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Research Article

Some Properties of Complex Harmonic Mapping

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We introduce new class of harmonic functions by using certain generalized differential operator of harmonic. Some results which generalize problems considered by many researchers are present. The main results are concerned with the starlikeness and convexity of certain class of harmonic functions.

1. Introduction

A continuous complex-valued function $f = u + iv$, defined in a simply-connected complex domain D, is said to be harmonic in D if both u and v are real harmonic in D. Such functions can be expressed as

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|\overline{h}(z)| > |\overline{g}(z)|$ for all z in D (see [1]). Let S_H be the class of functions of the form (1.1) that are harmonic univalent and sense-preserving in the unit disk $E = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_{H,}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_k z^k, \quad g(z) = \sum_{n=1}^{\infty} b_k z^k, \quad z \in E, \ |b_1| < 1. \tag{1.2}$$

In 1984, Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

In this paper, we aim at generalizing the respective results from the papers [2–5], that imply starlikeness and convexity of functions holomorphic in the unit disk.

Now, we will introduce generalized derivative operator for $f = h + \overline{g}$ given by (1.2). For fixed positive natural m, n, and $\lambda_2 \geq \lambda_1 \geq 0$,

$$D^{m,n}_{\lambda_1,\lambda_2} f(z) = D^{m,n}_{\lambda_1,\lambda_2} h(z) + D^{m,n}_{\lambda_1,\lambda_2} g(z), \quad z \in E, \tag{1.3}$$

where

$$\begin{aligned} D^{m,n}_{\lambda_1,\lambda_2} h(z) &= z + \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1)}{1 + \lambda_2(n-1)} \right)^m a_n z^n, \\ D^{m,n}_{\lambda_1,\lambda_2} g(z) &= \sum_{n=1}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1)}{1 + \lambda_2(n-1)} \right)^m b_n z^n. \end{aligned} \tag{1.4}$$

We note that by specializing the parameters, especially when $\lambda_1 = \lambda_2 = 0$, $D^{m,n}_{\lambda_1,\lambda_2}$ reduces to D^m which introduced by Sălăgean in [6].

Let $P = \{(\alpha, p) \in R^2 : 0 \leq \alpha \leq 1, p > 0\}$ and $U^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) = \alpha((1 + (\lambda_1 + \lambda_2)(n-1))/(1 + \lambda_2(n-1)))^{mp} + (1-\alpha)((1 + (\lambda_1 + \lambda_2)(n-1))/(1 + \lambda_2(n-1)))^{m(p+1)}$, $n = 2, 3, \ldots, (\alpha, p) \in P$.

For a fixed pair $(\alpha, p) \in P$, we denote by $HS^{m,n}_{\lambda_1,\lambda_2}(\alpha, p)$ the class of functions of the form (1.3) and such that

$$|b_1| + U^{m,n}_{\lambda_1,\lambda_2}(\alpha, p)(|a_n| + |b_n|) \leq 1, \quad |b_1| < 1. \tag{1.5}$$

Moreover,

$$HC^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) = \left\{ f \in HS^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) : b_1 = 0 \right\}. \tag{1.6}$$

The classes $HS^{1,n}_{0,0}(1,1) HC^{1,n}_{0,0}(1,1)$, $HS^{1,n}_{0,0}(1,2) HC^{1,n}_{0,0}(1,2)$ were studied in [2], and the classes $HS^{1,n}_{0,0}(1,p) HC^{1,n}_{0,0}(1,p) (p > 0)$ were investigated in [3]. It is known that each function of the class $HC^{1,n}_{0,0}(1,1)$ is starlike, and every function of the class $HC^{1,n}_{0,0}(1,2)$ is convex (see [2]). With respect to the following inequalities $U^{1,n}_{0,0}(1,p) = n^p \leq U^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) \leq n^{p+1} = U^{1,n}_{0,0}(0,p)$, $n = 2, 3, \ldots, (\alpha, p) \in P$, by condition (1.5) we have the following inclusions

$$\begin{aligned} HS^{1,n}_{0,0}(0,p) &\subset HS^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) \subset HS^{1,n}_{0,0}(1,p), \quad (\alpha, p) \in P, \\ HC^{1,n}_{0,0}(0,p) &\subset HC^{m,n}_{\lambda_1,\lambda_2}(\alpha, p) \subset HC^{1,n}_{0,0}(1,p), \quad (\alpha, p) \in P. \end{aligned} \tag{1.7}$$

2. Main Result

Directly from the definition of the class $HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)(HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p))$ we get the following.

Theorem 2.1. *Let $(\alpha,p)\in P$. If $f\in HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)(HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p))$, then functions*

$$z\longmapsto r^{-1}f(rz),\quad z\longmapsto e^{-it}f\left(e^{it}z\right),\quad z\in E,\ r\in(0,1),\ t\in R \tag{2.1}$$

also belong to $HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)(HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p))$.

Theorem 2.2. *If $0\le\alpha_1\le\alpha_2\le 1,\ p>0$, then*

$$HS^{m,n}_{\lambda_1,\lambda_2}(\alpha_1,p)\subset HS^{m,n}_{\lambda_1,\lambda_2}(\alpha_2,p),\quad HC^{m,n}_{\lambda_1,\lambda_2}(\alpha_1,p)\subset HC^{m,n}_{\lambda_1,\lambda_2}(\alpha_2,p). \tag{2.2}$$

If $\alpha\in[0,1]$ and $0<p_1\le p_2$, then

$$HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p_1)\supset HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p_2),\quad HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p_1)\supset HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p_2). \tag{2.3}$$

Theorem 2.3. *Let $(\alpha,p)\in p$. If $p\ge 1$, then every function $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ is univalent and maps the unit disk E onto a domain starlike with respect to the origin. If $p\ge 2$, then every function $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ is univalent and maps the unit disk E onto a convex domain.*

Proof. If $p\ge 1$, then $U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)\ge n$ for $n=2,3,\ldots,\ \alpha\in[0,1]$, so by the condition (1.5) we obtain

$$\sum_{n=2}^{\infty}n(|a_n|+|b_n|)\le 1. \tag{2.4}$$

Therefore (see [2]), f is univalent and starlike with respect to the origin. If $p\ge 2$, then by (1.5) we get

$$\sum_{n=2}^{\infty}n^2(|a_n|+|b_n|)\le 1. \tag{2.5}$$

Hence (see [2]), f is convex. □

Next, let $\alpha\in[0,1]$ and set $p_1(\alpha)=1-\log_2(2-\alpha),\ p_2(\alpha)=2-\log_2(2-\alpha)$, $\log_2 1=0$. We denote

$$\begin{aligned} D_1&=\{(\alpha,p)\in P: p\ge p_1(\alpha)\},\\ D_2&=\{(\alpha,p)\in P: p\ge p_2(\alpha)\}. \end{aligned} \tag{2.6}$$

The next theorem present results concerning starlikeness and convexity of functions of the class $HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ for arbitrary $(\alpha,p)\in D_1$ and $(\alpha,p)\in D_2$, respectively.

Theorem 2.4. *If $(\alpha,p)\in D_1$, then the functions of the class $HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ are starlike.*

Proof. We can check that the following inequality:

$$U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)\geq n,\quad (\alpha,p)\in D_1,\quad n=2,3,\ldots, \tag{2.7}$$

hold. If $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ for $(\alpha,p)\in D_1$, then in view of the inequality, the condition (1.5) and of the mentioned result from [2] it follows that f is a starlike function. □

Theorem 2.5. *Let $(\alpha,p)\in p\setminus D_1$. If $r\in(0,r_0(\alpha,p))$, where $r_0(\alpha,p)=2^{p-1}(2-\alpha)$, then each function $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ maps the disk E_r onto a domain starlike with respect to the origin. where $E_r=\{z\in C:|z|<r\}$, $r>0$, with $E_1=E$.*

Proof. For $(\alpha,p)\in p\setminus D_1$, we have $r_0(\alpha,p)<1$, let $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$, $(\alpha,p)\in p\setminus D_1$, and let $r\in(0,r_0(\alpha,p))$. By Theorem 2.1, the function f_r of the form $f_r(z)=r^{-1}f(rz)$ belongs to the class $HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ and we have

$$\sum_{n=2}^{\infty}n\left(\left|a_nr^{n-1}\right|+\left|b_nr^{n-1}\right|\right)=\sum_{n=2}^{\infty}nr^{n-1}(|a_n|+|b_n|). \tag{2.8}$$

In view of properties of elementary functions, we obtain

$$nr^{n-1}\leq n\left(r_0(\alpha,p)\right)^{n-1}\leq U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p),\quad n=2,3,\ldots. \tag{2.9}$$

Hence, $f_r\in HS^{1,n}_{0,0}(1,1)$ [2] for any $r\in(0,r_0(\alpha,p))$ maps the E onto a domain starlike with respect to the origin. □

Theorem 2.6. *Let $(\alpha,p)\in p\setminus D_2$. If $r\in(0,r_0^*(\alpha,p))$, where $r_0^*(\alpha,p)=2^{p-2}(2-\alpha)$, then each function $f\in HC^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$ maps the disk E_r onto a convex domain.*

Proof. For every $(\alpha,p)\in p\setminus D_2$ we have $r_0^*(\alpha,p)<1$. Further we proceed similarly as in the proof of Theorem 2.5, we have for any $r\in(0,r_0^*(\alpha,p))$

$$n^2r^{n-1}\leq U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p),\quad n=2,3,\ldots. \tag{2.10}$$

Hence $f_r\in HC^{1,n}_{0,0}(1,1)$ [2] for any $r\in(0,r_0^*(\alpha,p))$ maps the E onto a convex domain. □

Theorem 2.7. *Let $(\alpha,p)\in P$. If $f\in HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$, $z\in E$, $z\neq 0$, then*

$$\begin{aligned}\left|f(z)\right|&\leq(1+|b_1|)|z|+\frac{1-|b_1|}{2^p(2-\alpha)}|z|^2,\\ \left|f(z)\right|&\geq(1-|b_1|)|z|-\frac{1-|b_1|}{2^p(2-\alpha)}|z|^2.\end{aligned} \tag{2.11}$$

Proof. Let $f \in HS^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)$, $(\alpha,p) \in P$, f of the form (1.3) and fix $z \in E \setminus \{0\}$. Then the condition (1.5) holds, and after simple transformations we obtain

$$\sum_{n=2}^{\infty}(|a_n| + |b_n|) \leq \frac{1-|b_1|}{U^{m,2}_{\lambda_1,\lambda_2}(\alpha,p)} - \sum_{n=3}^{\infty}\left(\frac{U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p)}{U^{m,2}_{\lambda_1,\lambda_2}(\alpha,p)} - 1\right)(|a_n| + |b_n|). \tag{2.12}$$

Since $U^{m,n}_{\lambda_1,\lambda_2}(\alpha,p) \geq U^{m,2}_{\lambda_1,\lambda_2}(\alpha,p)$, $n = 3,4,\ldots,$ $(\alpha,p) \in P$, we have

$$\sum_{n=2}^{\infty}(|a_n| + |b_n|) \leq \frac{1-|b_1|}{U^{m,2}_{\lambda_1,\lambda_2}(\alpha,p)}. \tag{2.13}$$

Hence,

$$\left|f(z)\right| \leq \sum_{n=2}^{\infty}(|a_n| + |b_n|)|z|^n + (1+|b_1|)|z| \leq (1+|b_1|)|z| + \frac{1-|b_1|}{U^{m,2}_{\lambda_1,\lambda_2}(\alpha,p)}|z|^2, \tag{2.14}$$

that is, the upper estimate.

The lower estimate follows from (2.13) and the inequality:

$$\left|f(z)\right| \geq |z| - |b_1||z| - \sum_{n=2}^{\infty}(|a_n| + |b_n|)|z|^n. \tag{2.15}$$

□

Remark 2.8. Other works related to harmonic analytic functions can be read in [7–13].

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