_{n+1}(\tau - \gamma\alpha)]\|x_{n+1} - x_n\| + \|T_{n+1}x_n - T_nx_n\| + M_1|\alpha_{n+1} - \alpha_n|.
\end{aligned} \tag{2.7}$$

Combining (2.5) and (2.7), we conclude that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq [1 - \alpha_{n+1}(\tau - \gamma\alpha)]\|x_{n+1} - x_n\| + M_2(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) \\
&\quad + 2\sup\{\|T_{n+1}z - T_nz\| : z \in B\},
\end{aligned} \tag{2.8}$$

where $M_2 = \max\{M, M_1\}$. Using the assumption (B3), we obtain

$$\|x_{n+2} - x_{n+1}\| \leq [1 - \alpha_{n+1}(\tau - \gamma\alpha)]\|x_{n+1} - x_n\| + M_2o(\alpha_{n+1}) + \mu_n \tag{2.9}$$

with $\mu_n = M_2(\sigma_n + |\beta_{n+1} - \beta_n|) + 2\sup\{\|T_{n+1}z - T_nz\| : z \in B\}$. The various assumptions imply that $\sum_{n=1}^{\infty} \mu_n < \infty$. Now it follows from Lemma 2.4 that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Step 3. Next we show that

$$\lim_{n \rightarrow \infty} \|T_nx_n - y_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{2.10}$$

In fact, from Step 1, $\{\gamma f(x_n) - \mu FT_nx_n\}$ is bounded. Moreover, it follows from $\lim_{n \rightarrow \infty} \alpha_n = 0$ that

$$\|T_nx_n - y_n\| = \|\alpha_n\gamma f(x_n) - \alpha_n\mu FT_nx_n\| = \alpha_n\|\gamma f(x_n) - \mu FT_nx_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.11}$$

This is the first part. From condition (B2), we obtain, for n sufficiently large and $b < b_1 < 1$,

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \beta_n\|T_ny_n - y_n\| \\
&\leq \beta_n(\|T_ny_n - T_nx_n\| + \|T_nx_n - y_n\|) \\
&\leq b_1(\|y_n - x_n\| + \|T_nx_n - y_n\|) \\
&\leq b_1(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|T_nx_n - y_n\|),
\end{aligned} \tag{2.12}$$

which implies

$$\|x_{n+1} - y_n\| \leq \frac{b_1}{1 - b_1} (\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|). \quad (2.13)$$

Hence, using Step 2 and the above, we obtain $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0$ and thus $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Step 4. We show that $\lim_{n \rightarrow \infty} \|y_n - T_n y_n\| = 0$.

Indeed, we have

$$\begin{aligned} \|y_n - T_n y_n\| &\leq \|y_n - T_n x_n\| + \|T_n x_n - T_n y_n\| \\ &\leq \|y_n - T_n x_n\| + \|x_n - y_n\|, \end{aligned} \quad (2.14)$$

and now the statement follows from Step 3.

Step 5. We show that $\|T y_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Observe that

$$\begin{aligned} \|T y_n - y_n\| &\leq \|T y_n - T_n y_n\| + \|T_n y_n - y_n\| \\ &\leq \sup\{\|T z - T_n z\| : z \in B\} + \|T_n y_n - y_n\|, \end{aligned} \quad (2.15)$$

and now apply Step 4 and the fact that T_n converges uniformly on B to T (which is a consequence of (1.5)).

Step 6. We claim that $\limsup_{n \rightarrow \infty} \langle \gamma f(q) - \mu F q, y_n - q \rangle \leq 0$, where $q = \lim_{t \rightarrow 0} x_t$ with x_t defined by (2.1) and the existence of q being guaranteed by Lemma 2.2.

Assume the contrary. Then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that the limit

$$c := \lim_{k \rightarrow \infty} \langle \gamma f(q) - \mu F q, y_{n_k} - q \rangle \quad (2.16)$$

exists and is greater than zero. Since $\{y_n\}$ is bounded, there exists a subsequence of $\{y_{n_k}\}$, which converges weakly to some $z \in H$. We can assume without loss of generality that $\{y_{n_k}\}$ is this subsequence. From $\|T y_n - y_n\| \rightarrow 0$, we obtain that $T y_{n_k}$ converges weakly to z . Now Lemma 2.3 implies that $z \in \text{Fix}(T)$. Hence from weak convergence and Lemma 2.2,

$$\lim_{k \rightarrow \infty} \langle \gamma f(q) - \mu F q, y_{n_k} - q \rangle = \langle (\gamma f - \mu F) q, z - q \rangle \leq 0. \quad (2.17)$$

This is a contradiction.

Step 7. We show that $\{x_n\}$ converges strongly to q . From (1.10), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - \beta_n)(y_n - q) + \beta_n(T_n y_n - q)\|^2 \\ &\leq \|y_n - q\|^2 \\ &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_n x_n - q\|^2 \\ &= \|\alpha_n (\gamma f(x_n) - \mu F q) + (I - \alpha_n \mu F) T_n x_n - (I - \alpha_n \mu F) q\|^2. \end{aligned} \quad (2.18)$$

Using the general inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, we can further estimate the above by

$$\begin{aligned}
 & \leq \|(I - \alpha_n \mu F)T_n x_n - (I - \alpha_n \mu F)q\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu Fq, y_n - q \rangle \\
 & \leq (1 - \alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \langle \gamma f(x_n) - \gamma f(q), y_n - q \rangle + 2\alpha_n \langle \gamma f(q) - \mu Fq, y_n - q \rangle \\
 & \leq (1 - \alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \gamma \alpha \|x_n - q\| (\|y_n - x_n\| + \|x_n - q\|) + 2\alpha_n \langle \gamma f(q) - \mu Fq, y_n - q \rangle \\
 & \leq (1 - 2\alpha_n(\tau - \gamma \alpha)) \|x_n - q\|^2 + (\alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \gamma \alpha \|x_n - q\| \|y_n - x_n\| \\
 & \quad + 2\alpha_n \langle \gamma f(q) - \mu Fq, y_n - q \rangle.
 \end{aligned} \tag{2.19}$$

Introducing $M_3 = \sup\{\|x_n - q\| : n \geq 1\}$, $\delta_n = 2\alpha_n(\tau - \gamma \alpha) > 0$, and

$$\theta_n = \frac{\alpha_n \tau^2 M_3^2}{2(\tau - \gamma \alpha)} + \frac{\gamma \alpha M_3}{\tau - \gamma \alpha} \|y_n - x_n\| + \frac{1}{\tau - \gamma \alpha} \langle \gamma f(q) - \mu Fq, y_n - q \rangle, \tag{2.20}$$

we obtain

$$\|x_{n+1} - q\|^2 \leq (1 - \delta_n) \|x_n - q\|^2 + \delta_n \theta_n. \tag{2.21}$$

Because $\delta_n \rightarrow 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$, and $\limsup_{n \rightarrow \infty} \theta_n \leq 0$, it follows from Lemma 2.4 that the sequence $\{x_n\}$ converges strongly to q . This finishes the proof of the main part of the theorem.

From the definition of q as $q = \lim_{t \rightarrow 0} x_t$ (see Step 6) and from Lemma 2.2, we obtain $\langle (\mu F - \gamma f)q, q - z \rangle \leq 0$, for all $z \in \text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$. This completes the proof. $\square$

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