Research Article

A Rotation on Wiener Space with Applications

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We first investigate a rotation property of Wiener measure on the product of Wiener spaces. Next, using the concept of the generalized analytic Feynman integral, we define a generalized Fourier-Feynman transform and a generalized convolution product for functionals on Wiener space. We then proceed to establish a fundamental result involving the generalized transform and the generalized convolution product.

1. Introduction

Let $C_0[0, T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions x on $[0, T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. Then $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0, T]} F(x) dm(x). \quad (1.1)$$

In [1], Bearman gave a significant theorem for Wiener integral on product Wiener space. It can be summarized as follows.

Theorem 1.1 (Bearman's Rotation Theorem). *Let $G(w, z)$ be an $m \times m$ -integrable functional on $C_0^2[0, T]$, the product of 2 copies of $C_0[0, T]$, and let θ be a function of bounded variation on $[0, T]$. Let $T_\theta : C_0^2[0, T] \rightarrow C_0^2[0, T]$ be the transformation defined by $T_\theta(w, z) = (w', z')$ with*

$$w'(t) = \int_0^t \cos \theta(s) dw(s) - \int_0^t \sin \theta(s) dz(s),$$

$$z'(t) = \int_0^t \sin \theta(s) dw(s) + \int_0^t \cos \theta(s) dz(s). \quad (1.2)$$

Then the transform T_θ is measure preserving and

$$\int_{C_0^2[0,T]} G(w, z) d(m \times m)(w, z) = \int_{C_0^2[0,T]} G(T_\theta(w, z)) d(m \times m)(w, z). \quad (1.3)$$

As a special case of Theorem 1.1, one can obtain the following corollary.

Corollary 1.2. *Let F be Wiener integrable on $C_0[0, T]$. Then for any $\theta \in \mathbb{R}$, $F(w \sin \theta + z \cos \theta)$ is integrable on $C_0^2[0, T]$ and*

$$\int_{C_0^2[0,T]} F(w \sin \theta + z \cos \theta) d(m \times m)(w, z) = \int_{C_0[0,T]} F(x) dm(x). \quad (1.4)$$

The following more general case of Corollary 1.2 is due to Cameron and Storvick [2]. But we state the theorem with some assumption for our research.

Theorem 1.3. *Let F be Wiener measurable on $C_0[0, T]$. Assume that for any $\rho > 0$, $F(\rho \cdot)$ is Wiener integrable. Then for any $a, b \in \mathbb{R}$, $F(aw + bz)$ is integrable on $C_0^2[0, T]$ and*

$$\int_{C_0^2[0,T]} F(aw + bz) d(m \times m)(w, z) = \int_{C_0[0,T]} F(\sqrt{a^2 + b^2}x) dm(x). \quad (1.5)$$

In many papers, Theorem 1.3 is used to study relationships between analytic Fourier-Feynman transforms and convolution products of Feynman integrable functionals on Wiener space, see for instance [3–6]. In this paper, we will extend the result in Theorem 1.3 to a more general case for functionals of Gaussian processes given by (2.2) below. We then apply our rotation property of Wiener measure to establish a fundamental relationship between the generalized Fourier-Feynman transform and the generalized convolution product.

2. A Rotation on Wiener Space

The most important concepts we will employ in the statements and proofs of our results are the concepts of the scale-invariant measurability and the Paley-Wiener-Zygmund stochastic integral [7].

A subset B of $C_0[0, T]$ is said to be scale-invariant measurable [8] provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\{\phi_n\}$ be a complete orthonormal set in $L_2[0, T]$, each of whose elements is of bounded variation on $[0, T]$. Then for each $v \in L_2[0, T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n \rightarrow \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_2 \phi_j(t) dx(t) \quad (2.1)$$

for all $x \in C_0[0, T]$ for which the limit exists, where $(\cdot, \cdot)_2$ denotes the L_2 -inner product.

It was shown in [7] that for each $v \in L_2[0, T]$, the limit defining the PWZ integral $\langle v, x \rangle$ exists for m -a.e. $x \in C_0[0, T]$ and that this limit is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}$. It was also shown in [7] that if v is of bounded variation on $[0, T]$, then the PWZ integral $\langle v, x \rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t) dx(t)$ for m -a.e. $x \in C_0[0, T]$. In fact, the integrals are equal for s -a.e. $x \in C_0[0, T]$ and that for all $v \in L_2[0, T]$, $\langle v, x \rangle$ is a Gaussian random variable with mean 0 and variance $\|v\|_2^2$.

For any $h \in L_2[0, T]$ with $\|h\|_2 > 0$, let $\mathcal{Z}_h$ be the Gaussian process

$$\mathcal{Z}_h(x, t) = \int_0^t h(s) dx(s) = \langle v, x \rangle \quad (2.2)$$

introduced by Park and Skoug in [9] and used extensively since; see for example [5, 6, 10, 11]. Of course if $h(t) \equiv 1$ on $[0, T]$, then $\mathcal{Z}_h(x, t) = x(t)$.

It is easy to see that $\mathcal{Z}_h$ is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0, T]} \mathcal{Z}_h(x, s) \mathcal{Z}_h(x, t) dm(x) = \int_0^{\min\{s, t\}} h^2(u) du. \quad (2.3)$$

In addition, $\mathcal{Z}_h(\cdot, t)$ is stochastically continuous in t on $[0, T]$, and for any $h_1, h_2 \in L_2[0, T]$,

$$\int_{C_0[0, T]} \mathcal{Z}_{h_1}(x, s) \mathcal{Z}_{h_2}(x, t) dm(x) = \int_0^{\min\{s, t\}} h_1(u) h_2(u) du. \quad (2.4)$$

For any complete orthonormal set $\{\phi_n\}$ in $L_2[0, T]$ and for any $n \in \mathbb{N}$, define the projection map ρ_n from $L_2[0, T]$ into $\text{span}\{\phi_1, \dots, \phi_n\}$ by

$$\rho_n h(t) = \sum_{j=1}^n (h, \phi_j)_2 \phi_j(t). \quad (2.5)$$

Then for $h \in L_2[0, T]$ and $x \in C_0[0, T]$, we see that

$$\mathcal{Z}_h(x, t) = \lim_{n \rightarrow \infty} \int_0^t \rho_n h(s) dx(s) = \lim_{n \rightarrow \infty} \mathcal{Z}_{\rho_n h}(x, t), \quad (2.6)$$

that is, $\mathcal{Z}_{\rho_n h}(x, t)$ converges in $L_2(C_0[0, T])$ -mean to $\mathcal{Z}_h(x, t)$.

Throughout this paper, we will assume that each functional $F : C_0[0, T] \rightarrow \mathbb{C}$ we consider is scale-invariant measurable and that

$$\int_{C_0[0, T]} |F(\mathcal{Z}_h(x, \cdot))| dm(x) < +\infty \quad (2.7)$$

for all $h \in L_2[0, T]$.

We are now ready to state the main theorem of this paper.

Theorem 2.1. *Let F be a functional on $C_0[0, T]$. Then for any $h_1, h_2 \in L_2[0, T]$,*

$$\int_{C_0^2[0, T]} F(\mathcal{Z}_{h_1}(w, \cdot) + \mathcal{Z}_{h_2}(z, \cdot)) d(m \times m)(w, z) = \int_{C_0[0, T]} F(\mathcal{Z}_k(x, \cdot)) dm(x), \quad (2.8)$$

where h_1, h_2 , and k are related by

$$k(t) = \sum_{n=1}^{\infty} \sqrt{(h_1, \phi_n)_2^2 + (h_2, \phi_n)_2^2} \phi_n(t) \quad (2.9)$$

for some complete orthonormal set $\{\phi_n\}$ in $L_2[0, T]$, each of those elements is of bounded variation on $[0, T]$.

3. Proof of the Main Theorem

We begin this section with three lemmas in order to establish (2.8).

Lemma 3.1. *Let F be a functional on $C_0[0, T]$, and let ϕ be a function of bounded variation on $[0, T]$. Then for all $a, b \in \mathbb{R}$,*

$$\int_{C_0^2[0, T]} F(a\mathcal{Z}_\phi(w, \cdot) + b\mathcal{Z}_\phi(z, \cdot)) d(m \times m)(w, z) = \int_{C_0[0, T]} F(\sqrt{a^2 + b^2} \mathcal{Z}_\phi(x, \cdot)) dm(x). \quad (3.1)$$

Proof. We first note that for each $t \in [0, T]$,

$$a\mathcal{Z}_\phi(w, t) + b\mathcal{Z}_\phi(z, t) = \int_0^t \phi(s) d(aw(s) + bz(s)) = \mathcal{Z}_\phi(aw + bz, t). \quad (3.2)$$

We also note that $F(\mathcal{Z}(x, \cdot))$ is Wiener integrable as a functional of x . Hence, by (1.5), we obtain that for all $a, b \in \mathbb{R}$,

$$\begin{aligned}
 & \int_{C_0^2[0,T]} F(a\mathcal{Z}_\phi(w, \cdot) + b\mathcal{Z}_\phi(z, \cdot)) d(m \times m)(w, z) \\
 &= \int_{C_0^2[0,T]} F(\mathcal{Z}_\phi(aw + bz, \cdot)) d(m \times m)(w, z) \\
 &= \int_{C_0[0,T]} F(\mathcal{Z}_\phi(\sqrt{a^2 + b^2}x, \cdot)) dm(x) \\
 &= \int_{C_0[0,T]} F(\sqrt{a^2 + b^2}\mathcal{Z}_\phi(x, \cdot)) dm(x).
 \end{aligned} \tag{3.3}$$

Thus (3.1) is established. $\square$

Lemma 3.2. Let F be a functional on $C_0[0, T]$. Then for any $h_1, h_2 \in L_2[0, T]$ and each $n \in \mathbb{N}$,

$$\int_{C_0^2[0,T]} F(\mathcal{Z}_{\rho_n h_1}(w, \cdot) + \mathcal{Z}_{\rho_n h_2}(z, \cdot)) d(m \times m)(w, z) = \int_{C_0[0,T]} F(\mathcal{Z}_{\rho_n k}(x, \cdot)) dm(x), \tag{3.4}$$

where h_1, h_2 , and k are related by (2.9).

Proof. Since the addition is continuous in the uniform topology on $C_0[0, T]$, we can apply (3.1) to the functional $F(\sum_{j=1}^n \mathcal{Z}_{\phi_j}(x, \cdot))$. Thus using (2.5) and (3.1), we have

$$\begin{aligned}
 & \int_{C_0^2[0,T]} F(\mathcal{Z}_{\rho_n h_1}(w, \cdot) + \mathcal{Z}_{\rho_n h_2}(z, \cdot)) d(m \times m)(w, z) \\
 &= \int_{C_0^2[0,T]} F\left(\sum_{j=1}^n [(h_1, \phi_j)_2 \mathcal{Z}_{\phi_j}(w, \cdot) + (h_2, \phi_j)_2 \mathcal{Z}_{\phi_j}(z, \cdot)]\right) d(m \times m)(w, z) \\
 &= \int_{C_0[0,T]} F\left(\sum_{j=1}^n \sqrt{(h_1, \phi_j)_2^2 + (h_2, \phi_j)_2^2} \mathcal{Z}_{\phi_j}(x, \cdot)\right) dm(x) \\
 &= \int_{C_0[0,T]} F(\mathcal{Z}_{\rho_n k}(x, \cdot)) dm(x).
 \end{aligned} \tag{3.5}$$

Thus (3.4) is established. $\square$

Lemma 3.3. Let F be bounded and continuous on $C_0[0, T]$. Then for any $h_1, h_2 \in L_2[0, T]$,

$$\int_{C_0^2[0,T]} F(\mathcal{Z}_{h_1}(w, \cdot) + \mathcal{Z}_{h_2}(z, \cdot)) d(m \times m)(w, z) = \int_{C_0[0,T]} F(\mathcal{Z}_k(x, \cdot)) dm(x), \tag{3.6}$$

where h_1, h_2 , and k are related by (2.9) above.

Proof. We clearly see that F is Wiener integrable. We also note that $\{\rho_n h\}$ is a sequence of functions of bounded variation on $[0, T]$ such that $\rho_n h$ converges to h in the space $L_2[0, T]$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$ and $h \in L_2[0, T]$, let $F_n(\mathcal{Z}_h(x, \cdot)) = F(\mathcal{Z}_{\rho_n h}(x, \cdot))$. Since $\mathcal{Z}_{\rho_n h}$ converges to $\mathcal{Z}_h$ uniformly and F is continuous in the uniform topology, by (2.6),

$$F(\mathcal{Z}_h(x, \cdot)) = F\left(\lim_{n \rightarrow \infty} \mathcal{Z}_{\rho_n h}(x, \cdot)\right) = \lim_{n \rightarrow \infty} F_n(\mathcal{Z}_h(x, \cdot)). \quad (3.7)$$

Since F is bounded, by using the dominated convergence theorem and (3.4), we have

$$\begin{aligned} & \int_{C_0^2[0, T]} F(\mathcal{Z}_{h_1}(w, \cdot) + \mathcal{Z}_{h_2}(z, \cdot)) d(m \times m)(w, z) \\ &= \lim_{n \rightarrow \infty} \int_{C_0^2[0, T]} F_n(\mathcal{Z}_{h_1}(w, \cdot) + \mathcal{Z}_{h_2}(z, \cdot)) d(m \times m)(w, z) \\ &= \lim_{n \rightarrow \infty} \int_{C_0[0, T]} F_n(\mathcal{Z}_k(x, \cdot)) dm(x) \\ &= \int_{C_0[0, T]} F(\mathcal{Z}_k(x, \cdot)) dm(x), \end{aligned} \quad (3.8)$$

which concludes the proof of Lemma 3.3. $\square$

We are now ready to prove our main theorem.

Proof of Theorem 2.1. Let F be Wiener integrable. Suppose that the left-hand side of (2.8) exists. By usual arguments of integration theory, there exists a sequence $\{F_n\}$ of bounded and continuous functionals such that F_n converges to F . By Lemma 3.3 and the dominated convergence theorem, we can obtain the desired result. $\square$

Corollary 3.4. *Let F be a functional on $C_0[0, T]$. Then for all $h \in L_2[0, T]$ and all $a, b \in \mathbb{R}$,*

$$\int_{C_0^2[0, T]} F(a\mathcal{Z}_h(w, \cdot) + b\mathcal{Z}_h(z, \cdot)) d(m \times m)(w, z) = \int_{C_0[0, T]} F\left(\sqrt{a^2 + b^2} \mathcal{Z}_h(x, \cdot)\right) dm(x). \quad (3.9)$$

Proof. Simply choose $h_1 = ah$ and $h_2 = bh$ in (2.8) and use the linearity property of the PWZ stochastic integral. $\square$

Using similar arguments as in the proofs of Lemmas 3.1, 3.2, and 3.3 and Theorem 2.1 above, we can obtain the following theorems.

Theorem 3.5. *Let F be a functional on $C_0[0, T]$, and let $\{h_1, \dots, h_v\}$ be any subset of $L_2[0, T]$. Then*

$$\int_{C_0^v[0, T]} F\left(\sum_{j=1}^v \mathcal{Z}_{h_j}(x_j, \cdot)\right) dm^v(\vec{x}) = \int_{C_0[0, T]} F(\mathcal{Z}_{k_v}(x, \cdot)) dm(x), \quad (3.10)$$

where m^ν is the product Wiener measure on $C_0^\nu[0, T]$, the product of ν copies of $C_0[0, T]$, and

$$k_\nu(t) = \sum_{n=1}^{\infty} \sqrt{\sum_{j=1}^{\nu} (h_j, \phi_n)_2^2} \phi_n(t) \quad (3.11)$$

for some complete orthonormal set $\{\phi_n\}$ in $L_2[0, T]$.

Theorem 3.6. Let F be a functional on $C_0[0, T]$. Then for any h_1 and h_2 in $L_2[0, T]$,

$$\begin{aligned} \int_{C_0^2[0, T]} F(\mathcal{Z}_{h_2}(w, \cdot) - \mathcal{Z}_{h_1}(z, \cdot), \mathcal{Z}_{h_1}(w, \cdot) + \mathcal{Z}_{h_2}(z, \cdot)) d(m \times m)(w, z) \\ \stackrel{*}{=} \int_{C_0^2[0, T]} F(\mathcal{Z}_k(w, \cdot), \mathcal{Z}_k(z, \cdot)) d(m \times m)(w, z), \end{aligned} \quad (3.12)$$

where h_1 , h_2 , and k are related by (2.9).

Remark 3.7. Equations (2.8) and (3.12) are indeed very general formulas.

- (1) For any $\theta \in \mathbb{R}$, choosing $h_1(t) \equiv \sin \theta$ and $h_2(t) \equiv \cos \theta$ in (2.8) yields (1.4).
- (2) For any $a, b \in \mathbb{R}$, choosing $h_1(t) \equiv a$ and $h_2(t) \equiv b$ in (2.8) or choosing $h(t) \equiv 1$ in (3.9) yields (1.5).
- (3) For any function of bounded variation $\theta(\cdot)$, choosing $h_1(t) = \sin \theta(t)$ and $h_2(t) = \cos \theta(t)$ on $[0, T]$ in (3.12) yields (1.3).

4. Generalized Fourier-Feynman Transform and Generalized Convolution Product

In this section, we will apply our main theorem to the generalized analytic Fourier-Feynman transform and the convolution product theories.

In defining various analytic Feynman integrals, one usually starts, for $\lambda > 0$, with the Wiener integral

$$\int_{C_0[0, T]} F(\lambda^{-1/2} x) dm(x) \quad (4.1)$$

and then extends analytically in λ to the right-half complex plane. Here we start with the (generalized) Wiener integral

$$\int_{C_0[0, T]} F(\lambda^{-1/2} \mathcal{Z}_h(x, \cdot)) dm(x) = J(h; \lambda), \quad (4.2)$$

where $\mathcal{Z}_h$ is the Gaussian process given by (2.2) above.

Throughout this section, let $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively.

Let F be a complex-valued scale-invariant measurable functional on $C_0[0, T]$ such that $J(h; \lambda)$ given by (4.2) exists and is finite for all $\lambda > 0$. If there exists a function $J^*(h; \lambda)$ analytic on $\mathbb{C}_+$ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the generalized analytic Wiener integral (with respect to the process $\mathcal{Z}_h$) of F over $C_0[0, T]$ with parameter λ , and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) dm(x) = J^*(h; \lambda). \quad (4.3)$$

Let q be a nonzero real number and let F be a functional such that $\int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) dm(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0, T]}^{\text{anf}_q} F(\mathcal{Z}_h(x, \cdot)) dm(x) = \lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) dm(x), \quad (4.4)$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Note that if $h \equiv 1$ on $[0, T]$, then these definitions agree with the previous definitions of the analytic Wiener integral and the analytic Feynman integral [3, 4, 8, 12–14].

Next (see [5, 6, 15]) we state the definition of the generalized Fourier-Feynman transform (GFFT).

Definition 4.1. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$, let

$$T_{\lambda, h}(F)(y) = \int_{C_0[0, T]}^{\text{anw}_\lambda} F(y + \mathcal{Z}_h(x, \cdot)) dm(x). \quad (4.5)$$

Let q be a non-zero real number. For $p \in (1, 2]$, we define the L_p analytic GFFT with respect to $\mathcal{Z}_h$, $T_{q, h}^{(p)}(F)$ of F , by the formula ($\lambda \in \mathbb{C}_+$),

$$T_{q, h}^{(p)}(F)(y) = \text{l.i.m.}_{\lambda \rightarrow -iq} T_{\lambda, h}(F)(y) \quad (4.6)$$

if it exists; that is, for each $\rho > 0$,

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} \left| T_{\lambda, h}(F)(\rho y) - T_{q, h}^{(p)}(F)(\rho y) \right|^{p'} dm(y) = 0, \quad (4.7)$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_{q, h}^{(1)}(F)$ of F , by the formula ($\lambda \in \mathbb{C}_+$)

$$T_{q, h}^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} T_{\lambda, h}(F)(y) \quad (4.8)$$

if it exists.

We note that for $p \in [1, 2]$, $T_{q,h}^{(p)}(F)$ is defined only s-a.e. We also note that if $T_{q,h}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,h}^{(p)}(G)$ exists and $T_{q,h}^{(p)}(G) \approx T_{q,h}^{(p)}(F)$. One can see that for each $h \in L_2[0, T]$, $T_{q,h}^{(p)}(F) \approx T_{q,-h}^{(p)}(F)$ since

$$\int_{C_0[0,T]} F(x) dm(x) = \int_{C_0[0,T]} F(-x) dm(x). \quad (4.9)$$

Next we give the definition of the generalized convolution product (GCP).

Definition 4.2. Let F and G be scale-invariant measurable functionals on $C_0[0, T]$. For $\lambda \in \tilde{\mathbb{C}}_+$ and $h_1, h_2 \in L_2[0, T]$, we define their GCP with respect to $\{\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}\}$ (if it exists) by

$$\begin{aligned} (F * G)_\lambda^{(h_1, h_2)}(y) &= \begin{cases} \int_{C_0[0,T]}^{\text{anw}_\lambda} F\left(\frac{y + \mathcal{Z}_{h_1}(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - \mathcal{Z}_{h_2}(x, \cdot)}{\sqrt{2}}\right) dm(x), & \lambda \in \mathbb{C}_+, \\ \int_{C_0[0,T]}^{\text{anf}_q} F\left(\frac{y + \mathcal{Z}_{h_1}(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - \mathcal{Z}_{h_2}(x, \cdot)}{\sqrt{2}}\right) dm(x), & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases} \end{aligned} \quad (4.10)$$

When $\lambda = -iq$, we denote $(F * G)_\lambda^{(h_1, h_2)}$ by $(F * G)_q^{(h_1, h_2)}$.

Remark 4.3. Our definition of the GCP is different than the definition given by Huffman et al. in [5, 6] and used by Chang et al. in [15]. But if we choose $h_1 = h_2$ in (4.10), our GCP $(F * G)_q^{(h_1, h_2)}$ is the GCP used in [5, 6, 15].

We begin this section with a key lemma for a relationship between the GFFT and the GCP.

Lemma 4.4. Let $\{g_1, g_2, g_3, g_4\}$ be a subset of $L_2[0, T]$, and let $Y_{g_1, g_2}, Y_{g_3, g_4} : C_0^2[0, T] \times [0, T] \rightarrow \mathbb{R}$ be given by

$$\begin{aligned} Y_{g_1, g_2}(w, z; t) &= \mathcal{Z}_{g_1}(w, t) + \mathcal{Z}_{g_2}(z, t), \\ Y_{g_3, g_4}(w, z; t) &= \mathcal{Z}_{g_3}(w, t) - \mathcal{Z}_{g_4}(z, t), \end{aligned} \quad (4.11)$$

respectively. Then the following assertions are equivalent.

- (i) Y_{g_1, g_2} and Y_{g_3, g_4} are independent processes.
- (ii) $g_1 g_3 = g_2 g_4$.

Proof. Since the processes Y_{g_1, g_2} and Y_{g_3, g_4} are Gaussian with mean zero, we know that Y_{g_1, g_2} and Y_{g_3, g_4} are independent processes if and only if

$$\int_{C_0^2[0,T]} Y_{g_1, g_2}(w, z; s) Y_{g_3, g_4}(w, z; t) d(m \times m)(w, z) = 0 \quad (4.12)$$

for every $s, t \in [0, T]$. But, using (2.4), we have

$$\begin{aligned}
 & \int_{C_0^2[0, T]} Y_{g_1, g_2}(w, z; s) Y_{g_3, g_4}(w, z; t) d(m \times m)(w, z) \\
 &= \int_{C_0^2[0, T]} \{ \mathcal{Z}_{g_1}(w, s) \mathcal{Z}_{g_3}(w, t) - \mathcal{Z}_{g_1}(w, s) \mathcal{Z}_{g_4}(z, t) + \mathcal{Z}_{g_2}(z, s) \mathcal{Z}_{g_3}(w, t) - \mathcal{Z}_{g_2}(z, s) \mathcal{Z}_{g_4}(z, t) \} \\
 & \quad \times d(m \times m)(w, z) \\
 &= \int_0^{\min\{s, t\}} g_1(u) g_3(u) du - \int_0^{\min\{s, t\}} g_2(u) g_4(u) du.
 \end{aligned} \tag{4.13}$$

From this, we can obtain the desired result. $\square$

We are now ready to establish fundamental relationships between the GFFT and the GCP.

Lemma 4.5. *Let F and G be functionals on $C_0[0, T]$. Let $\{h_1, h_2, h_3\}$ be a subset of $L_2[0, T]$ such that $h_3^2 = h_1 h_2 \geq 0$ almost everywhere on $[0, T]$, and let*

$$\begin{aligned}
 k_1(t) &= 2^{-1/2} \sum_{n=1}^{\infty} \sqrt{(h_1, \phi_n)_2^2 + (h_3, \phi_n)_2^2} \phi_n(t), \\
 k_2(t) &= 2^{-1/2} \sum_{n=1}^{\infty} \sqrt{(h_2, \phi_n)_2^2 + (h_3, \phi_n)_2^2} \phi_n(t).
 \end{aligned} \tag{4.14}$$

Furthermore, assume that for all $\lambda \in \mathbb{C}_+$, $T_{\lambda, h_3}((F * G)_\lambda^{(h_1, h_2)})$, $T_{\lambda, k_1}(F)$ and $T_{\lambda, k_2}(G)$ all exist. Then

$$T_{\lambda, h_3}((F * G)_\lambda^{(h_1, h_2)})(y) = T_{\lambda, k_1}(F)\left(\frac{y}{\sqrt{2}}\right) T_{\lambda, k_2}(G)\left(\frac{y}{\sqrt{2}}\right) \tag{4.15}$$

for s-a.e. $y \in C_0[0, T]$.

Proof. We note that for all $\lambda > 0$,

$$\begin{aligned}
 & T_{\lambda, h_3}((F * G)_\lambda^{(h_1, h_2)})(y) \\
 &= \int_{C_0[0, T]} (F * G)_\lambda^{(h_1, h_2)}\left(y + \lambda^{-1/2} \mathcal{Z}_{h_3}(w, \cdot)\right) dm(w) \\
 &= \iint_{C_0[0, T]} F\left(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2\lambda}} (\mathcal{Z}_{h_3}(w, \cdot) + \mathcal{Z}_{h_1}(z, \cdot))\right) \\
 & \quad \times G\left(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2\lambda}} (\mathcal{Z}_{h_3}(w, \cdot) - \mathcal{Z}_{h_2}(z, \cdot))\right) d(m \times m)(w, z)
 \end{aligned}$$

$$\begin{aligned}
&= \int_{C_0^2[0,T]} F\left(\frac{y}{\sqrt{2}} + \lambda^{-1/2} \left(\frac{\mathcal{Z}_{h_3}(w, \cdot) + \mathcal{Z}_{h_1}(z, \cdot)}{\sqrt{2}}\right)\right) \\
&\quad \times G\left(\frac{y}{\sqrt{2}} + \lambda^{-1/2} \left(\frac{\mathcal{Z}_{h_3}(w, \cdot) - \mathcal{Z}_{h_2}(z, \cdot)}{\sqrt{2}}\right)\right) d(m \times m)(w, z).
\end{aligned} \tag{4.16}$$

But $h_3^2 = h_1 h_2$, and so $(\mathcal{Z}_{h_3}(w, \cdot) + \mathcal{Z}_{h_1}(z, \cdot))/\sqrt{2}$ and $(\mathcal{Z}_{h_3}(w, \cdot) - \mathcal{Z}_{h_2}(z, \cdot))/\sqrt{2}$ are independent processes by Lemma 4.4. Hence by (2.8), we obtain that for all $\lambda > 0$,

$$\begin{aligned}
&T_{\lambda, h_3} \left((F * G)_{\lambda}^{(h_1, h_2)} \right) (y) \\
&= \int_{C_0^2[0,T]} F\left(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2\lambda}} (\mathcal{Z}_{h_3}(w, \cdot) + \mathcal{Z}_{h_1}(z, \cdot))\right) d(m \times m)(w, z) \\
&\quad \times \int_{C_0^2[0,T]} G\left(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2\lambda}} (\mathcal{Z}_{h_3}(w, \cdot) - \mathcal{Z}_{h_2}(z, \cdot))\right) d(m \times m)(w, z) \\
&= \int_{C_0[0,T]} F\left(\frac{y}{\sqrt{2}} + \frac{\mathcal{Z}_{k_1}(x, \cdot)}{\sqrt{\lambda}}\right) dm(x) \int_{C_0[0,T]} G\left(\frac{y}{\sqrt{2}} + \frac{\mathcal{Z}_{k_2}(x, \cdot)}{\sqrt{\lambda}}\right) dm(x) \\
&= T_{\lambda, k_1}(F) \left(\frac{y}{\sqrt{2}} \right) T_{\lambda, k_2}(G) \left(\frac{y}{\sqrt{2}} \right).
\end{aligned} \tag{4.17}$$

Equation (4.15) holds for all $\lambda \in \mathbb{C}_+$ by analytic continuation. $\square$

In next theorem, we show that the GFFT of the GCP is the product of GFFTs.

Theorem 4.6. *Let $F, G, \{h_1, h_2, h_3\}, k_1$, and k_2 be as in Lemma 4.5. Furthermore, assume that for $p \in [1, 2], \lambda \in \mathbb{C}_+$ and $q \in \mathbb{R} - \{0\}$, $T_{\lambda, h_3}((F * G)_q^{(h_1, h_2)})$, $T_{q, h_3}^{(p)}((F * G)_q^{(h_1, h_2)})$, $T_{q, k_1}^{(p)}(F)$, and $T_{q, k_2}^{(p)}(G)$ all exist and that*

$$T_{q, h_3}^{(p)} \left((F * G)_q^{(h_1, h_2)} \right) = \begin{cases} \text{l.i.m.}_{\lambda \rightarrow -iq} T_{\lambda, h_3} \left((F * G)_{\lambda}^{(h_1, h_2)} \right) (y), & p \in (1, 2], \\ \lim_{\lambda \rightarrow -iq} T_{\lambda, h_3} \left((F * G)_{\lambda}^{(h_1, h_2)} \right) (y), & p = 1. \end{cases} \tag{4.18}$$

Then

$$T_{q, h_3}^{(p)} \left((F * G)_q^{(h_1, h_2)} \right) (y) = T_{q, k_1}^{(p)}(F) \left(\frac{y}{\sqrt{2}} \right) T_{q, k_2}^{(p)}(G) \left(\frac{y}{\sqrt{2}} \right) \tag{4.19}$$

for s-a.e. $y \in C_0[0, T]$.

Proof. Equation (4.19) follows from (4.15) by letting $\lambda \rightarrow -iq$, since all transforms in (4.18) and (4.19) exist. $\square$

Remark 4.7. We note that the hypotheses (and hence the conclusions) of Theorem 4.6 above are indeed satisfied by many of the functionals in the following large classes of functionals. These classes of functionals include;

- (i) the Banach algebra $\mathcal{S}$ defined by Cameron and Storvick in [16]: also see [3, 5, 14, 15],
- (ii) various spaces of functionals of the form

$$F(x) = \exp \left\{ \int_0^T f(t, x(t)) dt \right\} \quad (4.20)$$

for appropriate $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{C}$ as discussed in [4, 12, 13]; and

- (iii) various spaces of functionals of the form

$$F(x) = \exp \left\{ \iint_0^T f(s, t, x(s), x(t)) ds dt \right\} \quad (4.21)$$

for appropriate $f : [0, T]^2 \times \mathbb{R}^2 \rightarrow \mathbb{C}$ as discussed in [3].

Next five corollaries include the results of [3–6] by Huffman et al. The notations used in [3–6] are slightly different than ours.

Corollary 4.8. *Refer to Theorem 2.1 in [5].*

Proof. In our Lemma 4.5, simply choose $h_1 = h_2 = h_3 = k_1 = k_2 \equiv h \in L_2[0, T]$. □

Corollary 4.9. *Refer to Theorem 1 in [6].*

Proof. In our Theorem 4.6, simply choose $h_1 = h_2 = h_3 = k_1 = k_2 \equiv h \in L_2[0, T]$. □

Corollary 4.10. *Refer to Theorem 3.3 in [5].*

Proof. In our Theorem 4.6, simply choose $h_1 = h_2 = h_3 = k_1 = k_2 \equiv h \in L_\infty[0, T]$. □

Corollary 4.11. *Refer to Theorem 3.3 in [3].*

Proof. In our Theorem 4.6, simply choose $h_1 = h_2 = h_3 = k_1 = k_2 \equiv 1$. □

Corollary 4.12. *Refer to Lemma 4.1 and Theorems 4.1 and 4.2 in [4].*

Proof. In our Lemma 4.5 and Theorem 4.6, simply choose $h_1 = h_2 = h_3 = k_1 = k_2 \equiv 1$. □

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