Research Article

Ulam-Hyers-Rassias Stability of a Hyperbolic Partial Differential Equation

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We consider a nonlinear hyperbolic partial differential equation in a general form. Using a Gronwall-type lemma we prove results on the Ulam-Hyers stability and the generalised Ulam-Hyers-Rassias stability of this equation.

1. Introduction

Results on Ulam stability for the functional equations are well known (see, e.g., [1–4]). In this paper we consider different types of Ulam stability of a nonlinear hyperbolic partial differential equation. Following results from Rus [5] and Rus and Lungu [6], we present some Ulam-Hyers and Ulam-Hyers-Rassias stabilities for the following equation:

$$\frac{\partial^2 u}{\partial x \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right), \quad 0 \leq x < a, 0 \leq y < b, \quad (1.1)$$

where $f \in C([0, a) \times [0, b) \times \mathbb{B}^3, \mathbb{B})$ and $(\mathbb{B}, |\cdot|)$ is a real or complex Banach space.

2. Preliminaries

In this section we present some definitions and results on different types of Ulam stability for a hyperbolic partial differential equation. We also present a Gronwall type lemma, which will be useful in the proof of the main results.

Let $a, b \in (0, \infty]$, $\varepsilon > 0$, $\varphi \in C([0, a] \times [0, b], \mathbb{R}_+)$ and $(\mathbb{B}, |\cdot|)$ be a real or complex Banach space.

We consider the following hyperbolic partial differential equation:

$$\frac{\partial^2 u}{\partial x \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right), \quad 0 \leq x < a, 0 \leq y < b, \quad (2.1)$$

where $f \in C([0, a] \times [0, b] \times \mathbb{B}^3, \mathbb{B})$.

We also consider the following inequalities:

$$\left| \frac{\partial^2 v}{\partial x \partial y}(x, y) - f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) \right| \leq \varepsilon, \quad x \in [0, a], y \in [0, b], \quad (2.2)$$

$$\left| \frac{\partial^2 v}{\partial x \partial y}(x, y) - f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) \right| \leq \varphi(x, y), \quad x \in [0, a], y \in [0, b], \quad (2.3)$$

$$\left| \frac{\partial^2 v}{\partial x \partial y}(x, y) - f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) \right| \leq \varepsilon \varphi(x, y), \quad x \in [0, a], y \in [0, b]. \quad (2.4)$$

Definition 2.1. A function u is a solution to (2.1) if

$$u \in C^1([0, a] \times [0, b], \mathbb{B}), \quad \frac{\partial^2 u}{\partial x \partial y} \in C([0, a] \times [0, b], \mathbb{B}) \quad (2.5)$$

and u satisfies (2.1).

From [5, 6] we have the following definitions and results.

Definition 2.2. Equation (2.1) is Ulam-Hyers stable if there exist the real numbers C_f^1, C_f^2 , and $C_f^3 > 0$ such that for any $\varepsilon > 0$ and for any solution v to the inequality (2.2) there exists a solution u to (2.1) with

$$\begin{aligned} |v(x, y) - u(x, y)| &\leq C_f^1 \varepsilon, \quad \forall x \in [0, a], \forall y \in [0, b], \\ \left| \frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial x}(x, y) \right| &\leq C_f^2 \varepsilon, \quad \forall x \in [0, a], \forall y \in [0, b], \\ \left| \frac{\partial v}{\partial y}(x, y) - \frac{\partial u}{\partial y}(x, y) \right| &\leq C_f^3 \varepsilon, \quad \forall x \in [0, a], \forall y \in [0, b]. \end{aligned} \quad (2.6)$$

Definition 2.3. Equation (2.1) is generalised Ulam-Hyers-Rassias stable if there exist the real numbers $C_{f,\varphi}^1, C_{f,\varphi}^2$, and $C_{f,\varphi}^3 > 0$ such that for any $\varepsilon > 0$ and for any solution v to the inequality (2.3) there exists a solution u to (2.1) with

$$\begin{aligned} |v(x, y) - u(x, y)| &\leq C_{f,\varphi}^1 \varphi(x, y), \quad \forall x \in [0, a), \forall y \in [0, b), \\ \left| \frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial x}(x, y) \right| &\leq C_{f,\varphi}^2 \varphi(x, y), \quad \forall x \in [0, a), \forall y \in [0, b), \\ \left| \frac{\partial v}{\partial y}(x, y) - \frac{\partial u}{\partial y}(x, y) \right| &\leq C_{f,\varphi}^3 \varphi(x, y), \quad \forall x \in [0, a), \forall y \in [0, b). \end{aligned} \quad (2.7)$$

Remark 2.4. A function v is a solution to the inequality (2.2) if and only if, there exists a function $g \in C([0, a) \times [0, b), \mathbb{B})$, which depends on v , such that

- (i) For all $\varepsilon > 0$, $|g(x, y)| \leq \varepsilon$, for all $x \in [0, a)$, for all $y \in [0, b)$;
- (ii) For all $x \in [0, a)$, for all $y \in [0, b)$:

$$\frac{\partial^2 v}{\partial x \partial y}(x, y) = f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) + g(x, y). \quad (2.8)$$

Remark 2.5. A function v is a solution to the inequality (2.3) if and only if, there exists a function $g \in C([0, a) \times [0, b), \mathbb{B})$, which depends on v , such that

- (i) $|g(x, y)| \leq \varphi(x, y)$, for all $x \in [0, a)$, for all $y \in [0, b)$;
- (ii) For all $x \in [0, a)$, for all $y \in [0, b)$:

$$\frac{\partial^2 v}{\partial x \partial y}(x, y) = f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) + g(x, y). \quad (2.9)$$

Remark 2.6. A function v is a solution to the inequality (2.4) if and only if, there exists a function $g \in C([0, a) \times [0, b), \mathbb{B})$, which depends on v , such that

- (i) For all $\varepsilon > 0$, $|g(x, y)| \leq \varepsilon \varphi(x, y)$, for all $x \in [0, a)$, for all $y \in [0, b)$;
- (ii) For all $x \in [0, a)$, for all $y \in [0, b)$:

$$\frac{\partial^2 v}{\partial x \partial y}(x, y) = f\left(x, y, v(x, y), \frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)\right) + g(x, y). \quad (2.10)$$

Throughout this paper we denote

$$\begin{aligned} u_1(x, y) &= \frac{\partial u}{\partial x}(x, y), & u_2(x, y) &= \frac{\partial u}{\partial y}(x, y), \\ v_1(x, y) &= \frac{\partial v}{\partial x}(x, y), & v_2(x, y) &= \frac{\partial v}{\partial y}(x, y). \end{aligned} \quad (2.11)$$

Theorem 2.7. *If v is a solution to the inequality (2.2), then (v, v_1, v_2) satisfies the following system of integral inequalities:*

$$\begin{aligned} \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| &\leq \varepsilon xy, \\ \left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| &\leq \varepsilon y, \\ \left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| &\leq \varepsilon x, \end{aligned} \quad (2.12)$$

for all $x \in [0, a)$ and $y \in [0, b)$.

Proof. By Remark 2.4 we have that

$$\begin{aligned} v(x, y) &= v(x, 0) + v(0, y) - v(0, 0) \\ &\quad + \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt + \int_0^x \int_0^y g(s, t) ds dt \\ v_1(x, y) &= v_1(x, 0) + \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt + \int_0^y g(x, t) dt \\ v_2(x, y) &= v_2(0, y) + \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds + \int_0^x g(s, y) ds, \end{aligned} \quad (2.13)$$

and we have the following inequalities:

$$\begin{aligned} \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| \\ = \left| \int_0^x \int_0^y g(s, t) ds dt \right| \leq \int_0^x \int_0^y |g(s, t)| ds dt \leq \varepsilon xy, \\ \left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| \leq \int_0^y |g(x, t)| dt \leq \varepsilon y, \\ \left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| \leq \int_0^x |g(s, y)| ds \leq \varepsilon x. \end{aligned} \quad (2.14)$$

□

The following two theorems are obtained in a similar fashion.

Theorem 2.8. *If v is a solution to the inequality (2.3), then (v, v_1, v_2) satisfies the following system of integral inequalities:*

$$\begin{aligned}
 & \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| \\
 & \leq \int_0^x \int_0^y \varphi(s, t) ds dt, \\
 & \left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| \leq \int_0^y \varphi(x, t) dt, \\
 & \left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| \leq \int_0^x \varphi(s, y) ds,
 \end{aligned} \tag{2.15}$$

for all $x \in [0, a]$ and $y \in [0, b]$.

Theorem 2.9. *If v is a solution to the inequality (2.4), then (v, v_1, v_2) satisfies the following system of integral inequalities:*

$$\begin{aligned}
 & \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| \\
 & \leq \varepsilon \int_0^x \int_0^y \varphi(s, t) ds dt, \\
 & \left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| \leq \varepsilon \int_0^y \varphi(x, t) dt, \\
 & \left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| \leq \varepsilon \int_0^x \varphi(s, y) ds,
 \end{aligned} \tag{2.16}$$

for all $x \in [0, a]$ and $y \in [0, b]$.

The following Gronwall type lemma is an important tool in proving the main results of this paper.

Lemma 2.10 (see [7], also [8]). *One assumes that*

- (i) $u, v, h \in C(\mathbb{R}_+^n, \mathbb{R}_+)$;
- (ii) for any $t \geq t_0$ one has

$$u(t) \leq h(t) + \int_{t_0}^t v(s)u(s)ds; \tag{2.17}$$

- (iii) $h(t)$ is positive and nondecreasing.

Then,

$$u(t) \leq h(t) \exp \int_s^t v(r)dr, \quad \text{for any } t \geq t_0. \tag{2.18}$$

3. Ulam-Hyers Stability

In this section we present a result on the existence and uniqueness of the solution to (2.1) and derive a result on Ulam-Hyers stability for the same equation in the case of $a < \infty$ and $b < \infty$.

Theorem 3.1. *One assumes that*

- (i) $a < \infty, b < \infty$;
- (ii) $f \in C([0, a] \times [0, b] \times \mathbb{B}^3, \mathbb{B})$;
- (iii) *there exists $L_f > 0$ such that*

$$|f(x, y, z_1, z_2, z_3) - f(x, y, t_1, t_2, t_3)| \leq L_f \max\{|z_i - t_i|, i = 1, 2, 3\}, \quad (3.1)$$

for all $x \in [0, a], y \in [0, b]$ and $z_1, z_2, z_3, t_1, t_2, t_3 \in \mathbb{B}$.

Then,

- (a) *for $\phi \in C^1([0, a], \mathbb{B})$ and $\psi \in C^1([0, b], \mathbb{B})$ (2.1) has a unique solution, which satisfies*

$$\begin{aligned} u(x, 0) &= \phi(x), \quad \forall x \in [0, a], \\ u(0, y) &= \psi(y), \quad \forall y \in [0, b]; \end{aligned} \quad (3.2)$$

- (b) *(2.1) is Ulam-Hyers stable.*

Proof. (a) This is a known result (see, e.g. [9, 10]).

(b) Let v be a solution to the inequality (2.2) and let u be the unique solution to (2.1), which satisfies the following conditions:

$$\begin{aligned} u(x, 0) &= v(x, 0), \quad \forall x \in [0, a], \\ u(0, y) &= v(0, y), \quad \forall y \in [0, b]. \end{aligned} \quad (3.3)$$

From Theorem 2.7, the hypothesis (iii), and Gronwall Lemma 2.10, it follows that

$$\begin{aligned} &|v(x, y) - u(x, y)| \\ &\leq \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right. \\ &\quad \left. + \int_0^x \int_0^y |f(s, t, v(s, t), v_1(s, t), v_2(s, t)) - f(s, t, u(s, t), u_1(s, t), u_2(s, t))| ds dt \right| \\ &\leq \varepsilon xy + L_f \int_0^x \int_0^y \max_{i \in \{1, 2, 3\}} |v_i(s, t) - u_i(s, t)| ds dt \\ &\leq \varepsilon ab \exp(L_f ab) = C_f^1 \varepsilon, \quad \text{where } C_f^1 := ab \exp(L_f ab). \end{aligned} \quad (3.4)$$

Similarly we have

$$\begin{aligned}
& |v_1(x, y) - u_1(x, y)| \\
& \leq \left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| \\
& \quad + \int_0^y |f(x, t, v(x, t), v_1(x, t), v_2(x, t)) - f(x, t, u(x, t), u_1(x, t), u_2(x, t))| dt \\
& \leq \varepsilon y + L_f \int_0^y \max_{i \in \{1, 2, 3\}} |v_i(x, t) - u_i(x, t)| dt \\
& \leq \varepsilon b \exp(L_f b) = C_f^2 \varepsilon, \quad \text{where } C_f^2 := b \exp(L_f b), \\
& |v_2(x, y) - u_2(x, y)| \\
& \leq \left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| \\
& \quad + \int_0^x |f(s, y, v(s, y), v_1(s, y), v_2(s, y)) - f(s, y, u(s, y), u_1(s, y), u_2(s, y))| ds \\
& \leq \varepsilon x + L_f \int_0^x \max_{i \in \{1, 2, 3\}} |v_i(s, y) - u_i(s, y)| ds \\
& \leq \varepsilon a \exp(L_f a) = C_f^3 \varepsilon, \quad \text{where } C_f^3 := a \exp(L_f a).
\end{aligned} \tag{3.5}$$

□

Remark 3.2. In general, if $a = \infty$ or $b = \infty$, then (2.1) is not Ulam-Hyers stable.

4. Generalised Ulam-Hyers-Rassias Stability

In this section we prove the generalised Ulam-Hyers-Rassias stability of the hyperbolic partial differential equation (2.1). We consider (2.1) and the inequality (2.3) in the case $a = \infty$ and $b = \infty$.

Theorem 4.1. *One assumes that*

- (i) $C([0, \infty) \times [0, \infty) \times \mathbb{B}^3, \mathbb{B})$;
- (ii) *there exists $l_f \in C^1([0, \infty) \times [0, \infty), \mathbb{R}_+)$ such that*

$$|f(x, y, z_1, z_2, z_3) - f(x, y, t_1, t_2, t_3)| \leq l_f(x, y) \max\{|z_i - t_i|, i = 1, 2, 3\}, \tag{4.1}$$

for all $x, y \in [0, \infty)$;

(iii) there exist $\lambda_\varphi^1, \lambda_\varphi^2, \lambda_\varphi^3 > 0$ such that

$$\begin{aligned} \int_0^x \int_0^y \varphi(s, t) ds dt &\leq \lambda_\varphi^1 \varphi(x, y), \quad \forall x, y \in [0, \infty), \\ \int_0^y \varphi(x, t) dt &\leq \lambda_\varphi^2 \varphi(x, y), \quad \forall x, y \in [0, \infty), \\ \int_0^x \varphi(s, y) ds &\leq \lambda_\varphi^3 \varphi(x, y), \quad \forall x, y \in [0, \infty); \end{aligned} \quad (4.2)$$

(iv) $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing.

Then (2.1) ($a = \infty$ and $b = \infty$) is generalised Ulam-Hyers-Rassias stable.

Proof. Let v be a solution to the inequality (2.3). Denote by u the unique solution to the Darboux problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y}(x, y) &= f(x, y, u(x, y), u_1(x, y), u_2(x, y)), \quad \forall x, y \in [0, \infty), \\ u(x, 0) &= v(x, 0), \quad \forall x \in [0, \infty), \\ u(0, y) &= v(0, y), \quad \forall y \in [0, \infty). \end{aligned} \quad (4.3)$$

If u is a solution to the Darboux problem (4.3), then (u, u_1, u_2) is a solution to the following system:

$$\begin{aligned} u(x, y) &= v(x, 0) + v(0, y) - v(0, 0) + \int_0^x \int_0^y f(s, t, u(s, t), u_1(s, t), u_2(s, t)) ds dt, \\ u_1(x, y) &= v_1(x, 0) + \int_0^y f(x, t, u(x, t), u_1(x, t), u_2(x, t)) dt, \\ u_2(x, y) &= v_2(0, y) + \int_0^x f(s, y, u(s, y), u_1(s, y), u_2(s, y)) ds. \end{aligned} \quad (4.4)$$

From Theorem 2.8 and the hypothesis (iii), it follows that

$$\begin{aligned} &\left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| \\ &\leq \int_0^x \int_0^y \varphi(s, t) ds dt \leq \lambda_\varphi^1 \varphi(x, y), \quad \forall x, y \in [0, \infty), \\ &\left| v_1(x, y) - v_1(x, 0) - \int_0^y f(x, t, v(x, t), v_1(x, t), v_2(x, t)) dt \right| \\ &\leq \int_0^y \varphi(x, t) dt \leq \lambda_\varphi^2 \varphi(x, y), \quad \forall x, y \in [0, \infty), \\ &\left| v_2(x, y) - v_2(0, y) - \int_0^x f(s, y, v(s, y), v_1(s, y), v_2(s, y)) ds \right| \\ &\leq \int_0^x \varphi(s, y) ds \leq \lambda_\varphi^3 \varphi(x, y), \quad \forall x, y \in [0, \infty). \end{aligned} \quad (4.5)$$

Using (4.5) gives us

$$\begin{aligned}
& |v(x, y) - u(x, y)| \\
& \leq \left| v(x, y) - v(x, 0) - v(0, y) + v(0, 0) - \int_0^x \int_0^y f(s, t, v(s, t), v_1(s, t), v_2(s, t)) ds dt \right| \\
& \quad + \int_0^x \int_0^y |f(s, t, v(s, t), v_1(s, t), v_2(s, t)) - f(s, t, u(s, t), u_1(s, t), u_2(s, t))| ds dt \\
& \leq \lambda_\varphi^1 \varphi(x, y) + \int_0^x \int_0^y l_f(s, t) \max_{i \in \{1, 2, 3\}} (|v_i(s, t) - u_i(s, t)|) ds dt.
\end{aligned} \tag{4.6}$$

From Lemma 2.10 it follows that

$$\begin{aligned}
|v(x, y) - u(x, y)| & \leq \lambda_\varphi^1 \exp\left(\int_0^\infty \int_0^\infty l_f(s, t) ds dt\right) \varphi(x, y), \quad \forall x, y \in [0, \infty) \text{ or} \\
|v(x, y) - u(x, y)| & \leq C_{f, \varphi}^1 \varphi(x, y), \quad \forall x, y \in [0, \infty),
\end{aligned} \tag{4.7}$$

where $C_{f, \varphi}^1 := \lambda_\varphi^1 \exp \int_0^\infty \int_0^\infty l_f(s, t) ds dt$.

Similarly, we have

$$|v_1(x, y) - u_1(x, y)| \leq \lambda_\varphi^2 \varphi(x, y) + \int_0^y l_f(x, t) \max_{i \in \{1, 2, 3\}} (|v_i(x, t) - u_i(x, t)|) dt, \tag{4.8}$$

and from Lemma 2.10 we get

$$\begin{aligned}
|v_1(x, y) - u_1(x, y)| & \leq \lambda_\varphi^2 \exp\left(\int_0^\infty l_f(x, t) dt\right) \varphi(x, y), \quad \forall x, y \in [0, \infty) \text{ or} \\
|v_1(x, y) - u_1(x, y)| & \leq C_{f, \varphi}^2 \varphi(x, y), \quad \forall x, y \in [0, \infty),
\end{aligned} \tag{4.9}$$

where $C_{f, \varphi}^2 := \lambda_\varphi^2 \exp \int_0^\infty l_f(x, t) dt$.

Also,

$$|v_2(x, y) - u_2(x, y)| \leq \lambda_\varphi^3 \varphi(x, y) + \int_0^x l_f(s, y) \max_{i \in \{1, 2, 3\}} (|v_i(s, y) - u_i(s, y)|) ds. \tag{4.10}$$

By using Lemma 2.10 we obtain

$$\begin{aligned}
|v_2(x, y) - u_2(x, y)| & \leq \lambda_\varphi^3 \exp\left(\int_0^\infty l_f(s, y) ds\right) \varphi(x, y), \quad \forall x, y \in [0, \infty) \text{ or} \\
|v_2(x, y) - u_2(x, y)| & \leq C_{f, \varphi}^3 \varphi(x, y), \quad \forall x, y \in [0, \infty),
\end{aligned} \tag{4.11}$$

where $C_{f, \varphi}^3 := \lambda_\varphi^3 \exp \int_0^\infty l_f(s, y) ds$.

So, (2.1) is generalised Ulam-Hyers-Rassias stable. $\square$

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