Research Article

A New Efficient Method for Solving Two-Dimensional Burgers' Equation

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We introduce a new hybrid of the Laplace transform method and new homotopy perturbation method (LTNHPM) that efficiently solves nonlinear two-dimensional Burgers' equation. Three examples are given to demonstrate the efficiency of the new method.

1. Introduction

The system of partial differential equation of the following form

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{R} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{1}{R} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ (x, y) \in \Omega, \quad t > 0,\end{aligned}\quad (1)$$

subject to the initial conditions:

$$\begin{aligned}u(x, y, 0) &= f(x, y), \quad (x, y) \in \Omega, \\ v(x, y, 0) &= g(x, y), \quad (x, y) \in \Omega,\end{aligned}\quad (2)$$

is called the system of two-dimensional Burgers' equation, where $\Omega = \{(x, y) | a \leq x \leq b, a \leq y \leq b\}$, $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined, f and g are known functions, and R is the Reynolds number. The Burgers model of turbulence is a very important fluid dynamic model, and the study of this model and the theory of shock waves have been considered by many authors, both to obtain a conceptual understanding of a class of physical flows and for testing various numerical methods. The mathematical properties of Burgers' equation have been studied by Burgers [1]. Nonlinear phenomena play a crucial role in applied mathematics and physics. The

importance of obtaining the exact or approximate solutions of PDEs in physics and mathematics is still a hot topic as regards seeking new methods for obtaining new exact or approximate solutions [2–5]. For that purpose, different methods have been put forward for seeking various exact solutions of multifarious physical models described using nonlinear PDEs. A well-known model was first introduced by Bateman [6], who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers [1] as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf [7] and Cole [8]. They also illustrated independently that the Burgers equation can be solved exactly for an arbitrary initial condition. Benton and Platzman [9] have surveyed the analytical solutions of the one-dimensional Burgers equation. It can be considered as a simplified form of the Navier-Stokes equation [10] due to the form of the nonlinear convection term and the occurrence of the viscosity term. The numerical solution of the Burgers equation is of great importance due to the application of the equation in the approximate theory of flow through a shock wave, travelling in a viscous fluid [8] and in the Burgers model of turbulence [11]. It can be solved analytically for arbitrary initial conditions [7]. Numerical methods such as finite difference, finite element, and classical ones like Fourier series, Fourier integral, and Laplace transformation commonly used for solving these methods either need a lot of computations and have less convergence speed and accuracy or solve only certain types of problems. Therefore,

science and engineering researchers attempt to propose new methods for solving functional equations.

In this paper, we propose a new hybrid of Laplace transform method and new homotopy perturbation method [12] to obtain exact and numerical solutions of the system of two-dimensional Burgers' equation. Finally, three examples are given to illustrate the proposed approach.

2. Analysis of the Method

For solving system of two-dimensional Burgers' equation by LTNHPM, we construct the following homotopy:

$$U_t - u_0(x, y, t) + p \left(u_0(x, y, t) + UU_x + VU_y - \frac{1}{R} (U_{xx} + U_{yy}) \right) = 0, \quad (3)$$

$$V_t - v_0(x, y, t) + p \left(v_0(x, y, t) + UV_x + VV_y - \frac{1}{R} (V_{xx} + V_{yy}) \right) = 0,$$

where $p \in [0, 1]$ is an embedding parameter and $u_0(x, y, t)$ and $v_0(x, y, t)$ are initial approximation of solution of (1).

By applying Laplace transform on both sides of (3), we have

$$\begin{aligned} & \mathcal{L} \left\{ U_t - u_0(x, y, t) \right. \\ & \quad \left. + p \left(u_0(x, y, t) + UU_x + VU_y - \frac{1}{R} (U_{xx} + U_{yy}) \right) \right\} = 0, \\ & \mathcal{L} \left\{ V_t - v_0(x, y, t) \right. \\ & \quad \left. + p \left(v_0(x, y, t) + UV_x + VV_y - \frac{1}{R} (V_{xx} + V_{yy}) \right) \right\} = 0. \end{aligned} \quad (4)$$

Using the differential property of Laplace transform we have

$$\begin{aligned} s\mathcal{L}\{U\} - U(x, y, 0) &= \mathcal{L} \left\{ u_0(x, y, t) - p \left(u_0(x, y, t) + UU_x + VU_y - \frac{1}{R} (U_{xx} + U_{yy}) \right) \right\}, \\ s\mathcal{L}\{V\} - V(x, y, 0) &= \mathcal{L} \left\{ v_0(x, y, t) - p \left(v_0(x, y, t) + UV_x + VV_y - \frac{1}{R} (V_{xx} + V_{yy}) \right) \right\}, \end{aligned} \quad (5)$$

or

$$\begin{aligned} \mathcal{L}\{U\} &= \frac{1}{s} \left(U(x, y, 0) + \mathcal{L} \left\{ u_0(x, y, t) - p \left(u_0(x, y, t) + UU_x + VU_y - \frac{1}{R} (U_{xx} + U_{yy}) \right) \right\} \right), \\ \mathcal{L}\{V\} &= \frac{1}{s} \left(V(x, y, 0) + \mathcal{L} \left\{ v_0(x, y, t) - p \left(v_0(x, y, t) + UV_x + VV_y - \frac{1}{R} (V_{xx} + V_{yy}) \right) \right\} \right). \end{aligned} \quad (6)$$

By applying inverse Laplace transform on both sides of (6), we have

$$\begin{aligned} U(x, y, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left(U(x, y, 0) \right. \right. \\ & \quad \left. \left. + \mathcal{L} \left\{ u_0(x, y, t) \right. \right. \right. \\ & \quad \left. \left. - p \left(u_0(x, y, t) + UU_x + VU_y \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1}{R} (U_{xx} + U_{yy}) \right) \right\} \right\}, \end{aligned}$$

$$\begin{aligned} V(x, y, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left(V(x, y, 0) \right. \right. \\ & \quad \left. \left. + \mathcal{L} \left\{ v_0(x, y, t) \right. \right. \right. \\ & \quad \left. \left. - p \left(v_0(x, y, t) + UV_x + VV_y \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1}{R} (V_{xx} + V_{yy}) \right) \right\} \right\}. \end{aligned} \quad (7)$$

According to the HPM, we use the embedding parameter p as a small parameter and assume that the solutions of (7) can be represented as a power series in p as

$$\begin{aligned} U &= U_0 + pU_1 + p^2U_2 + \dots, \\ V &= V_0 + pV_1 + p^2V_2 + \dots. \end{aligned} \quad (8)$$

TABLE 1: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 0.5$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $
(0.1, 0.1)	0.6249902344	$9E - 6$	0.6245117213	$4E - 4$	0.6230474455	$1E - 3$
(0.5, 0.1)	0.6234278173	$9E - 6$	0.6229494028	$2E - 3$	0.6214857057	$3E - 3$
(0.9, 0.1)	0.6218658937	$3E - 3$	0.6213877271	$3E - 3$	0.6199250641	$5E - 3$
(0.3, 0.3)	0.6249902344	$9E - 6$	0.6245117213	$4E - 4$	0.6230474455	$1E - 3$
(0.7, 0.3)	0.6234278173	$1E - 3$	0.6229494028	$2E - 3$	0.6214857057	$3E - 3$
(0.1, 0.5)	0.6265526528	$1E - 3$	0.6260741906	$1E - 3$	0.6246097925	$3E - 4$
(0.5, 0.5)	0.6249902344	$9E - 6$	0.6245117213	$4E - 4$	0.6230474455	$3E - 3$
(0.9, 0.5)	0.6234278173	$1E - 3$	0.6229494028	$2E - 3$	0.6214857057	$3E - 3$
(0.3, 0.7)	0.6265526528	$1E - 3$	0.6260741906	$1E - 3$	0.62460987925	$3E - 4$
(0.7, 0.7)	0.6249902344	$9E - 6$	0.6245117213	$4E - 4$	0.6230474455	$1E - 3$
(0.1, 0.9)	0.6281145910	$3E - 3$	0.6276363292	$2E - 3$	0.6261722650	$1E - 3$
(0.5, 0.9)	0.6265526528	$1E - 3$	0.6260741906	$1E - 3$	0.6246097925	$3E - 3$
(0.9, 0.9)	0.6249902344	$9E - 6$	0.6245117213	$4E - 4$	0.6230474455	$1E - 3$

TABLE 2: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 0.5$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $
(0.1, 0.1)	0.8750097656	$9E - 6$	0.8754882787	$4E - 4$	0.8769525545	$1E - 3$
(0.5, 0.1)	0.8765721828	$1E - 3$	0.8770505973	$2E - 3$	0.8785142944	$3E - 3$
(0.9, 0.1)	0.8781341113	$3E - 3$	0.8786122779	$3E - 3$	0.8800749409	$5E - 3$
(0.3, 0.3)	0.8750097656	$9E - 6$	0.8754882787	$4E - 4$	0.8769525545	$1E - 3$
(0.7, 0.3)	0.8765721828	$1E - 3$	0.8770505973	$2E - 3$	0.8785142944	$3E - 3$
(0.1, 0.5)	0.8734473429	$1E - 3$	0.8739258051	$1E - 3$	0.8753902032	$3E - 4$
(0.5, 0.5)	0.8750097656	$9E - 6$	0.8754882787	$4E - 4$	0.8769525545	$1E - 3$
(0.9, 0.5)	0.8765721828	$1E - 3$	0.8770505973	$2E - 3$	0.8785142944	$3E - 3$
(0.3, 0.7)	0.8734473429	$1E - 3$	0.8739258051	$1E - 3$	0.8753902032	$3E - 4$
(0.7, 0.7)	0.8750097656	$9E - 6$	0.8754882787	$4E - 4$	0.8769525545	$1E - 3$
(0.1, 0.9)	0.8753902032	$3E - 3$	0.8723636741	$2E - 3$	0.8738277383	$1E - 3$
(0.5, 0.9)	0.8734473429	$3E - 10$	0.8739258051	$1E - 3$	0.8753902032	$3E - 4$
(0.9, 0.9)	0.8750097656	$1E - 3$	0.8754882787	$4E - 4$	0.8769525545	$1E - 3$

TABLE 3: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 1$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $
(0.1, 0.1)	0.6249804688	$1E - 10$	0.6240234576	$2E - 10$	0.6210993749	$4E - 6$
(0.5, 0.1)	0.6218561341	$2E - 9$	0.6208999113	$2E - 9$	0.6179805324	$4E - 6$
(0.9, 0.1)	0.6187357207	$1E - 10$	0.6177814780	$2E - 10$	0.6148703830	$4E - 6$
(0.3, 0.3)	0.6249804688	$1E - 10$	0.6240234576	$2E - 10$	0.6210993749	$4E - 6$
(0.7, 0.3)	0.6218561341	$2E - 9$	0.6208999113	$2E - 9$	0.6179805324	$4E - 6$
(0.1, 0.5)	0.6281048314	$1E - 9$	0.6271482276	$1E - 9$	0.6242230488	$4E - 6$
(0.5, 0.5)	0.6249804688	$1E - 10$	0.6240234576	$2E - 10$	0.6210993749	$4E - 6$
(0.9, 0.5)	0.6218561341	$2E - 9$	0.6208999113	$2E - 9$	0.6179805324	$4E - 6$
(0.3, 0.7)	0.6281048314	$1E - 9$	0.6271482276	$1E - 9$	0.6242230488	$4E - 6$
(0.7, 0.7)	0.6249804688	$1E - 10$	0.6240234576	$2E - 10$	0.6210993749	$4E - 6$
(0.1, 0.9)	0.6312253142	$0E - 0$	0.6302703114	$2E - 10$	0.6273476542	$4E - 6$
(0.5, 0.9)	0.6281048314	$1E - 9$	0.6271482276	$1E - 9$	0.6242230488	$4E - 6$
(0.9, 0.9)	0.6249804688	$1E - 10$	0.6240234576	$2E - 10$	0.6210993749	$4E - 6$

TABLE 4: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 1$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $
(0.1, 0.1)	0.8750195312	$1E - 10$	0.8759765424	$2E - 10$	0.8789006251	$4E - 6$
(0.5, 0.1)	0.8781438713	$3E - 9$	0.8791000941	$3E - 9$	0.8820194730	$4E - 6$
(0.9, 0.1)	0.8812642793	$1E - 10$	0.8822185220	$2E - 10$	0.8851296170	$4E - 6$
(0.3, 0.3)	0.8750195312	$1E - 10$	0.8759765424	$2E - 10$	0.8789006251	$4E - 6$
(0.7, 0.3)	0.8781438713	$3E - 9$	0.8791000941	$3E - 9$	0.8820194730	$4E - 6$
(0.1, 0.5)	0.8718951722	$2E - 9$	0.8728517760	$2E - 9$	0.8757769548	$4E - 6$
(0.5, 0.5)	0.8750195312	$1E - 10$	0.8759765424	$2E - 10$	0.8789006251	$4E - 6$
(0.9, 0.5)	0.8781438713	$3E - 9$	0.8791000941	$3E - 9$	0.8820194730	$4E - 6$
(0.3, 0.7)	0.8718951722	$2E - 10$	0.8728517760	$2E - 9$	0.8757769548	$4E - 6$
(0.7, 0.7)	0.8750195312	$1E - 10$	0.8759765424	$2E - 10$	0.8789006251	$4E - 6$
(0.1, 0.9)	0.8687746864	$6E - 10$	0.8697296892	$4E - 10$	0.8726523464	$4E - 6$
(0.5, 0.9)	0.8718951722	$1E - 9$	0.8728517760	$2E - 9$	0.8757769548	$4E - 6$
(0.9, 0.9)	0.8750195312	$1E - 10$	0.8759765424	$2E - 10$	0.8789006251	$4E - 6$

TABLE 5: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 100$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $
(0.1, 0.1)	-0.01433515980	$4E - 10$	-0.01146399722	$3E - 6$	-0.04985414081	$4E - 2$
(0.5, 0.1)	0.01932053175	$0E - 0$	0.01517417083	$3E - 6$	-0.04626741859	$5E - 2$
(0.9, 0.1)	-0.01718352263	$4E - 19$	-0.01319841673	$1E - 5$	0.06125231399	$6E - 2$
(0.3, 0.3)	0.01130347869	$1E - 11$	0.009435085671	$1E - 6$	-0.01389095422	$1E - 2$
(0.7, 0.3)	0.02532577158	$9E - 9$	0.01777109572	$2E - 4$	0.1210568093	$1E - 1$
(0.1, 0.5)	-0.03913650004	$8E - 11$	-0.03227952068	$2E - 5$	0.2751624426	$2E - 1$
(0.5, 0.5)	0.06252255411	$1E - 11$	0.04901387388	$5E - 5$	-0.8092220313	$1E - 1$
(0.3, 0.7)	0.01130347871	$3E - 11$	0.009435085675	$1E - 6$	-0.01389095417	$1E - 2$
(0.7, 0.7)	0.02532586111	$9E - 8$	0.01777138161	$2E - 4$	0.1210567369	$1E - 1$
(0.1, 0.9)	-0.01433515851	$8E - 10$	-0.01146399707	$3E - 6$	-0.04985413728	$4E - 2$
(0.5, 0.9)	0.01932053170	$0E - 0$	0.01517417079	$3E - 6$	-0.04626741849	$5E - 2$
(0.9, 0.9)	-0.01718352694	$2E - 10$	-0.01319842063	$1E - 5$	0.06125228267	$6E - 2$

TABLE 6: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 100$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $
(0.1, 0.1)	-0.01602719682	$1E - 9$	-0.01281338076	$5E - 7$	-0.01158528430	$5E - 3$
(0.5, 0.1)	-5.521281362E - 16	$5E - 16$	-7.159309431E - 6	$7E - 6$	-0.01817103931	$1E - 2$
(0.9, 0.1)	0.01921176621	$7E - 10$	0.01476978811	$2E - 7$	-0.06185015166	$6E - 2$
(0.3, 0.3)	-0.01263767338	$1E - 11$	-0.01054980461	$7E - 7$	2.61506638E - 3	$8E - 3$
(0.7, 0.3)	0.2831506768	$5E - 9$	0.01969169193	$7E - 5$	0.6129399566	$6E - 1$
(0.1, 0.5)	-0.0000000000	$0E - 0$	-0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.5, 0.5)	0.0000000000	$0E - 0$	0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.9, 0.5)	-0.0000000000	$0E - 0$	-0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.3, 0.7)	0.01263767338	$0E - 0$	0.01054980461	$7E - 7$	-2.61506637E - 3	$8E - 3$
(0.7, 0.7)	-0.02831513946	$7E - 8$	-0.019969205679	$7E - 5$	-0.6129371973	$6E - 1$
(0.1, 0.9)	0.01602719556	$1E - 10$	0.01281338024	$5E - 7$	0.01158528474	$5E - 3$
(0.5, 0.9)	-5.521281351E - 16	$5E - 16$	7.159309412E - 6	$7E - 6$	0.01817103920	$1E - 2$
(0.9, 0.9)	-0.01146399722	$3E - 6$	-0.01476978755	$2E - 7$	0.06185014690	$6E - 2$

TABLE 7: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 500$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $	$u^*(x, y, t)$	$ u^* - u $
(0.1, 0.1)	-0.002877429822	$3E - 10$	-0.002752396466	$2E - 9$	-0.002412373414	$1E - 5$
(0.5, 0.1)	0.003879391382	$0E - 0$	0.003696240199	$3E - 9$	0.003158290011	$2E - 5$
(0.9, 0.1)	-0.003451655816	$3E - 10$	-0.003273279771	$3E - 9$	-0.002774708749	$1E - 5$
(0.3, 0.3)	0.002267155545	$2E - 12$	0.002188809320	$1E - 9$	0.001947448603	$1E - 5$
(0.7, 0.3)	0.005097690552	$5E - 9$	0.004717986784	$1E - 9$	0.003624160352	$1E - 4$
(0.1, 0.5)	-0.007851234710	$2E - 12$	-0.007561582238	$1E - 8$	-0.006599038031	$1E - 4$
(0.5, 0.5)	0.01255397423	$1E - 11$	0.01196126856	$2E - 8$	0.01020345914	$1E - 4$
(0.9, 0.5)	-0.01437772613	$1E - 8$	-0.01343545467	$1E - 7$	-0.01201493924	$1E - 3$
(0.3, 0.7)	0.002267155545	$3E - 12$	0.002188809324	$1E - 9$	0.001947448606	$1E - 5$
(0.7, 0.7)	0.005097745644	$4E - 8$	0.004718074710	$8E - 8$	0.003624161959	$1E - 4$
(0.1, 0.9)	-0.002877429571	$9E - 11$	-0.002752396375	$2E - 9$	-0.002412373226	$1E - 5$
(0.5, 0.9)	0.003879391372	$0E - 0$	0.0036962440189	$4E - 9$	0.003158290003	$2E - 5$
(0.9, 0.9)	-0.003451655844	$3E - 9$	-0.003273279804	$3E - 9$	-0.002774708685	$1E - 5$

TABLE 8: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 500$.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2$	
	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $	$v^*(x, y, t)$	$ v^* - v $
(0.1, 0.1)	-0.003217063991	$2E - 11$	-0.003077270449	$5E - 10$	-0.002683638014	$4E - 7$
(0.5, 0.1)	-4.350477708E - 20	$4E - 20$	-6.738702226E - 10	$6E - 10$	-4.937502413E - 6	$4E - 6$
(0.9, 0.1)	-0.003859067764	$1E - 9$	0.003659637690	$4E - 9$	0.003090544099	$2E - 5$
(0.3, 0.3)	-0.002534756956	$0E - 0$	-0.002447164709	$5E - 10$	-0.002188737874	$2E - 6$
(0.7, 0.3)	0.005699386966	$1E - 8$	0.005274890097	$2E - 8$	0.004315287928	$1E - 4$
(0.1, 0.5)	-0.0000000000	$0E - 0$	-0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.5, 0.5)	0.0000000000	$0E - 0$	0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.9, 0.5)	-0.0000000000	$0E - 0$	-0.0000000000	$0E - 0$	0.0000000000	$0E - 0$
(0.3, 0.7)	0.002534756959	$0E - 0$	0.002447164718	$5E - 10$	0.002188737877	$2E - 6$
(0.7, 0.7)	-0.005699453379	$5E - 8$	-0.005275019330	$1E - 7$	-0.004315322934	$1E - 4$
(0.1, 0.9)	0.003217064054	$9E - 11$	0.003077270418	$4E - 10$	0.002683637890	$4E - 7$
(0.5, 0.9)	4.350477662E - 20	$4E - 20$	6.738702149E - 10	$6E - 10$	4.937502361E - 6	$4E - 6$
(0.9, 0.9)	-0.003859067867	$1E - 9$	-0.003659637235	$4E - 9$	-0.003090543816	$2E - 5$

Substituting (8) into (7) and equating the terms with the identical powers of p lead to

$$\begin{aligned}
p^0 : & \begin{cases} U_0 = L^{-1} \left\{ \frac{1}{s} (U(x, y, 0) + L\{u_0(x, y, t)\}) \right\}, \\ V_0 = L^{-1} \left\{ \frac{1}{s} (V(x, y, 0) + L\{v_0(x, y, t)\}) \right\}, \end{cases} \\
p^1 : & \begin{cases} U_1 = L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ u_0(x, y, t) + U_0(U_0)_x + V_0(U_0)_y - \frac{1}{R} ((U_0)_{xx} + (U_0)_{yy}) \right\} \right) \right\}, \\ V_1 = L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ v_0(x, y, t) + U_0(V_0)_x + V_0(V_0)_y - \frac{1}{R} ((V_0)_{xx} + (V_0)_{yy}) \right\} \right) \right\}, \end{cases} \\
p^2 : & \begin{cases} U_2 = L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(U_1)_x + U_1(U_0)_x + V_0(U_1)_y + V_1(U_0)_y - \frac{1}{R} ((U_1)_{xx} + (U_1)_{yy}) \right\} \right) \right\}, \\ V_2 = L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(V_1)_x + U_1(V_0)_x + V_0(V_1)_y + V_1(V_0)_y - \frac{1}{R} ((V_1)_{xx} + (V_1)_{yy}) \right\} \right) \right\}, \end{cases} \\
& \vdots
\end{aligned}$$

$$\begin{aligned}
p^j : \left\{ \begin{aligned} U_j &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ \sum_{k=0}^{j-1} \left(U_k (U_{j-k-1})_x + V_k (U_{j-k-1})_y \right) - \frac{1}{R} \left((U_{j-1})_{xx} + (U_{j-1})_{yy} \right) \right\} \right) \right\}, \\ V_j &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ \sum_{k=0}^{j-1} \left(U_k (V_{j-k-1})_x + V_k (V_{j-k-1})_y \right) - \frac{1}{R} \left((V_{j-1})_{xx} + (V_{j-1})_{yy} \right) \right\} \right) \right\}, \end{aligned} \right. \\
&\vdots
\end{aligned} \tag{9}$$

Suppose that the initial approximation has the form $U(x, y, 0) = u_0(x, y, t) = f(x, y)$ and $V(x, y, 0) = v_0(x, y, t) = g(x, y)$; therefore the exact solution may be obtained as follows

$$\begin{aligned}
u(x, y, t) &= \lim_{p \rightarrow 1} U(x, y, t) = U_0(x, y, t) + U_1(x, y, t) + \cdots, \\
v(x, y, t) &= \lim_{p \rightarrow 1} V(x, y, t) = V_0(x, y, t) + V_1(x, y, t) + \cdots.
\end{aligned} \tag{10}$$

subject to the initial condition

$$\begin{aligned}
u(x, y, 0) &= x + y, \\
v(x, y, 0) &= x - y.
\end{aligned} \tag{12}$$

3. Examples

Example 1. Consider the following homogeneous form of a coupled Burgers equation [13]:

$$\begin{aligned}
u_t + uu_x + vu_y &= \frac{1}{R} (u_{xx} + u_{yy}), \\
v_t + uv_x + vv_y &= \frac{1}{R} (v_{xx} + v_{yy}),
\end{aligned} \tag{11}$$

The exact solution of this equation is $u(x, y, t) = (x + y - 2xt)/(1 - 2t^2)$ and $v(x, y, t) = (x - y - 2yt)/(1 - 2t^2)$. Starting with $U(x, y, 0) = u_0 = x + y$, $V(x, y, 0) = v_0 = x - y$ and using (9), we obtain

$$\begin{aligned}
U_0 &= L^{-1} \left\{ \frac{1}{s} (x + y + L\{x + y\}) \right\} = (x + y)(1 + t), \\
V_0 &= L^{-1} \left\{ \frac{1}{s} (x - y + L\{x - y\}) \right\} = (x - y)(1 + t), \\
U_1 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ x + y + U_0(U_0)_x + V_0(U_0)_y - \frac{1}{R} ((U_0)_{xx} + (U_0)_{yy}) \right\} \right) \right\} \\
&= -(3x + y)t - 2xt^2 - \frac{2}{3}xt^3, \\
V_1 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ x - y + U_0(V_0)_x + V_0(V_0)_y - \frac{1}{R} ((V_0)_{xx} + (V_0)_{yy}) \right\} \right) \right\} \\
&= -(x + y)t - 2yt^2 - \frac{2}{3}yt^3, \\
U_2 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(U_1)_x + U_1(U_0)_x + V_0(U_1)_y + V_1(U_0)_y - \frac{1}{R} ((U_1)_{xx} + (U_1)_{yy}) \right\} \right) \right\} \\
&= (4x + 2y)t^2 + \left(4x + \frac{8}{3}y\right)t^3 + \left(\frac{4}{3}x + \frac{4}{3}y\right)t^4 + \left(\frac{4}{15}x + \frac{4}{15}y\right)t^5, \\
V_2 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(V_1)_x + U_1(V_0)_x + V_0(V_1)_y + V_1(V_0)_y - \frac{1}{R} ((V_1)_{xx} + (V_1)_{yy}) \right\} \right) \right\} \\
&= 2xt^2 + \left(\frac{8}{3}x - \frac{4}{3}y\right)t^3 + \left(\frac{4}{3}x - \frac{4}{3}y\right)t^4 + \left(\frac{4}{15}x - \frac{4}{15}y\right)t^5,
\end{aligned}$$

$$\begin{aligned}
U_3 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(U_2)_x + U_1(U_1)_x + U_2(U_0)_x + V_0(U_2)_y + V_1(U_1)_y + V_2(U_0)_y - \frac{1}{R} \left((U_2)_{xx} + (U_2)_{yy} \right) \right\} \right) \right\} \\
&= - \left(\frac{22}{3}x + \frac{8}{3}y \right) t^3 - \left(\frac{28}{3}x + \frac{8}{3}y \right) t^4 - \left(\frac{16}{3}x + \frac{4}{5}y \right) t^5 - \frac{68}{45}xt^6 - \frac{68}{315}xt^7, \\
V_3 &= L^{-1} \left\{ -\frac{1}{s} \left(L \left\{ U_0(V_2)_x + U_1(V_1)_x + U_2(V_0)_x + V_0(V_2)_y + V_1(V_1)_y + V_2(V_0)_y - \frac{1}{R} \left((V_2)_{xx} + (V_2)_{yy} \right) \right\} \right) \right\} \\
&= - \left(\frac{8}{3}x + 2y \right) t^3 - \left(\frac{8}{3}x + 4y \right) t^4 - \left(\frac{4}{5}x + \frac{56}{15}y \right) t^5 - \frac{68}{45}yt^6 - \frac{68}{315}yt^7 \\
&\vdots
\end{aligned} \tag{13}$$

Therefore we gain the solution of (11) as

$$\begin{aligned}
u(x, y, t) &= U_0(x, y, t) + U_1(x, y, t) + U_3(x, y, t) + \dots \\
&= x + y - 2xt + 2xt^2 + 2yt^2 - 4xt^3 \\
&\quad + 4xt^4 + 4yt^4 - 8xt^5 + \dots \\
&= x(1 + 2t^2 + 4t^4 + \dots) + y(1 + 2t^2 + 4t^4 + \dots) \\
&\quad - 2xt(1 + 2t^2 + 4t^4 + \dots) \\
&= \frac{x + y - 2xt}{1 - 2t^2}, \\
v(x, y, t) &= V_0(x, y, t) + V_1(x, y, t) + V_3(x, y, t) + \dots \\
&= x - y - 2yt + 2xt^2 - 2yt^2 - 4yt^3 \\
&\quad + 4xt^4 - 4yt^4 - 8yt^5 + \dots \\
&= x(1 + 2t^2 + 4t^4 + \dots) \\
&\quad - y(1 + 2t^2 + 4t^4 + \dots) \\
&\quad - 2yt(1 + 2t^2 + 4t^4 + \dots) \\
&= \frac{x - y - 2yt}{1 - 2t^2}
\end{aligned} \tag{14}$$

which is exact solution.

Example 2. Let us consider system of Burgers' equations (1), with the following initial conditions [14]:

$$\begin{aligned}
u(x, y, 0) &= \frac{3}{4} - \frac{1}{4[1 + \exp(y - x)R/8]}, \\
v(x, y, 0) &= \frac{3}{4} + \frac{1}{4[1 + \exp(y - x)R/8]},
\end{aligned} \tag{15}$$

for which exact solutions are

$$\begin{aligned}
u(x, y, t) &= \frac{3}{4} - \frac{1}{4[1 + \exp(4y - 4x - t)R/32]}, \\
v(x, y, t) &= \frac{3}{4} + \frac{1}{4[1 + \exp(4y - 4x - t)R/32]}.
\end{aligned} \tag{16}$$

To solve system (1) by LTNHPM, following the same procedure discussed in Section 2 and Example 1, we obtain the iterative relations (9); in this example we take initial approximations (15). The accuracy of LTNHPM for the system of two-dimensional Burgers' equation agrees good with the exact solution, and absolute errors are very small for the present choice of x , y , and t . These results are listed in Tables 1, 2, 3, and 4 for $R = 0.5$ and $R = 1$.

Example 3. Let us consider system of Burgers' equations (8), with the following initial conditions [14]:

$$\begin{aligned}
u(x, y, 0) &= -\frac{4\pi \cos(2\pi x) \sin(\pi y)}{R(2 + \sin(2\pi x) \sin(\pi y))}, \\
v(x, y, 0) &= -\frac{2\pi \sin(2\pi x) \cos(\pi y)}{R(2 + \sin(2\pi x) \sin(\pi y))},
\end{aligned} \tag{17}$$

for which exact solutions are

$$\begin{aligned}
u(x, y, t) &= -\frac{4\pi \exp(-5\pi^2 t/R) \cos(2\pi x) \sin(\pi y)}{R(2 + \exp(-5\pi^2 t/R) \sin(2\pi x) \sin(\pi y))}, \\
v(x, y, t) &= -\frac{2\pi \exp(-5\pi^2 t/R) \sin(2\pi x) \cos(\pi y)}{R(2 + \exp(-5\pi^2 t/R) \sin(2\pi x) \sin(\pi y))}.
\end{aligned} \tag{18}$$

To solve system (1) by LTNHPM, following the same procedure discussed in Section 2 and Example 1, we obtain the iterative relations (9); in this example we take initial approximations (17). The accuracy of LTNHPM for the system of two-dimensional Burgers' equation agrees good with the exact solution, and absolute errors are very small for the present choice of x , y , and t . These results are listed in Tables 5, 6, 7, and 8 for $R = 100$ and $R = 500$.

4. Conclusions

In this work, we considered a new hybrid of Laplace transform method and homotopy perturbation method (LTNHPM) for solving system of two-dimensional Burgers' equation. Using this method we obtained new efficient relations to solve these systems. New method is a powerful straightforward method. The LTNHPM is apt to be utilized as an alternative approach to current techniques being employed to a wide variety of mathematical problems.

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