Research Article

On Maximum Principles for m -Metaharmonic Equations

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We study homogeneous linear elliptic partial differential equations of even order. Several maximum principle results are deduced for such equations as well as a priori bounds for certain boundary value problems.

1. Introduction

The P -function technique for obtaining subharmonic functions defined on the solution of certain partial differential equations of order ≥ 4 is well established. In [1] Miranda shows that the functional $P = u_{,i}u_{,i} - u\Delta u$ is subharmonic where u is a solution to the biharmonic equation $\Delta^2 u = 0$. Later, in [2], Payne uses functionals containing the square of the second gradient of the solution to semilinear equations of the form

$$\Delta^2 u = f(u) \quad (1.1)$$

to deduce integral bounds on $(\Delta u)^2$.

Other works such as [3, 4] develop maximum principle results for the sixth-order equations of the form

$$\Delta^3 u - a\Delta^2 u + b\Delta u - cu = 0 \quad (1.2)$$

with constant coefficients. In [5, 6], P -functions containing the squares of terms of the form $\Delta^i u$ are used to obtain a priori that bounds for solutions to the constant coefficient m -metaharmonic equation

$$\Delta^m u + a_{m-1} \Delta^{m-1} u + \cdots + a_0 u = 0. \quad (1.3)$$

Most recently the authors in [7] obtain maximum principles results for the more general variable coefficient m -metaharmonic equation

$$\Delta^m u - a_{m-1}(x) \Delta^{m-1} u + a_{m-2}(x) \Delta^{m-2} u - \cdots + (-1)^m a_0(x) u = 0 \quad (1.4)$$

using P -functions containing terms of the form $(\Delta^i u)^2$. For three special cases, namely, when $m = 3$, $m = 4$, and when (1.4) reduces to the equation $\Delta^m u = 0$ for any integer $m \geq 2$, a more complicated class of P -functions containing the squares of certain gradient terms is used. An open question arising from [7] is whether maximum principle results for (1.4) can be obtained for say any integer $m > 4$ for the latter class of P -functions. In this work, we establish such results by requiring that certain bounds and differential inequalities for the coefficient functions $a_0(x), \dots, a_{m-2}(x)$ hold. Then we obtain integral bounds on various gradient terms.

2. Assumptions and Results

Throughout this work we assume that Ω is a bounded domain in $\mathbf{R}^n$ whose boundary $\partial\Omega$ is sufficiently smooth, that the integer $n \geq 2$ is even (without loss of generality), and that the integer $m > 4$. We identify the products of the first, second, and third gradients of the functions v and w as follows:

$$v_{,i} w_{,i} \equiv \nabla v \cdot \nabla w, v_{,ij} w_{,ij} \equiv \nabla^2 v : \nabla^2 w, v_{,ijk} w_{,ijk} \equiv \nabla^3 v : \nabla^3 w, \quad (2.1)$$

where the commas denote partial differentiation.

For functions $a_i(x) \in C^2(\Omega)$ we impose the boundedness conditions

$$\sum_{i=0}^{m-3} a_i^2 \leq \beta, \quad \sum_{i=0}^{m-1} \nabla a_i \cdot \nabla a_i \leq \gamma, \quad (2.2)$$

for constants $\gamma \geq 0, \beta > 0$.

Finally, Δ denotes the Laplace operator, $\Delta^m \equiv \Delta(\Delta^{m-1})$, and $\Delta^0 \equiv I$.

Now we derive two general maximum principle results for (1.4).

Theorem 2.1. Suppose that $u \in C^{2m+1}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ is a solution of (1.4). Furthermore for $n > 4$ one defines

$$\begin{aligned} P = & \nabla^2(\Delta^{m-2}u) \cdot \nabla^2(\Delta^{m-2}u) - \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-1}u) \\ & + \frac{a_{m-1}}{2} \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-2}u) + \frac{a_{m-3}}{2} \nabla(\Delta^{m-3}u) \cdot \nabla(\Delta^{m-3}u) \\ & + \sum_{i=0}^{m-2} \phi_i (\Delta^i u)^2 + a_{m-2} \left(\frac{n-4}{n+2} \right) (\Delta^{m-2}u)^2 - \frac{(4-n)}{2(n+2)} (\Delta^{m-1}u)^2, \end{aligned} \quad (2.3)$$

where the functions $\phi_0(x), \dots, \phi_{m-2}(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\sum_{i=0}^{m-2} \phi_i^2 + 1 \leq \alpha$ for some positive constant α .

Additionally, one imposes the conditions

$$\begin{aligned} \phi_i & \geq \frac{\beta}{2} \quad (i = 0, \dots, m-2), \\ a_{m-2} & \geq 1, \quad \left(a_{m-1} - \frac{1}{2} \right) \geq \frac{\gamma}{2} \left(\frac{n+2}{n-4} \right), \\ \frac{\Delta a_i}{2} - \frac{\nabla a_{m-i} \cdot \nabla a_{m-i}}{a_{m-i}} & \geq 0 \quad (i = 1, 3), \\ \Delta a_{m-2} - 4 \frac{\nabla a_{m-2} \cdot \nabla a_{m-2}}{a_{m-2}} & \geq 0, \\ \frac{\Delta \phi_i}{3} & \geq \max \left\{ \frac{\beta}{2} \left(\frac{n-4}{n+2} \right) + \frac{\gamma}{2}, \alpha, 4 \frac{\nabla \phi_i \cdot \nabla \phi_i}{\phi_i} \right\} \quad (i = 0, \dots, m-2). \end{aligned} \quad (2.4)$$

Then, P is subharmonic in Ω .

Proof. A straightforward calculation yields

$$\begin{aligned} \Delta P = & 2\nabla^3(\Delta^{m-2}u) : \nabla^3(\Delta^{m-2}u) - \left(\frac{6}{n+2} \right) \nabla(\Delta^{m-1}u) \cdot \nabla(\Delta^{m-1}u) \\ & - \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^m u) - \left(\frac{4-n}{n+2} \right) \Delta^{m-1}u \Delta^m u \\ & + a_{m-1} \nabla^2(\Delta^{m-2}u) : \nabla^2(\Delta^{m-2}u) + a_{m-1} \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-1}u) \\ & + a_{m-3} \nabla^2(\Delta^{m-3}u) : \nabla^2(\Delta^{m-3}u) + a_{m-3} \nabla(\Delta^{m-3}u) \cdot \nabla(\Delta^{m-2}u) \\ & + \sum_{i=0}^{m-2} \left(\Delta \phi_i (\Delta^i u)^2 + 4 \nabla(\phi_i) \cdot \nabla(\Delta^i u) \Delta^i u + 2 \phi_i \nabla(\Delta^i u) \cdot \nabla(\Delta^i u) \right) \end{aligned}$$

$$\begin{aligned}
& + a_{m-2} \left(\frac{n-4}{n+2} \right) \left[\nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \Delta^{m-2} u \Delta^{m-1} u \right] \\
& + \left(\frac{n-4}{n+2} \right) \Delta a_{m-2} (\Delta^{m-2} u)^2 + 4 \left(\frac{n-4}{n+2} \right) \nabla a_{m-2} \cdot \nabla \left(\Delta^{m-2} u \right) \Delta^{m-2} u \\
& + \frac{1}{2} \Delta a_{m-1} \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \frac{1}{2} \Delta a_{m-3} \nabla \left(\Delta^{m-3} u \right) \cdot \nabla \left(\Delta^{m-3} u \right) \\
& + 2 \nabla a_{m-1} \nabla \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + 2 \nabla a_{m-3} \nabla \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
& + \sum_{i=0}^{m-2} 2 \phi_i \Delta^i u \Delta^{i+1} u.
\end{aligned} \tag{2.5}$$

Using the well-known inequality (see [2])

$$\nabla^3 w : \nabla^3 w \geq \frac{3}{n+2} \nabla(\Delta w) \cdot \nabla(\Delta w), \quad \text{for all functions } w \in C^3(\Omega), \tag{2.6}$$

and (1.4) we deduce the inequality

$$\begin{aligned}
\Delta P & \geq -\nabla \left(\Delta^{m-2} u \right) \cdot \left[\sum_{i=0}^{m-1} (-1)^{i+1} \nabla(a_i) \Delta^i u + \sum_{i=0}^{m-1} (-1)^{i+1} a_i \nabla \left(\Delta^i u \right) \right] \\
& + \left(\frac{n-4}{n+2} \right) \Delta^{m-1} u \left[\sum_{i=0}^{m-1} (-1)^{i+1} a_i \Delta^i u \right] \\
& + a_{m-1} \nabla^2 \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + a_{m-1} \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-1} u \right) \\
& + a_{m-3} \nabla^2 \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) + a_{m-3} \nabla \left(\Delta^{m-3} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) \\
& + \sum_{i=0}^{m-2} \left(\Delta \phi_i (\Delta^i u)^2 + 4 \nabla(\phi_i) \cdot \nabla \left(\Delta^i u \right) \Delta^i u + 2 \phi_i \nabla \left(\Delta^i u \right) \cdot \nabla \left(\Delta^i u \right) \right) \\
& + a_{m-2} \left(\frac{n-4}{n+2} \right) \left[\nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \Delta^{m-2} u \Delta^{m-1} u \right] \\
& + \left(\frac{n-4}{n+2} \right) \Delta a_{m-2} (\Delta^{m-2} u)^2 + 4 \left(\frac{n-4}{n+2} \right) \nabla a_{m-2} \cdot \nabla \left(\Delta^{m-2} u \right) \Delta^{m-2} u \\
& + \frac{1}{2} \Delta a_{m-1} \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \frac{1}{2} \Delta a_{m-3} \nabla \left(\Delta^{m-3} u \right) \cdot \nabla \left(\Delta^{m-3} u \right) \\
& + 2 \nabla a_{m-1} \nabla \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + 2 \nabla a_{m-3} \nabla \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
& + \sum_{i=0}^{m-2} 2 \phi_i \Delta^i u \Delta^{i+1} u.
\end{aligned} \tag{2.7}$$

The right side of (2.7) is

$$\begin{aligned}
&= a_{m-2} \left(1 + \frac{n-4}{n+2} \right) \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + a_{m-1} \left(\frac{n-4}{n+2} \right) \left(\Delta^{m-1} u \right)^2 \\
&\quad + a_{m-1} \nabla^2 \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + a_{m-3} \nabla^2 \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
&\quad + \sum_{i=0}^{m-2} \left(\Delta \phi_i \left(\Delta^i u \right)^2 + 4 \nabla(\phi_i) \cdot \nabla \left(\Delta^i u \right) \Delta^i u + 2 \phi_i \nabla \left(\Delta^i u \right) \cdot \nabla \left(\Delta^i u \right) \right) \\
&\quad + \left(\frac{n-4}{n+2} \right) \Delta^{m-1} u \sum_{i=0}^{m-3} (-1)^{i+1} a_i \Delta^i u - \nabla \left(\Delta^{m-2} u \right) \cdot \sum_{i=0}^{m-1} (-1)^{i+1} \nabla(a_i) \Delta^i u \\
&\quad + \left(\frac{n-4}{n+2} \right) \Delta a_{m-2} \left(\Delta^{m-2} u \right)^2 + 4 \left(\frac{n-4}{n+2} \right) \nabla a_{m-2} \cdot \nabla \left(\Delta^{m-2} u \right) \Delta^{m-2} u \\
&\quad + \frac{1}{2} \Delta a_{m-1} \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \frac{1}{2} \Delta a_{m-3} \nabla \left(\Delta^{m-3} u \right) \cdot \nabla \left(\Delta^{m-3} u \right) \\
&\quad + 2 \nabla a_{m-1} \nabla \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + 2 \nabla a_{m-3} \nabla \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
&\quad - \nabla \left(\Delta^{m-2} u \right) \cdot \sum_{i=0}^{m-4} (-1)^{i+1} a_i \nabla \left(\Delta^i u \right) + \sum_{i=0}^{m-2} 2 \phi_i \Delta^i u \Delta^{i+1} u.
\end{aligned} \tag{2.8}$$

Subsequently, we obtain

$$\begin{aligned}
\Delta P &\geq \left[a_{m-2} \left(1 + \frac{n-4}{n+2} \right) - 1 \right] \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) \\
&\quad + a_{m-1} \nabla^2 \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + a_{m-3} \nabla^2 \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
&\quad + \left[a_{m-1} - \frac{1}{2} \right] \left(\frac{n-4}{n+2} \right) \left(\Delta^{m-1} u \right)^2 \\
&\quad - \frac{1}{2} \left(\sum_{i=0}^{m-1} (-1)^{i+1} \nabla(a_i) \Delta^i u \right)^2 - \frac{1}{2} \left[\frac{n-4}{n+2} \right] \left(\sum_{i=0}^{m-3} (-1)^{i+1} a_i \left(\Delta^i u \right) \right)^2 \\
&\quad - \frac{1}{2} \left(\sum_{i=0}^{m-4} (-1)^{i+1} a_i \nabla \left(\Delta^i u \right) \right)^2 \\
&\quad + \sum_{i=0}^{m-2} \Delta \phi_i \left(\Delta^i u \right)^2 + 4 \nabla(\phi_i) \cdot \nabla \left(\Delta^i u \right) \Delta^i u + 2 \phi_i \nabla \left(\Delta^i u \right) \cdot \nabla \left(\Delta^i u \right) \\
&\quad + \left(\frac{n-4}{n+2} \right) \Delta a_{m-2} \left(\Delta^{m-2} u \right)^2 + 4 \left(\frac{n-4}{n+2} \right) \nabla a_{m-2} \cdot \nabla \left(\Delta^{m-2} u \right) \Delta^{m-2} u \\
&\quad + \frac{1}{2} \Delta a_{m-1} \nabla \left(\Delta^{m-2} u \right) \cdot \nabla \left(\Delta^{m-2} u \right) + \frac{1}{2} \Delta a_{m-3} \nabla \left(\Delta^{m-3} u \right) \cdot \nabla \left(\Delta^{m-3} u \right) \\
&\quad + 2 \nabla a_{m-1} \nabla \left(\Delta^{m-2} u \right) : \nabla^2 \left(\Delta^{m-2} u \right) + 2 \nabla a_{m-3} \nabla \left(\Delta^{m-3} u \right) : \nabla^2 \left(\Delta^{m-3} u \right) \\
&\quad + \sum_{i=0}^{m-2} 2 \phi_i \Delta^i u \Delta^{i+1} u.
\end{aligned} \tag{2.9}$$

We now state a series of inequalities to demonstrate that $\Delta P \geq 0$. First we note

$$\sum_{i=0}^{m-2} 2\phi_i \Delta^i u \Delta^{i+1} u \geq - \left(\sum_{i=0}^{m-2} (\phi_i)^2 + 1 \right) \sum_{i=0}^{m-2} (\Delta^i u)^2. \quad (2.10)$$

Similarly, we obtain

$$\begin{aligned} -\frac{1}{2} \left(\sum_{i=0}^{m-1} (-1)^{i+1} \nabla(a_i) \Delta^i u \right)^2 &\geq -\frac{1}{2} \left(\sum_{i=0}^{m-1} [\nabla(a_i) \cdot \nabla(a_i)] \right) (\Delta^{m-1} u)^2 \\ &\quad - \frac{1}{2} \left(\sum_{i=0}^{m-1} [\nabla(a_i) \cdot \nabla(a_i)] \right) \sum_{i=0}^{m-2} \Delta^i u \Delta^i u, \end{aligned} \quad (2.11)$$

$$-\frac{1}{2} \left[\frac{n-4}{n+2} \right] \left(\sum_{i=0}^{m-3} (-1)^{i+1} a_i \Delta^i u \right)^2 \geq -\frac{1}{2} \left[\frac{n-4}{n+2} \right] \sum_{i=0}^{m-3} a_i^2 \sum_{i=0}^{m-3} (\Delta^i u)^2, \quad (2.12)$$

$$-\frac{1}{2} \left(\sum_{i=0}^{m-4} (-1)^{i+1} a_i \nabla(\Delta^i u) \right)^2 \geq -\frac{1}{2} \sum_{i=0}^{m-4} a_i^2 \sum_{i=0}^{m-4} \nabla(\Delta^i u) \cdot \nabla(\Delta^i u), \quad (2.13)$$

$$\begin{aligned} &+ \sum_{i=0}^{m-2} (\Delta \phi_i) (\Delta^i u)^2 + 4 \nabla(\phi_i) \cdot \nabla(\Delta^i u) \Delta^i u + \phi_i \nabla(\Delta^i u) \cdot \nabla(\Delta^i u) \\ &\geq \sum_{i=0}^{m-2} (\Delta^i u)^2 \left[\frac{\Delta \phi_i}{3} - \frac{4 \nabla(\phi_i) \cdot \nabla(\phi_i)}{\phi_i} \right]. \end{aligned} \quad (2.14)$$

We also have that

$$\begin{aligned} &\frac{1}{2} \Delta a_{m-1} \nabla(\Delta^{m-2} u) \cdot \nabla(\Delta^{m-2} u) + 2 \nabla a_{m-1} \nabla(\Delta^{m-2} u) : \nabla^2(\Delta^{m-2} u) \\ &\quad + a_{m-1} \nabla^2(\Delta^{m-2} u) : \nabla^2(\Delta^{m-2} u) \\ &\geq \nabla(\Delta^{m-2} u) \cdot \nabla(\Delta^{m-2} u) \left[\frac{\Delta a_{m-1}}{2} - \frac{\nabla a_{m-1} \cdot \nabla a_{m-1}}{a_{m-1}} \right]. \end{aligned} \quad (2.15)$$

Additional inequalities analogous to (2.15) can be developed for the remaining terms in (2.9) involving Δa_{m-2} and Δa_{m-3} .

From (2.2), (2.10)–(2.15) and upon inclusion of the aforementioned additional inequalities we deduce that

$$\begin{aligned}
\Delta P \geq & \left(\left[a_{m-1} - \frac{1}{2} \right] \left(\frac{n-4}{n+2} \right) - \frac{1}{2} \sum_{i=0}^{m-1} \nabla(a_i) \cdot \nabla(a_i) \right) (\Delta^{m-1} u)^2 \\
& + \sum_{i=0}^{m-2} \left[\frac{\Delta \phi_i}{3} - \frac{\beta}{2} \left(\frac{n-4}{n+2} \right) - \frac{\gamma}{2} \right] (\Delta^i u)^2 \\
& + \sum_{i=0}^{m-2} \left[\phi_i - \frac{\beta}{2} \right] (\Delta^i u)^2 + \sum_{i=0}^{m-2} \left[\frac{\Delta \phi_i}{3} - \alpha \right] (\Delta^i u)^2 \\
& + \sum_{i=0}^{m-2} (\Delta^i u)^2 \left[\frac{\Delta \phi_i}{3} - \frac{4 \nabla(\phi_i) \cdot \nabla(\phi_i)}{\phi_i} \right] \\
& + (a_{m-2} - 1) \nabla(\Delta^{m-2} u) \cdot \nabla(\Delta^{m-2} u) \\
& + \nabla(\Delta^{m-2} u) \cdot \nabla(\Delta^{m-2} u) \left[\frac{\Delta a_{m-1}}{2} - \frac{\nabla a_{m-1} \cdot \nabla a_{m-1}}{a_{m-1}} \right] \\
& + \nabla(\Delta^{m-3} u) \cdot \nabla(\Delta^{m-3} u) \left[\frac{\Delta a_{m-3}}{2} - \frac{\nabla a_{m-3} \cdot \nabla a_{m-3}}{a_{m-3}} \right] \\
& + \left(\frac{n-4}{n+2} \right) (\Delta^{m-2} u)^2 \left[\Delta a_{m-2} - 4 \frac{\nabla a_{m-2} \cdot \nabla a_{m-2}}{a_{m-2}} \right].
\end{aligned} \tag{2.16}$$

Utilizing (2.4) the conclusion follows. $\square$

Now we handle the case where $n \leq 4$ in the following theorem.

Theorem 2.2. Suppose that $u \in C^{2m+1}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ is a solution of (1.4) where $n \leq 4$. For $\phi_0(x), \dots, \phi_{m-1}(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying the inequality $\sum_{i=0}^{m-2} \phi_i^2 + 1 \leq \alpha$ for some positive constant α , one defines

$$P = \nabla^2(\Delta^{m-2} u) : \nabla^2(\Delta^{m-2} u) - \nabla(\Delta^{m-2} u) \cdot \nabla(\Delta^{m-1} u) + \sum_{i=0}^{m-1} \phi_i (\Delta^i u)^2 + \frac{1}{2} (\Delta^{m-2} u)^2. \tag{2.17}$$

Then if

$$\begin{aligned}
\phi_i & \geq \frac{\alpha}{2}, \quad \text{for } i = 0, \dots, m-1, \\
\frac{\Delta \phi_{m-i}}{3} - \frac{1}{2} & \geq 0, \quad \text{for } i = 1, 2, \\
\frac{\Delta \phi_i}{3} - \frac{4 \nabla \phi_i \cdot \nabla \phi_i}{\phi_i} & \geq 0, \quad \text{for } i = 0, \dots, m-1,
\end{aligned} \tag{2.18}$$

P is subharmonic in Ω .

Proof. A calculation similar to that of Theorem 2.1 yields

$$\begin{aligned}
\Delta P &= 2\nabla^3(\Delta^{m-2}u) : \nabla^3(\Delta^{m-2}u) - \nabla(\Delta^{m-1}u) \cdot \nabla(\Delta^{m-1}u) \\
&\quad - \nabla(\Delta^{m-2}u) \cdot \sum_{i=0}^{m-1} (-1)^{i+1} \nabla a_i \Delta^i u + \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-2}u) \\
&\quad - \nabla(\Delta^{m-2}u) \cdot \sum_{i=0}^{m-1} \left[(-1)^{i+1} a_i \nabla(\Delta^i u) \right] + \Delta^{m-2}u \Delta^{m-1}u \\
&\quad + \sum_{i=0}^{m-1} \Delta \phi_i (\Delta^i u)^2 + 4 \nabla(\phi_i) \cdot \nabla(\Delta^i u) \Delta^i u \\
&\quad + \sum_{i=0}^{m-1} 2 \phi_i \nabla(\Delta^i u) \cdot \nabla(\Delta^i u) + 2 \phi_i \Delta^i u \Delta^{i+1} u.
\end{aligned} \tag{2.19}$$

Using (2.6), (2.10), (2.11), (2.13), and (2.14) we obtain

$$\begin{aligned}
\Delta P &\geq \sum_{i=0}^{m-1} (\Delta^i u)^2 \left[\frac{\Delta \phi_i}{3} - \frac{4 \nabla \phi_i \cdot \nabla \phi_i}{\phi_i} \right] + \sum_{i=0}^{m-1} \frac{2 \Delta \phi_i}{3} (\Delta^i u)^2 \\
&\quad + \sum_{i=0}^{m-1} \phi_i \nabla(\Delta^i u) \cdot \nabla(\Delta^i u) - \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-2}u) \\
&\quad - \frac{1}{2} (\Delta^{m-2}u)^2 - \frac{1}{2} (\Delta^{m-1}u)^2 + \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-2}u) \\
&\quad - \frac{1}{2} \left(\sum_{i=0}^{m-1} \nabla a_i \cdot \nabla a_i \right) \sum_{i=0}^{m-1} (\Delta^i u)^2 - \frac{1}{2} \left(\sum_{i=0}^{m-1} a_i^2 \right) \sum_{i=0}^{m-1} \nabla(\Delta^i u) \cdot \nabla(\Delta^i u) \\
&\geq \sum_{i=0}^{m-1} (\Delta^i u)^2 \left[\frac{\Delta \phi_i}{3} - \frac{4 \nabla \phi_i \cdot \nabla \phi_i}{\phi_i} \right] + \sum_{i=0}^{m-1} \left[\phi_i - \frac{\alpha}{2} \right] \nabla(\Delta u) \cdot \nabla(\Delta u) + \sum_{i=0}^{m-1} \left[\frac{\Delta \phi_i}{3} - \frac{\gamma}{2} \right] (\Delta^i u)^2 \\
&\quad + \left(\frac{\Delta \phi_{m-2}}{3} - \frac{1}{2} \right) (\Delta^{m-2}u)^2 + \left(\frac{\Delta \phi_{m-1}}{3} - \frac{1}{2} \right) (\Delta^{m-1}u)^2.
\end{aligned} \tag{2.20}$$

Hence by (2.18), we conclude that $\Delta P \geq 0$. $\square$

3. Applications

Here we briefly indicate how theorem 1 and theorem 2 can be used to obtain integral bounds on the square of the second gradient of $\Delta^{m-2}u$. suppose that the hypotheses of theorem 1 are satisfied and that the m conditions

$$\begin{aligned}
\Delta^i u &= 0, \quad \text{for } i = 0, \dots, m-5, \\
\Delta^{m-2}u &= \frac{\Delta^{m-2}u}{\partial n} = 0, \quad \Delta^{m-3}u = \frac{\Delta^{m-3}u}{\partial n} = 0
\end{aligned} \tag{3.1}$$

hold on $\partial\Omega$. Let A denote the area of Ω . As a consequence of integration by parts, Theorem 2.1 implies that

$$\int_{\Omega} \left| \nabla^2 (\Delta^{m-2} u) \right|^2 dx \leq \frac{A}{2} \max_{\partial\Omega} \left[\left| \nabla^2 (\Delta^{m-2} u) \right|^2 + \phi_{m-4}(x) (\Delta^{m-4} u)^2 + \frac{n-4}{2(n+2)} (\Delta (\Delta^{m-2} u))^2 \right]. \quad (3.2)$$

Now, if the hypotheses of Theorem 2.2 hold and if we impose the m conditions

$$\begin{aligned} \Delta^i u &= 0, \quad \text{for } i = 0, \dots, m-3, \\ \Delta^{m-2} u &= \frac{\Delta^{m-2} u}{\partial n} = 0 \end{aligned} \quad (3.3)$$

on $\partial\Omega$ we can deduce the inequality

$$\int_{\Omega} \left| \nabla^2 (\Delta^{m-2} u) \right|^2 dx \leq \frac{A}{2} \max_{\partial\Omega} \left[\left| \nabla^2 (\Delta^{m-2} u) \right|^2 + \phi_{m-1}(x) (\Delta (\Delta^{m-2} u))^2 \right], \quad (3.4)$$

from Theorem 2.2.

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