Research Article

A Gelfand Model for Weyl Groups of Type D_{2n}

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A Gelfand model for a finite group G is a complex representation of G , which is isomorphic to the direct sum of all irreducible representations of G . When G is isomorphic to a subgroup of $GL_n(\mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers, it has been proved that each G -module over $\mathbb{C}$ is isomorphic to a G -submodule in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, and taking the space of zeros of certain G -invariant operators in the Weyl algebra, a finite-dimensional G -space $\mathcal{N}_G$ in $\mathbb{C}[x_1, \dots, x_n]$ can be obtained, which contains all the simple G -modules over $\mathbb{C}$. This type of representation has been named polynomial model. It has been proved that when G is a Coxeter group, the polynomial model is a Gelfand model for G if, and only if, G has not an irreducible factor of type D_{2n} , E_7 , or E_8 . This paper presents a model of Gelfand for a Weyl group of type D_{2n} whose construction is based on the same principles as the polynomial model.

1. Introduction

Gelfand models for a finite group are complex representations whose character is the sum of all irreducible characters of the given group. In this sense, Bernstein et al. have presented Gelfand models for semisimple compact Lie groups, see [1]. Since then, Gelfand models have been developed in several articles; see [2–12], among these there are two types of models that can be associated with reflection groups: *the involution model* and *the polynomial model*.

Parallel works, made by Klyachko, on one side, and by Inglis, Richardson, and Saxl, on the other, showed an identity that describes a Gelfand model associated with the symmetric group. The identity is given by

$$\chi_{C_k} \uparrow \mathfrak{S}_n = \sum \chi_\lambda, \quad (1.1)$$

where C_k is the centralizer of an involution in $\mathfrak{S}_n$ with exactly k fixed points, χ_{C_k} is a linear character of C_k , and χ_λ is an irreducible character of $\mathfrak{S}_n$ associated with the partition λ of n with exactly k odd terms. From this identity, it follows immediately that

$$\sum_k \chi_{C_k} \uparrow \mathfrak{S}_n = \sum_\lambda \chi_\lambda, \quad (1.2)$$

where the centralizers C_k are in correspondence with the conjugacy classes of involutions in $\mathfrak{S}_n$.

Later on, this type of models was called an *involution model* by Baddeley [6]. He also proved that if H is a finite group that admits an involution model, then so does the semidirect product $H^n \rtimes_s \mathfrak{S}_n$.

Baddeley's result implies the existence of involution models for classic Weyl groups, with the exception of the group of type D_{2n} . An involution model for a Weyl group of type A_n is presented in [8] by Inglis et al. and for a Weyl group of type B_n an involution model is shown in [6, 13]. In [6], Baddeley presents an involution model for a Weyl group of type D_{2n+1} , and in [14] it is proved that there is no involution model for a Weyl group of type D_{2n} with $n \geq 2$. In [6], it is mentioned that is not difficult to prove that there is an involution model for a Weyl group of type G_2 and that it has been checked using computers the non existence of involution models for exceptional Weyl groups of type F_4 , E_6 , E_7 , and E_8 . In [15], Vinroot does some research about involution models for irreducible non crystallographic Coxeter groups. He proves the existence of an involution model for groups of type I_2^n ($n \geq 3, n \neq 6$) and H_3 and presents a conceptual demonstration of the no existence of an involution model for the group of type H_4 .

More recently, in [16] the *generalized involution model* has been studied in order to include some cases of unitary reflection groups.

A reflection group G comes equipped with a canonical representation called the *geometric representation* of G . The geometric representation induces a natural action of G on the space of polynomial functions.

Chevalley [17], Shephard and Todd [18], Steinberg [19], and others studied the corresponding action on the space $\mathcal{H}_G$ of G -harmonic polynomials proving that $\mathcal{H}_G$ is isomorphic to the regular representation of G , and thus $\mathcal{H}_G$ contains a Gelfand model for G . On the other hand, Macdonald found irreducible representations of a Weyl group associated with the root systems of the reflection subgroups that can be naturally realized in the G -harmonic polynomial space. These representations are known as Macdonald representations see [20].

More recently, Araujo and Aguado in [21] have associated with each finite subgroup $G \subset GL_n(\mathbb{C})$ a subspace $\mathcal{N}_G$ of the algebra of polynomials $\mathbb{C}[x_1, \dots, x_n]$, defined as zeros of certain G -invariant differential operators, and have shown $\mathcal{N}_G$ contains a Gelfand model of G . This space, called the *polynomial model*, is a Gelfand model for some Weyl groups. In [3–5], it was proved that $\mathcal{N}_G$ is a Gelfand model for Weyl groups of type A_n , B_n and D_{2n+1} . Garge and Oesterlé in [22], using the computation of fake degrees of the irreducible characters of a Coxeter group G , determined that $\mathcal{N}_G$ is a Gelfand model of G if, and only if, G has not irreducible factors of type D_{2n} , E_7 , or E_8 . The fake degrees have been determined due to works of Steinberg [23], when G is of type A_n , Lusztig [24], when G is of type B_n or D_n , Beynon and Lusztig [25], when G is an exceptional Weyl group, Alvis and Lusztig [26], when G is of type H_4 , and Macdonald, when G is of type F_4 (unpublished). The remaining cases are not difficult.

For the case of Weyl groups of type D_{2n} , neither the polynomial model nor the involution model provides a Gelfand model.

In this paper the construction of a Gelfand model for a Weyl group of type D_{2n} will be presented. It will be built upon a light modification of the polynomial model.

2. Polynomial Model

The notation introduced in this section will be used in the remaining of this paper.

G will denote a finite subgroup of $GL_n(\mathbb{C})$ and $\mathcal{P}$ the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$.

Let $\mathbf{I}_n = \{1, \dots, n\}$ be the set of the first n natural numbers and $\mathcal{M}_n$ the set of multi-index functions:

$$\mathcal{M}_n = \{\alpha : \mathbf{I}_n \longrightarrow \mathbb{N}_0\} \quad (2.1)$$

For each $\alpha \in \mathcal{M}_n$ the following notation will be used in the rest of this paper:

$$\alpha_i = \alpha(i), \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \sum_{i=1}^n \alpha_i. \quad (2.2)$$

Let $\mathcal{A} = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ be the Weyl algebra of $\mathbb{C}$ -linear differential operators generated by the multiplication operators x_i and partial differential operators $\partial_i = \partial/\partial x_i$ with $1 \leq i \leq n$.

It is known that each $D \in \mathcal{A}$ has a unique expression as a finite sum (see [27]):

$$D = \sum \lambda_{\alpha, \beta} x^\alpha \partial^\beta, \quad (2.3)$$

where $\alpha, \beta \in \mathcal{M}_n$, $\lambda_{\alpha, \beta} \in \mathbb{C}$, and

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \\ \partial^\beta &= \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}. \end{aligned} \quad (2.4)$$

The degree of D is defined by

$$\deg(D) = \max \left\{ \sum_i (\alpha_i - \beta_i) : \lambda_{\alpha, \beta} \neq 0 \right\}. \quad (2.5)$$

The Weyl algebra is a graduated algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$, where

$$\mathcal{A}_i = \left\{ \sum_{\alpha, \beta \in \mathcal{M}_n} \lambda_{\alpha, \beta} x^\alpha \partial^\beta : |\alpha| - |\beta| = i \right\}. \quad (2.6)$$

The action of G on $\mathcal{P}$ induces an action of G on the endomorphism ring $\text{End}_{\mathbb{C}}(\mathcal{P})$, which is defined by

$$(g \cdot D)(p) = (gDg^{-1})(p) \quad (g \in G, D \in \text{End}_{\mathbb{C}}(\mathcal{P})). \quad (2.7)$$

This action can be restricted to the Weyl algebra $\mathcal{A}$ noting that each $\mathcal{A}_i$ is invariant under the action of G .

Let $\mathcal{O}_G$ be the subalgebra of G -invariant operators in $\mathcal{A}$, that is,

$$\mathcal{O}_G = \{D \in \mathcal{A} : g \cdot D = D, \forall g \in G\}. \quad (2.8)$$

Notice that $\mathcal{O}_G$ is contained in the centralizer of G in $\text{End}_{\mathbb{C}}(\mathcal{P})$.

Let $\mathcal{O}_G^-$ be the subspace of the Weyl algebra, formed by the G -invariant operators with negative degree

$$\mathcal{O}_G^- = \{D \in \mathcal{O}_G : \deg(D) < 0\}. \quad (2.9)$$

Definition 2.1. Let $\mathcal{N}_G$ be the subspace of $\mathcal{P}$ defined by

$$\mathcal{N}_G = \{p \in \mathcal{P} : D(p) = 0, \forall D \in \mathcal{O}_G^-\}. \quad (2.10)$$

$\mathcal{N}_G$ is named the polynomial model of G .

Notice that $\mathcal{N}_G$ is a G -module.

Below, some properties of $\mathcal{N}_G$ will be mentioned.

Theorem 2.2. $\mathcal{N}_G$ is a finite-dimensional G -module, and every simple G -module has a copy in $\mathcal{N}_G$.

Proof. See [21, page 38]. □

The analysis of the polynomial model for Coxeter groups has been completely solved by the following theorem.

Theorem 2.3. Let G be a finite irreducible Coxeter group, and let W be its realization as a reflection group. Then, the polynomial model $\mathcal{N}_W$ is a Gelfand model for G if, and only if, W is not a Weyl group of type D_{2n} , E_7 , or E_8 .

Proof. See [22, page 7]. □

In the following sections it will be presented a characterization of the polynomial model for the classical Weyl groups of type A_n , B_n and D_n .

2.1. Polynomial Model for a Weyl Group of Type A_n

Let G be a Weyl group of type A_{n-1} . It is known that G can be presented as the symmetric group $\mathfrak{S}_n$.

The symmetric group $\mathfrak{S}_n$ acts on the set of multi-index functions $\mathcal{M}_n$ by

$$\sigma \cdot \alpha = \alpha \circ \sigma^{-1} \quad (\sigma \in \mathfrak{S}_n, \alpha \in \mathcal{M}_n). \quad (2.11)$$

This action induces a natural homomorphism from $\mathfrak{S}_n$ in $\text{Aut}(\mathcal{P})$ given by

$$\sigma \left(\sum_{\alpha \in \mathcal{M}_n} \lambda_\alpha x^\alpha \right) = \sum_{\alpha \in \mathcal{M}_n} \lambda_\alpha x^{\sigma \cdot \alpha} \quad (\lambda_\alpha \in \mathbb{C}). \quad (2.12)$$

2.1.1. $\mathfrak{S}_n$ -Minimal Orbit

Let $\mathcal{O}_n$ be the orbit space of $\mathfrak{S}_n$ in $\mathcal{M}_n$. It is clear that if two multi-indexes α and β belong to the same orbit γ , then $|\alpha|$ and $|\beta|$ take the same value, where $|\alpha|$ and $|\beta|$ are defined by (2.2), and this value will be denoted by $|\gamma|$.

Definition 2.4. Two orbits γ and δ will be called **$\mathfrak{S}_n$ -equivalent**, denoted by $\gamma \sim_{\mathfrak{S}_n} \delta$, if there exists a bijection $\varphi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that

$$\delta = \{\varphi \circ \alpha : \alpha \in \gamma\}. \quad (2.13)$$

Definition 2.5. An orbit γ will be called **$\mathfrak{S}_n$ -minimal** if $|\gamma| \leq |\delta|$ for all $\delta \in \mathcal{O}_n$ such that $\gamma \sim_{\mathfrak{S}_n} \delta$.

Proposition 2.6. *An orbit γ is $\mathfrak{S}_n$ -minimal if, and only if, for each $\alpha \in \gamma$, there exists a nonnegative integer h such that*

- (1) $\text{Im } (\alpha) = \{0, 1, \dots, h-1\}$,
- (2) $|\alpha^{-1}(i)| \geq |\alpha^{-1}(i+1)|$ for all $0 \leq i \leq h-1$ ($|\alpha^{-1}(i)|$ being the cardinal of the set $\alpha^{-1}(i)$).

Proof. See [4, page 1845]. □

Definition 2.7. For each $\gamma \in \mathcal{O}_n$, let S_γ be the subspace of $\mathcal{P}$ defined by

$$S_\gamma = \left\{ \sum_{\alpha \in \gamma} \lambda_\alpha x^\alpha : \lambda_\alpha \in \mathbb{C} \right\}. \quad (2.14)$$

2.1.2. The Space S_γ^∂

Let ∂ be the operator defined by $\partial = \sum_{i=1}^n \partial_i$, where ∂_i are the partial differential operators as above. For each $\gamma \in \mathcal{O}_n$, let S_γ^∂ be the subspace defined by

$$S_\gamma^\partial = \{P \in S_\gamma : \partial(P) = 0\}. \quad (2.15)$$

2.1.3. The Structure of $\mathcal{N}_{\mathfrak{S}_n}$

Below the main theorem regarding $\mathcal{N}_{\mathfrak{S}_n}$ is announced without proof. For further details see [4, page 1850].

Theorem 2.8. S_γ^∂ is an irreducible G -module, and $\mathcal{N}_{\mathfrak{S}_n}$ can be decomposed as

$$\mathcal{N}_{\mathfrak{S}_n} = \bigoplus_{\gamma \in \mathcal{O}_{\mathfrak{S}_n}\text{-minimal}} S_\gamma^\partial. \quad (2.16)$$

Moreover, $\mathcal{N}_{\mathfrak{S}_n}$ is a Gelfand model of $\mathfrak{S}_n$.

2.2. Polynomial Model for a Weyl Group of Type B_n

The Gelfand model for a Weyl group of type B_n will be described using the same ideas as the previous section.

Let $C_2 = \{1, -1\} \subset \mathbb{C}^*$ be the subgroup of order two. The Weyl group $\mathcal{B}_n$, of type B_n , can be presented as the semidirect product

$$\mathcal{B}_n = C_2^n \times_s \mathfrak{S}_n, \quad (2.17)$$

where $C_2^n = C_2 \times \cdots \times C_2$ and the semidirect product is induced by the natural action of $\mathfrak{S}_n$ on C_2^n :

$$\sigma \cdot (w_1, \dots, w_n) = (w_{\sigma(1)}, \dots, w_{\sigma(n)}) \quad (\sigma \in \mathfrak{S}_n, (w_1, \dots, w_n) \in C_2^n). \quad (2.18)$$

The action of $\mathfrak{S}_n$ on $\mathcal{M}_n$ induces a natural homomorphism from $\mathcal{B}_n$ on $\text{Aut}(\mathcal{P})$ given by

$$(w, \sigma) \left(\sum_{\alpha \in \mathcal{M}_n} \lambda_\alpha x^\alpha \right) = \sum_{\alpha \in \mathcal{M}_n} \lambda_\alpha (wx)^{\sigma\alpha} \quad (\lambda_\alpha \in \mathbb{C}) \quad (2.19)$$

with

$$(wx)^{\sigma\alpha} = \prod_{i=1}^n (w_i x_i)^{(\sigma\alpha)_i}. \quad (2.20)$$

2.2.1. $\mathcal{B}_n$ -Minimal Orbit

Let $\mathcal{O}_n$ be the orbit space of $\mathfrak{S}_n$ on $\mathcal{M}_n$, as above.

Definition 2.9. Two orbits, γ and δ , will be called $\mathcal{B}_n$ -**equivalent**, denoted by $\gamma \sim_{\mathcal{B}_n} \delta$, if there exists a bijection $\varphi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that $\varphi(k)$ and k have the same parity for all $k \in \mathbb{N}_0$,

Definition 2.10. An orbit γ will be called $\mathcal{B}_n$ -**minimal** if $|\gamma| \leq |\delta|$ for all $\delta \in \mathcal{O}_n$ such that $\gamma \sim_{\mathcal{B}_n} \delta$.

Proposition 2.11. An orbit γ is $\mathcal{B}_n$ -minimal if, and only if, for each $\alpha \in \gamma$ and each pair $i, j \in \mathbb{N}_0$ with the same parity, one has $|\alpha^{-1}(i)| \geq |\alpha^{-1}(j)|$ with $0 \leq i < j$ ($|\alpha^{-1}(i)|$ being the cardinal of the set $\alpha^{-1}(i)$).

Proof. See [3, page 365]. □

2.2.2. The Space S_Y^Δ

Let Δ be the Laplacian operator defined by $\Delta = \sum_{i=1}^n \partial_i^2$, where ∂_i are the partial differential operators mentioned above. For each $\gamma \in \mathcal{O}_n$, let S_Y^Δ be the subspace defined by

$$S_Y^\Delta = \{P \in S_Y : \Delta(P) = 0\}. \quad (2.21)$$

2.2.3. The Structure of $\mathcal{N}_{\mathcal{B}_n}$

Below the main theorem regarding $\mathcal{N}_{\mathcal{B}_n}$ is announced without proof. See references.

Theorem 2.12. S_Y^Δ is an irreducible G -module, and $\mathcal{N}_{\mathcal{B}_n}$ can be decomposed as

$$\mathcal{N}_{\mathcal{B}_n} = \bigoplus_{\gamma \in \mathcal{O}_{\mathcal{B}_n} \text{-minimal}} S_Y^\Delta. \quad (2.22)$$

Moreover $\mathcal{N}_{\mathcal{B}_n}$ is a Gelfand model of $\mathcal{B}_n$.

Proof. See [3, page 371]. □

2.3. Polynomial Model for a Weyl Group of Type D_n

Let $\mathfrak{D}_n$ be the Weyl group of type D_n naturally included in $\mathcal{B}_n$. Using the previous notation, for $\alpha \in \mathcal{M}_n$ the following sets are considered:

$$E_\alpha = \{i \in I_n : \alpha_i \text{ is even}\}, \quad O_\alpha = \{i \in I_n : \alpha_i \text{ is odd}\}. \quad (2.23)$$

It is easy to check that the cardinals $|E_\alpha|$ and $|O_\alpha|$ are equal for all elements in the same orbit γ . Therefore, these values will be denoted by $|E_\gamma|$ and $|O_\gamma|$, respectively.

2.3.1. $\mathfrak{D}_n$ -Minimal Orbit

Definition 2.13. Two orbits γ and δ will be called **$\mathfrak{D}_n$ -equivalent**, denoted by $\gamma \sim_{\mathfrak{D}_n} \delta$, if there exists a bijection $\varphi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that

- (1) $\forall k \in \mathbb{N}_0$, $\varphi(k)$ and k have the same parity or $\varphi(k)$ and k have different parities,
- (2) $\delta = \{\varphi \circ \alpha : \alpha \in \gamma\}$.

Definition 2.14. An orbit γ will be called **$\mathfrak{D}_n$ -minimal** if $|\gamma| \leq |\delta|$ for all $\delta \in \mathcal{O}_n$ such that $\gamma \sim_{\mathfrak{D}_n} \delta$.

Proposition 2.15. Let γ be an orbit, and then the following statements are true

- (1) γ is $\mathfrak{D}_n$ -minimal if, and only if, the following statements are verified:
 - (a) given $\alpha \in \gamma$ and $i < j \in \mathbb{N}_0$ with the same parity, then $|\alpha^{-1}(i)| \geq |\alpha^{-1}(j)|$,
 - (b) $|E_\gamma| \leq |O_\gamma|$.

(2) Let $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be the involution given by $\pi(2i) = 2i + 1$ and $\pi(2i + 1) = 2i$. The following assertions are equivalent:

- (a) γ and $\pi \circ \gamma$ are $\mathfrak{D}_n$ -minimal orbits,
- (b) γ is $\mathfrak{B}_n$ -minimal,
- (c) $\pi \circ \gamma$ is $\mathfrak{B}_n$ -minimal.

- (3) There are at most two $\mathfrak{D}_n$ -minimal orbits equivalent to γ .
- (4) If n is odd, there is only one $\mathfrak{D}_n$ -minimal orbit equivalent to γ .
- (5) γ and $\pi \circ \gamma$ are $\mathfrak{D}_n$ -minimal orbits if, and only if, $|E_\gamma| = |O_\gamma|$.

Proof. See [5, page 106]. □

Proposition 2.16. *Let n be odd, and then the following statements are true.*

- (1) If γ is $\mathfrak{D}_n$ -minimal, then

$$\mathcal{N}_{\mathfrak{B}_n} \cap S_\gamma = \mathcal{N}_{\mathfrak{D}_n} \cap S_\gamma \quad (2.24)$$

and $\mathcal{N}_{\mathfrak{D}_n} \cap S_\gamma$ is a simple $\mathfrak{D}_n$ -module.

- (2) $\mathcal{N}_{\mathfrak{D}_n}$ is a Gelfand model for $\mathfrak{D}_n$.
- (3) Every simple $\mathfrak{B}_n$ -module remains simple when it is considered as a $\mathfrak{D}_n$ -module by restriction.
- (4) By considering $\mathcal{N}_{\mathfrak{B}_n}$ as a $\mathfrak{D}_n$ -module by restriction, $\mathcal{N}_{\mathfrak{B}_n}$ is isomorphic to $\mathcal{N}_{\mathfrak{D}_n} \oplus \mathcal{N}_{\mathfrak{D}_n}$.

Proof. See [5, page 110]. □

Also in [5] it has been proved that if n is even, $\mathcal{N}_{\mathfrak{D}_n}$ is not a Gelfand model for a Weyl group of type D_n . But it does happen that if M is a simple $\mathfrak{D}_n$ -module, then $\mathcal{N}_{\mathfrak{D}_n}$ contains a copy of this, and the multiplicity of M in $\mathcal{N}_{\mathfrak{D}_n}$ is

- (1) two, if M is isomorphic to $\mathcal{N}_{\mathfrak{D}_n} \cap S_\gamma$, γ being a $\mathfrak{D}_n$ -minimal orbit such that $\gamma \neq \pi \circ \gamma$ and $|E_\gamma| = |O_\gamma|$; in this case, as before, $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is the involution given by $\pi(2i) = 2i + 1$ and $\pi(2i + 1) = 2i$,
- (2) one, otherwise.

3. Gelfand Model for a Weyl Group of Type D_{2n}

As before, let $\mathcal{M}_n = \{\alpha : \mathbf{I}_n \rightarrow \mathbb{N}_0\}$ be the set of multi-index functions. Every $\alpha \in \mathcal{M}_n$ has an associated vector $\hat{\alpha} \in \mathbb{N}_0^n$, which is obtained by reordering α as follows.

$$\hat{\alpha} = (\alpha_{i_1}, \dots, \alpha_{i_n}) \quad \text{such that } \alpha_{i_1} \geq \dots \geq \alpha_{i_n}. \quad (3.1)$$

Thus, there is defined an order relationship $\leq$ in $\mathcal{M}_n$ given by for all $\alpha, \beta \in \mathcal{M}_n$, $\alpha \leq \beta$ if, and only if, $\hat{\alpha} = \hat{\beta}$ or there exists s ($1 \leq s \leq n$) such that

$$\hat{\alpha}_1 = \hat{\beta}_1, \dots, \hat{\alpha}_{s-1} = \hat{\beta}_{s-1}, \quad \hat{\alpha}_s < \hat{\beta}_s, \quad (3.2)$$

$\hat{\alpha}_i$ and $\hat{\beta}_i$ being the coordinates of the vectors $\hat{\alpha}$ and $\hat{\beta}$, respectively. Notice that this is the lexicographic order for $\mathbb{N}_0^n$.

Proposition 3.1. *Let $\gamma \in \mathcal{O}_n$ and $\alpha, \beta \in \mathcal{M}_n$, and then $\alpha, \beta \in \gamma$ if, and only if, $\hat{\alpha} = \hat{\beta}$.*

Proof. Let $\alpha, \beta \in \gamma$, and therefore there exists $\sigma \in \mathfrak{S}_n$ such that $\beta = \sigma\alpha$, which implies $\beta_i = \alpha_{\sigma^{-1}(i)}$ with $1 \leq i \leq n$. Thus, it is easy to see that $\hat{\alpha} = \hat{\beta}$.

On the other hand, let $\alpha, \beta \in \mathcal{M}_n$ and $\hat{\alpha} = \hat{\beta}$, say,

$$\alpha_{i_1} = \beta_{j_1}, \dots, \alpha_{i_n} = \beta_{j_n}. \quad (3.3)$$

Let $\sigma \in \mathfrak{S}_n$ be given by

$$\sigma^{-1}(i_k) = j_k \quad (1 \leq k \leq n). \quad (3.4)$$

Then, $\beta = \sigma\alpha$, and hence both multi-indexes belong to the same orbit. $\square$

From this proposition it is clear that $\leq$ induces a total order in $\mathcal{O}_n$, which is defined by

$$\gamma \leq \delta \iff \alpha \leq \beta \quad (\gamma, \delta \in \mathcal{O}_n, \alpha \in \gamma, \beta \in \delta). \quad (3.5)$$

Since the vector $\hat{\alpha}$ is independent of the choice α in γ , it will be denoted by $\hat{\gamma}$.

Proposition 3.2. *Let $\alpha \in \mathcal{M}_n$ be defined by $\alpha_i = i-1$, and let γ be the orbit of α . Then, γ is the $\leq$ -maximum of the $\mathfrak{S}_n$ -minimal orbits and $\hat{\gamma} = (n-1, \dots, 1, 0)$.*

Proof. From the previous considerations it is clear that γ is an $\mathfrak{S}_n$ -minimal orbit and $\hat{\gamma} = (n-1, \dots, 1, 0)$.

Now it will be proved that γ is the $\leq$ -maximum in the set of $\mathfrak{S}_n$ -minimal orbits. Let δ be an orbit such that $\delta \neq \gamma$ and $\gamma \leq \delta$. Then, it should exist an $s \in \mathbf{I}_n$ satisfying

$$\hat{\gamma}_i = \hat{\delta}_i, \quad \forall i < s, \quad \hat{\gamma}_s < \hat{\delta}_s, \quad (3.6)$$

that is,

$$\hat{\delta}_1 = n-1 \cdots \hat{\delta}_{s-1} = n-(s-1), \quad \hat{\delta}_s > n-s. \quad (3.7)$$

Thus, it occurs that $\hat{\delta}_s = n-(s-1)$, and from the minimality of δ every number less than $n-(s-1)$ must appear at least twice, which is a contradiction. And therefore γ is the maximum. $\square$

From now on, for a finite set A , $\mathfrak{S}_A$ will denote the symmetric group of A , $\mathcal{M}_A$ will denote the set of multi-index functions with domain A ,

$$\mathcal{M}_A = \{\alpha : A \longrightarrow \mathbb{N}_0\}, \quad (3.8)$$

and $\mathbf{1}$ will denote the function in $\mathcal{M}_A$ given by

$$\mathbf{1}(i) = 1, \quad \forall i \in A. \quad (3.9)$$

As in the case of $\mathfrak{S}_n$, there is a natural action from the symmetric group $\mathfrak{S}_A$ in the set of multi-index functions $\mathcal{M}_A$, defined by

$$\sigma \cdot \alpha = \alpha \circ \sigma^{-1} \quad (\sigma \in \mathfrak{S}_A, \alpha \in \mathcal{M}_A). \quad (3.10)$$

It is possible to extend the concept of $\mathfrak{S}_A$ -minimal orbit.

For each $\alpha \in \mathcal{M}_n$, let us consider the sets

$$E_\alpha = \{i \in \mathbf{I}_n : \alpha_i \text{ is even}\} \quad O_\alpha = \{i \in \mathbf{I}_n : \alpha_i \text{ is odd}\} \quad (3.11)$$

as defined in the previous section. Then, it is clear that α can be determined from its restrictions α_E and α_O to the sets E_α and O_α , respectively. Observe that $\alpha_E \in \mathcal{M}_{E_\alpha}$ and $\alpha_O \in \mathcal{M}_{O_\alpha}$.

Proposition 3.3. *Let $\alpha \in \mathcal{M}_n$ such that α is $\mathcal{B}_n$ -minimal, and then $\alpha_E/2$ is $\mathfrak{S}_{E_\alpha}$ -minimal and $(\alpha_O - 1)/2$ is $\mathfrak{S}_{O_\alpha}$ -minimal.*

Proof. It follows from Proposition 4 in [3] and the identities

$$\begin{aligned} \left| \left(\frac{\alpha_E}{2} \right)^{-1}(i) \right| &= |\alpha(i)| \quad (\forall i \in |E_\alpha|), \\ \left| \left(\frac{\alpha_O - 1}{2} \right)^{-1}(j) \right| &= |\alpha(j)| \quad (\forall j \in |O_\alpha|). \end{aligned} \quad (3.12)$$

□

Notation 1. Let $\mathcal{K}$ be the subset of $\mathcal{M}_n$ given by

$$\mathcal{K} = \left\{ \alpha \in \mathcal{M}_n : \alpha \text{ is } \mathfrak{D}_n\text{-minimal, } |E_\alpha| = |O_\alpha| \text{ and } \frac{\alpha_E}{2} < \frac{\alpha_O - 1}{2} \right\}. \quad (3.13)$$

It will be denoted by $\mathcal{F}$ the subset of the polynomial ring $\mathcal{P}$:

$$\mathcal{F} = \left\{ \sum_{\alpha} \lambda_{\alpha} x^{\alpha} : \lambda_{\alpha} = 0 \text{ if } \alpha \in \mathcal{K} \right\}. \quad (3.14)$$

Note that if n is odd, $\mathcal{F}$ is equal to $\mathcal{P}$.

Proposition 3.4. *$\mathcal{F}$ is a $\mathfrak{D}_n$ -submodule of the polynomial ring $\mathcal{P}$.*

Proof. It follows from the action of $\mathfrak{D}_n$ given by

$$(w, \sigma) \left(\sum_{\alpha} \lambda_{\alpha} x^{\alpha} \right) = \sum_{\alpha} \lambda_{\alpha} (wx)^{\sigma\alpha} = \sum_{\alpha} \pm \lambda_{\alpha} x^{\sigma\alpha} \quad (3.15)$$

and the fact that $\alpha \in \mathcal{K}$ if, and only if, $\sigma\alpha \in \mathcal{K}$, and it results that $\mathcal{F}$ is a $\mathfrak{D}_n$ -module of $\mathcal{P}$. $\square$

Proposition 3.5. $\mathcal{F}$ contains a Gelfand model for the Weyl group of type $\mathfrak{D}_n$.

Proof. It is sufficient to prove that $\mathcal{F}$ contains a submodule equivalent to the regular module $\mathfrak{D}_n$. Effectively, let us consider the polynomial:

$$P(x_1, \dots, x_n) = \frac{\prod_{i=2}^n [x_i(x_i + i - 1) \prod_{j=1}^{i-2} (x_i^2 - j^2)]}{2^{n-1} [(n-1)!]^2 \prod_{i=2}^n \prod_{j=1}^{i-2} (i-1+j)(i-1-j)}. \quad (3.16)$$

Thus P is the interpolating polynomial of the orbit of the regular vector $v = (0, 1, \dots, n-1)$, which satisfies

$$P(v) = 1, \quad P(\tau v) = 0 \quad (\forall \tau \in \mathfrak{D}_n, \tau \neq e). \quad (3.17)$$

It will be proved that P belongs to $\mathcal{F}$. Let $\lambda_{\alpha} x^{\alpha}$ be not a null term of P such that it is $\mathfrak{D}_n$ -minimal, $|E_{\alpha}| = |O_{\alpha}|$, and $\alpha_E/2 < (\alpha_O - 1)/2$. As P was defined in (3.16), it is easy to determine that $\alpha_1 = 0$ and $\alpha_j > 0$ for $1 < j \leq n$. Then, $(\alpha_E/2)_1 = 0$, and as $\alpha_E/2$ is $\mathfrak{S}_{|E_{\alpha}|}$ -minimal, it is obtained that

$$\widehat{\frac{\alpha_E}{2}} = (|E_{\alpha}| - 1, |E_{\alpha}| - 2, \dots, 0), \quad (3.18)$$

which by Proposition 3.2 is maximal, which is a contradiction.

Since $\mathcal{F}$ is a $\mathfrak{D}_n$ -module, $\mathcal{F}$ contains the module generated by the orbit of P , which is isomorphic to the regular module. Hence, $\mathcal{F}$ contains a Gelfand model. $\square$

Notation 2. Let $\mathcal{T}$ be a subset of the polynomial ring $\mathcal{P}$ and G a finite subgroup of $GL_n(\mathbb{C})$; we will denote by $\mathcal{T}^0$ the subset of $\mathcal{T}$ defined by

$$\mathcal{T}^0 = \{p \in \mathcal{T} : D(p) = 0, \forall D \in \mathcal{D}_G^-\}, \quad (3.19)$$

$\mathcal{D}_G^-$ being the set of differential operators invariant in the algebra of Weyl as it has been defined in (2.9).

Proposition 3.6. Let G be a finite subgroup of $GL_n(\mathbb{C})$ and $\mathcal{T}$ a G -module of the polynomial ring $\mathcal{P}$ such that $\mathcal{T}$ contains a model of G , and then $\mathcal{T}^0$ also contains a model of G .

Proof. Let $S \subset \mathcal{T}$ be a simple G -module and suppose that $S \not\subset \mathcal{T}^0$. Then, there exists $D \in \mathcal{D}_G^-$ such that $D(S) \neq 0$. Because S is simple and D not null, it follows that D is injective. Thus, $D(S) \simeq S$. If $D(S) \subset \mathcal{T}^0$, the proposition is proved; otherwise the procedure will be repeated.

As D is an operator of the Weyl algebra $\mathcal{A}$ with negative degree, the procedure is finite, that is to say, there exists $m \in \mathbb{N}$ such that $D^m(S) \subset \mathcal{T}^0$ and $D^m(S) \simeq S$. $\square$

Remark 3.7. An immediate consequence of this proposition is that the $\mathfrak{D}_n$ -module

$$\mathcal{F}^0 = \left\{ f \in \mathcal{F} : D(f) = 0, \forall D \in \mathcal{D}_{\mathfrak{D}_n}^- \right\} \quad (3.20)$$

contains a Gelfand model because $\mathcal{F}$ is a $\mathfrak{D}_n$ -module containing a Gelfand model.

Remark 3.8. Notice that if n is odd, then $\mathcal{F}^0 = \mathcal{N}_{\mathfrak{D}_n}$; instead, if n is even,

$$\mathcal{F}^0 \subseteq \mathcal{F} \cap \mathcal{N}_{\mathfrak{D}_n} = \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta \quad (3.21)$$

that is

$$\mathcal{F}^0 \subseteq \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| > |O_\gamma|}} S_\gamma^\Delta. \quad (3.22)$$

Using the result established in item 4 of Proposition 2.16 for decomposing $\mathcal{N}_{\mathfrak{D}_n}$, it follows that

$$\mathcal{N}_{\mathfrak{D}_n} = \bigoplus_{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal}} S_\gamma^\Delta = \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| > |O_\gamma|}} S_\gamma^\Delta. \quad (3.23)$$

Moreover if $|E_\gamma| = |O_\gamma|$

$$\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta = \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| > |O_\gamma|}} S_\gamma^\Delta. \quad (3.24)$$

As a consequence of this decomposition the next lemma follows.

Lemma 3.9. *The dimension of $\mathcal{N}_{\mathfrak{D}_n}$ is equal to*

$$2 \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta \right) + 2 \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| = |O_\gamma|}} S_\gamma^\Delta \right) + \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathfrak{D}_n}\text{-minimal} \\ |E_\gamma| > |O_\gamma|}} S_\gamma^\Delta \right). \quad (3.25)$$

Proof. It results from considering the identity

$$\dim(S_\gamma^\Delta) = \dim(S_{\pi \circ \gamma}^\Delta) \quad (3.26)$$

for each $\gamma \in \mathcal{O}_n$. This identity occurs from the fact that S_π^Δ and $S_{\pi \circ \gamma}^\Delta$ are isomorphic as $\mathfrak{D}_n$ -modules, see [5]. $\square$

Let G be a group; from now on we will denote by

$$\text{Inv}(G) = \{g \in G : g^2 = e\} \quad (3.27)$$

the set of involutions of the group G .

Lemma 3.10. *Let G be a Coxeter group and $\mathcal{M}$ a Gelfand model for G . Then,*

$$\dim(\mathcal{M}) = |\text{Inv}(G)|. \quad (3.28)$$

Proof. It is a consequence from the Fröbenius-Schur indicator and the fact that the representations of a Coxeter group can be realized over the real numbers, see [28]. $\square$

With the purpose to establish the central result of this work, a relationship between the number of involutions of $\mathcal{B}_n$ and the number of involutions of $\mathfrak{D}_n$ will be given. This will be used in the next theorem.

Lemma 3.11. *If n is even ($n = 2k$), then*

$$2|\text{Inv}(\mathfrak{D}_n)| - |\text{Inv}(\mathcal{B}_n)| = \frac{(2k)!}{k!}. \quad (3.29)$$

Proof. If $\sigma = (w, \pi) \in \mathcal{B}_n$, with $w \in \mathcal{C}_2^n$, and $\pi \in \mathfrak{S}_n$ is an involution, then the cyclic structure of σ looks like

$$(\pm i_1, \pm j_1) \cdots (\pm i_r, \pm j_r)(\pm k_1) \cdots (\pm k_s), \quad (3.30)$$

where $\mathbf{I}_n = \{i_1, \dots, i_r, j_1, \dots, j_r, k_1, \dots, k_s\}$, $\pi = (i_1, j_1) \cdots (i_r, j_r)(k_1) \cdots (k_s)$ is the decomposition of π as product of disjoint cycles, and $w_{i_l} = w_{j_l}$ for $1 \leq l \leq r$. Thus, the number of involutions of $\mathcal{B}_n$ is

$$|\text{Inv}(\mathcal{B}_n)| = \sum_{r=0}^k \frac{\prod_{j=0}^{r-1} \binom{n-2j}{2}}{r!} 2^r 2^{n-2r} \quad (3.31)$$

If $r < k$, half of the elements belong to $\mathfrak{D}_n$ and the other half to $\mathcal{B}_n - \mathfrak{D}_n$, and therefore

$$|\text{Inv}(\mathfrak{D}_n)| = \frac{1}{2} \sum_{r=0}^{k-1} \frac{\prod_{j=0}^{r-1} \binom{n-2j}{2}}{r!} 2^r 2^{n-2r} + \frac{\prod_{j=0}^{k-1} \binom{n-2j}{2}}{k!} 2^k 2^{n-2k}. \quad (3.32)$$

Then,

$$2|\text{Inv}(\mathfrak{D}_n)| - |\text{Inv}(\mathcal{B}_n)| = \frac{\prod_{j=0}^{k-1} \binom{n-2j}{2}}{k!} 2^k 2^{n-2k} = \frac{(2k)!}{k!}. \quad (3.33)$$

□

Theorem 3.12. *The G -module $\mathcal{F}^0$ is a Gelfand model for the group $\mathfrak{D}_n$.*

Proof. As it has been mentioned above, when n is odd, $\mathcal{F}^0$ is equal to $\mathcal{N}_{\mathfrak{D}_n}$, and in [5] it has been proved that $\mathcal{N}_{\mathfrak{D}_n}$ is a Gelfand model for the group $\mathfrak{D}_n$.

When n is even, from the fact $\mathcal{F}^0$ contains a Gelfand model, only it is necessary to prove that $\dim(\mathcal{F}^0) \leq |\text{Inv}(\mathfrak{D}_n)|$. From identity (3.22), it results that

$$\dim(\mathcal{F}^0) \leq \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathcal{B}_n} \text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\widehat{(\gamma_O-1)}/2 < \gamma_E/2} S_\gamma^\Delta + \bigoplus_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right). \quad (3.34)$$

By Lemma 3.10, it follows that the dimension of the model $\mathcal{N}_{\mathcal{B}_n}$ is equal to the number of involutions of the group $\mathcal{B}_n$, and thus by the Lemma 3.9 it results that

$$\begin{aligned} |\text{Inv}(\mathcal{B}_n)| &= 2 \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathcal{B}_n} \text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta \right) + 2 \dim \left(\bigoplus_{\gamma_E/2 < \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right) + \dim \left(\bigoplus_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right), \\ |\text{Inv}(\mathcal{B}_n)| + \dim \left(\bigoplus_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right) &= 2 \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathcal{B}_n} \text{-minimal} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{\widehat{(\gamma_O-1)}/2 \leq \gamma_E/2} S_\gamma^\Delta \right), \\ \frac{1}{2} \left[|\text{Inv}(\mathcal{B}_n)| + \dim \left(\bigoplus_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right) \right] &= \dim \left(\bigoplus_{\substack{v \in \mathcal{O}_{\mathcal{B}_n} \text{-minimal} \\ |E_v| < |O_v|}} S_v^\Delta + \bigoplus_{\widehat{(\gamma_O-1)}/2 \leq \gamma_E/2} S_\gamma^\Delta \right), \\ \dim \left(\bigoplus_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} S_\gamma^\Delta \right) &= \sum_{\gamma_E/2 = \widehat{(\gamma_O-1)}/2} \dim(S_\gamma^\Delta) = \sum_{\chi \in \mathfrak{S}_n} \binom{2n}{n} \chi^2(1) = \frac{(2n)!}{n!} \end{aligned} \quad (3.35)$$

and then

$$\frac{1}{2} \left[|\text{Inv}(\mathcal{B}_n)| + \frac{(2n)!}{n!} \right] = \dim \left(\bigoplus_{\substack{\gamma \in \mathcal{O}_{\mathcal{B}_n\text{-minimal}} \\ |E_\gamma| < |O_\gamma|}} S_\gamma^\Delta + \bigoplus_{(\gamma_O - 1)/2 \leq \gamma_E/2} S_\gamma^\Delta \right). \quad (3.36)$$

On the other hand, from identity established in Lemma 3.11, it results that

$$|\text{Inv}(\mathfrak{D}_n)| = \frac{1}{2} \left[|\text{Inv}(\mathcal{B}_n)| + \frac{(2n)!}{n!} \right], \quad (3.37)$$

and using identities (2.2) and (3.36), it is obtained that

$$\dim(\mathfrak{F}^0) \leq |\text{Inv}(\mathfrak{D}_n)|. \quad (3.38)$$

Therefore, it has been proved that $\mathfrak{F}^0$ is a Gelfand model for $\mathfrak{D}_n$. $\square$

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