Research Article

Basics Polyakov's Quantum Surface Theory on the Formalism of Functional Integrals and Applications

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We propose an alternative view to the covariant Polyakov's string path integral. In our new approach we clarified the role of the Liouville model on string theory for Q.C.D. and Dual Models. In the appendices, we present additional material, difficult to find in the specialized path integral literature on the detailed evaluation of the (Q.C.D.) fermion determinant, the path integral proof of the Atiyah-Singer index theorem on Riemannian Manifolds, and the role of the $1/D$ expansion on Polyakov's String path integral, all containing important new results, insights on these topics.

1. Introduction

Since its inception seventy years ago, nonabelian gauge theories have been shown to be the most promising mathematical formalism for a realistic description of strong interactions and even formulated on its supersymmetric version; it became an attractive attempt for unify Physics.

In strong interaction Physics the picture image of a mesonic quantum excitation is, for instance, a wave quantum mechanical functional assigned to a classical configuration of a space-time trajectory of a pair quark-antiquark bounding a space-time nonabelian gluon flux surface (of all topological genera) connecting both particle pairs: the famous t'Hooft-Feynman planar diagrams [1].

It appears thus appealing for mathematical formulations to be considered directly as dynamical variables or wave functions in this Faraday line framework for nonabelian gauge theories, the famous quantum Wilson Loop, or (quantum) holonomy factor associated to

a given space-time Feynman quark-antiquark closed trajectory C in a $SU(N_c)$ Yang-Mills quantum field theory. Namely,

$$\psi[C] = \langle W[C] \rangle = \left\langle \frac{1}{N_c} \left\{ \text{Tr}_{SU(N_c)}^{(c)} \left[\mathbb{P} \left(\exp i \oint_C A_\mu dx_\mu \right) \right] \right\} \right\rangle. \quad (1.1)$$

Here $\langle \dots \rangle$ is defined by the Yang-Mills quantum path integral and $A_\mu(x) = A_\mu^c(x) \lambda_a$ denotes the quantized Yang-Mills (Gluon) connection.

It thus searched loop space dynamical equations (at least on the formal level without considering those famous ultraviolet “renormalization problems”) for the new wave function equation (1.1). However, Lattice Yang-Mills theories have indicated that “string solutions” with a symbolic structure form

$$\psi[C] = \sum_{S, \partial S = C} \left\{ \left[\exp \left(-\frac{1}{2\pi\alpha'} A(S) \right) \right] \Phi^{QCD}[C] \right\} \quad (1.2)$$

with $\sum_{S, \partial S = C}$ denoting the Feynman-Wiener continuous sum over all surfaces S , with all topological genera and bounded by C ; α' is the fundamental constant of strong force physics called the Regge slope parameter [1]; $A(S)$ is the area functional “evaluated” at the sample surface S ; $\Phi[C]$ is a functional related probably to the existence of a (neutral) nonabelian 2D intrinsic fermions structure on the surface S , called the Elfin fermionic functional [1].

It is a consequence of (1.1)-(1.2) that nonabelian gauge theories (even on supersymmetric versions [2]) should be better reformulated as a dynamics of random surfaces (strings theories).

Our aim in this paper is to present in full technical details, added with our original improvements, the work of A. M. Polyakov [1] to give a precise path integral meaning for (1.2) in a 2D gravitational context and only considering the case of trivial surface topology.

We give (also in details) the path integral meaning for the usual case of Nambu-Goto for (1.2) in the pure surface context, our new results [1].

This paper is organized as follows: in Section 2 we survey some basic results of classical surface theory. In Section 3, we expose the physical toy model exactly soluble Polyakov’s framework. In Section 4, we expose the Nambu-Goto theory (our results).

2. Elementary Results on the Classical (Bosonic) Surfaces Theory

A given surface S , with a boundary being a curve C and embedded into euclidean space time R^D , is usually described by a vector field $X_\mu(\xi_1, \xi_2)$ ($\mu = 1, \dots, D$) and a given two-dimensional domain Ω , compatible with the topology of S in the case of the surface’s hypothesis smoothness. It is imposed also that such vector field, called from now on surface vector position field, when restricts to the boundary of Ω should be a reparameterization of $C(X_\mu(\xi_1, \xi_2))_{\partial\Omega} = C$.

The Nambu-Goto area functional associated to a given surface vector position $(X_\mu(\xi_1, \xi_2))$ is given by usual Riemann integral $((\xi_1, \xi_2) := \xi)$:

$$A(S_C) = \int_{\Omega} (\det\{h_{ab}(\xi)\})^{1/2} d^2\xi \quad (2.1)$$

with $h_{ab}(\xi) = (\partial_a X^\mu \partial_b X_\mu)(\xi)$ denoting the metric tensor induced on the surface S_c (rigorously induced on the surface-manifold tangent bundle) by the parameterization $\{X_\mu(\xi)\}$.

The most important property of the functional equation (2.1) is its invariance under the (formal) group of the reparameterizations of S . Namely,

$$\begin{aligned} (\xi_1, \xi_2) &= \varphi_1(\xi'_1, \xi'_2); \varphi_2(\xi'_1, \xi'_2), \\ X'_\mu(\xi') &= X_\mu(\varphi(\xi')). \end{aligned} \quad (2.2)$$

$\varphi(\xi') =: (\varphi_1(\xi'), \varphi_2(\xi'))$ Here $\varphi(\xi')$ denotes a two-dimensional intrinsic C_1 vector field on Ω with everywhere nonzero Jacobian.

Formal Euler-Lagrange equations associated to the surface action functional equation (2.1) can be easily written and produce the boundary value problem on Ω for each μ -component ($\mu = 1, \dots, D$):

$$\begin{cases} \Delta_h X^\mu(\xi) = 0, \\ X^\mu(\xi)|_{\xi \in \partial\Omega} = C^\mu, \end{cases} \quad (2.3)$$

where Δ_h denotes the second-order elliptic operator called Laplace-Beltrami associated to the metric $(h_{ab}(\xi))(h(\xi) = \det\{h_{ab}(\xi)\})$:

$$\Delta_h = \left(\frac{1}{\sqrt{h}} \partial_a \left(\sqrt{h} h^{ab} \partial_b \right) \right) (\xi). \quad (2.4)$$

If one now choose the conformal gauge for the surface $h_{ab}(\xi) = e^{\varphi(\xi)} \delta_{ab}$, formally one obtains that the surface vector position satisfies the Dirichlet problem in Ω :

$$\begin{cases} \Delta_{h=\delta_{ab}} X^\mu(\xi) = 0, \\ X^\mu(\xi)|_{\xi \in \partial D} = C^\mu. \end{cases} \quad (2.5)$$

The solution of the above-mentioned potential problem can be exactly given by conformal complex variable theory methods ([2], Chapter 1).

Note that (2.5) produces minimal surfaces (i.e., classical surfaces which minimize the area functional locally) as the associated classical surfaces actions on the Nambu-Goto theory (2.1). Note that all the above-displayed discussion remains correct to the case of general topological genera (i.e., $\chi(S) = (1/4\pi) \int_\Omega (\sqrt{h} R(h)) d\xi^2 \neq 1$; here $R(h)$ denoting the scalar of curvature associated to the induced metric $h_{ab}(\xi)$).

However, at the classical level, there is a quadratic area functional due to Howe-Brink-Polyakov equivalent to the above result related to the classical aspects Nambu-Goto action. It is the functional associated to a theory of D scalar classical massless fields on Ω but interacting with intrinsic (news) dynamical degrees of freedom given by the infinite-dimensional manifold of two-dimensional metrical structures on Ω .

The classically equivalent area action function is now given by the massless fields $\{X^\mu(\xi)\}$ gravitation content:

$$\overline{A}(S_C) := \frac{1}{2} \int_{\Omega} \left(\sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \right) (\xi) d^2 \xi, \quad (2.6)$$

where $\{g_{ab}(\xi)\}_{a=1,2 \atop b=1,2}$ belongs to the infinite-dimensional space manifold of all possible metrics admissible by the surface S the famous exactly soluble string Polyakov' model.

That more rich (from a dynamically point of view) surface action functional is now invariant (formally) under the extended group of surface reparameterizations (the group of local diffeomorphism of the surface S):

$$\begin{aligned} \xi'_a - \xi_a &= \delta \xi_a = \mathcal{E}_a(\xi), \\ \delta X_\mu(\xi) &= \mathcal{E}_a(\xi) \partial^a X_\mu(\xi), \\ \delta g_{ab}(\xi) &= (\nabla_a \mathcal{E}_b + \nabla_b \mathcal{E}_a)(\xi). \end{aligned} \quad (2.7)$$

Here the surface's reparameterization vector fields $\{\mathcal{E}_a(\xi)\}$ are such that $(\nabla^a \mathcal{E}_a)(\xi) = 0$, a necessary restriction in order to take out from the above written reparameterizations, all those which act on as a simply metrical rescalings. Note that we have imposed further the vector field $\mathcal{E}_a(\xi)$ vanishing at the boundary of S (i.e., $\mathcal{E}_a(\xi)|_{\partial\Omega} \equiv 0$). The covariant derivative is usually defined by the Christoffel-Schwartz objects associated to the given metric tensor $\{g_{ab}(\xi)\}$:

$$(\nabla_a \mathcal{E}_b)(\xi) = \left(g_{aa'} \nabla^{a'} \right) (\mathcal{E}_b)(\xi) = (\partial_a \mathcal{E}_b - \Gamma_{ab}^c \mathcal{E}_c)(\xi), \quad (2.8a)$$

$$\Gamma_{ab}^c(\xi) = \frac{1}{2} \left\{ g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \right\} (\xi). \quad (2.8b)$$

In the important case of the metric field $g_{ab}(\xi)$ has the conformal form $g_{ab}(\xi) = e^{\varphi(\xi)} \delta_{ab}$; the above written objects take the simple forms written below:

$$\begin{aligned} \Gamma_{ab}^c(\xi) &= \frac{1}{2} \delta^{cd} e^{-\varphi} [\partial_a \varphi e^{\varphi} \delta_{bd} + \partial_b \varphi e^{\varphi} \delta_{ad} - \partial_d \varphi e^{\varphi} \delta_{ab}] \\ &= \frac{1}{2} (\partial_a \varphi) \delta^{cb} + \frac{1}{2} (\partial_b \varphi) \delta^{ca} - \frac{1}{2} \partial_c \varphi \delta_{ab}, \end{aligned} \quad (2.9a)$$

$$\nabla_a \mathcal{E}_b = \partial_a \mathcal{E}_b - \frac{1}{2} (\partial_c \varphi) \mathcal{E}^c \delta_{ab} - \frac{1}{2} [(\partial_a \varphi) \mathcal{E}_b - (\partial_b \varphi) \mathcal{E}_a]. \quad (2.9b)$$

Note that one has in addition to the usual reparameterization invariance one further pivotal (point dependent) new symetry called the Weyl conformal symetry which acts solely on the new degree of freedom (with $\lambda(\xi) > 0$), crucial for the exactly model solubility:

$$g_{ab}(\xi) \longrightarrow \lambda(\xi) g_{ab}(\xi). \quad (2.10)$$

The classical motion equations associated to the Howe-Brink-Polyakov functional are easily written down:

$$\begin{aligned}\Delta_{g_{ab}} X^\mu(\xi) &= 0, \\ X^\mu(\xi)|_{\partial\Omega} &= C^\mu, \\ T_{ab}(\xi) &= \left(\partial_a X_\mu \partial_b X^\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu \right)(\xi) = 0.\end{aligned}\tag{2.11}$$

From the last classical motion equation (2.11) for the intrinsic metric field, one obtains the result

$$g_{ab}(\xi) = \bar{\lambda}(\xi) (\partial_a X_\mu \partial_b X^\mu)(\xi),\tag{2.12}$$

where the unknown scale $\bar{\lambda}(\xi)$ can be fixed to be the function $\bar{\lambda}(\xi) = 1$. Note that there is the additional boundary condition

$$(g_{ab}(\xi))_{\partial\Omega} = h_{ab}(\xi)|_{\partial\Omega} = (\partial_a X_\mu \partial_b X^\mu)|_{\partial\Omega}.\tag{2.13}$$

Another important classical surface function, probably related to the still not completely understood functional $\Phi^{ACD}[C]$ on (1.2), is the so-called extrinsic functional which is defined by the square of the surface mean curvature (see (2.4)):

$$\left(\mathcal{H}^2(S) \right) = (\nabla_h X^\mu)^2 = \left\{ -\frac{1}{\sqrt{h}} \partial_a \left(\sqrt{h} h^{ab} \partial_b \right) (X^\mu) \right\}^2.\tag{2.14}$$

Namely,

$$\mathcal{F}_{\text{extr}}(S) = \int_{\Omega} \sqrt{h} \left((\Delta_h X^\mu)^2(\xi) \right) d^2 \xi.\tag{2.15}$$

In the Polyakov's version of two-dimensional Ω quantum gravity, the above functional can be replaced by the fourth-order scalar field action:

$$\begin{aligned}\mathcal{F}_{\text{ext}}[X^\mu(\xi), g_{ab}(\xi)] &= \int_{\Omega} \left(\sqrt{g} (\Delta_g X^\mu)^2(\xi) \right) d^2 \xi \\ &+ \int_{\Omega} \left(\sqrt{g} \lambda^{ab}(\xi) (g_{ab}(\xi) - h_{ab}(\xi, X^\mu)) \right)(\xi),\end{aligned}\tag{2.16}$$

where $\lambda^{ab}(\xi)$ denote lagrange multipliers insuring that at the classical level $g_{ab}(\xi) = (\partial_a X^\mu \partial_b X_\mu)(\xi)$.

Boundary conditions to be imposed on the complete action $A(S) + \mathcal{F}_{\text{ext}}(S)$ are mathematical subtle in order to guarantee the mathematical good property of strong elliptic of the associated boundary value problem ([3], Chapter 5).

For instance, if one considers the case $g_{ab} = \delta_{ab}$ and the associated biharmonic operator

$$(\Delta X^\mu)^2 = \frac{\partial^4 X^\mu}{\partial \xi_1^4} + 2 \frac{\partial^4 X^\mu}{\partial \xi_1^2 \partial \xi_2^2} + \frac{\partial^4 X^\mu}{\partial \xi_2^4}, \quad (2.17a)$$

one can associate the following system of boundary operators:

$$\begin{aligned} X^\mu(\xi)|_{\partial\Omega} &= C^\mu(\xi), \\ -\frac{\partial}{\partial n} X^\mu(\xi) \Big|_{\partial\Omega} &= M^\mu(\xi). \end{aligned} \quad (2.17b)$$

Here the meaning of M^μ is not so clear as an independent dynamical parameter of the loop C (string).

In the following discussions we will always regard the fourth-order actions as effective actions coming from the integrating out fermionic intrinsic degrees of freedom [1].

We left to our readers to prove the Green's formula for the fourth-order problem equations (2.17a) and (2.17b):

$$\begin{aligned} \int_{\Omega} (f \Delta^2 g) d^2 \xi &= \int_{\Omega} (g \Delta^2 f) d^2 \xi \\ &+ \int_{\partial\Omega} \left(-f \partial \frac{\Delta g}{\partial n} + \frac{\partial f}{\partial n} \Delta g + g \frac{\partial}{\partial n} \Delta g - \frac{\partial g}{\partial n} \Delta g \right) ds. \end{aligned} \quad (2.18)$$

As a further comment and just for the reader's curiosity, one has an analogue of representing (at least locally) two-dimensional harmonic functions on Ω by analytical complex variable functions; one still has the following representation (not unique!) for biharmonic functions [1, 5]:

$$\mathcal{U}(\xi_1, \xi_2) = \text{Re} \{ \bar{z} \varphi(z) + \rho(z) \}, \quad (2.19)$$

where $z = \xi_1 + i\xi_2$ and $(\varphi(z), \rho(z))$, a pair of complex variable analytic functions on Ω .

If the surface S is on R^3 and given in "nonparametric form" $X^3 = f(X^1, X^2)$, we can introduce parametric coordinates $\xi^1 = X^1$, $\xi^2 = X^2$ and obtain a result convenient for computations (exercises for the diligent reader); the Gauss curvature is given explicitly by

$$R(h_{ab}) = \left(1 + f_{X^1}^2 + f_{X^2}^2 \right)^2 \cdot \det \{ f_{ab}(X^1, X^2) \}. \quad (2.20a)$$

and for the mean curvature, we have the following result:

$$H = \left(1 + f_{X^1}^2 + f_{X^2}^2 \right)^{-3/2} \left\{ \left(1 + f_{X^2}^2 \right) f_{X_1 X_1} - 2 f_{X_1} f_{X_2} f_{X_1 X_2} + \left(1 + f_{X_1}^2 \right) f_{X_2 X_2} \right\}. \quad (2.20b)$$

It is worth to recall that in the important case of a surface S embedded in R^3 (useful in Surface's Statistical Physics), but with the general parameterization:

$$X^1 = X^1(\xi_1, \xi_2); \quad X^2 = X^2(\xi_1, \xi_2); \quad X^3 = X^3(\xi_1, \xi_2), \quad (2.21)$$

the expression for the area functional and the surface extrinsic functional can be given explicitly by (exercise for our diligent readers!)

$$A(S) = \int_{\Omega} \left(\sqrt{EG - F^2} \right) (\xi) d^2\xi, \quad (2.22a)$$

$$\mathcal{F}_{\text{ext}}(S) = \int_{\Omega} H^2(\xi) d^2\xi, \quad (2.22b)$$

where the surface objects are given by

$$\begin{aligned} E &= \left(\frac{\partial X^\mu}{\partial \xi_1} \right), \left(\frac{\partial X_\mu}{\partial \xi_1} \right); \quad F = \left(\frac{\partial X^\mu}{\partial \xi_1} \right), \left(\frac{\partial X_\mu}{\partial \xi_2} \right); \quad G = \left(\frac{\partial X^\mu}{\partial \xi_2} \right), \left(\frac{\partial X_\mu}{\partial \xi_2} \right); \\ H &= \frac{1}{(EG - F^2)} (ED'' - 2D'F + DG); \\ D &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{\xi_1 \xi_1}^1 & X_{\xi_1 \xi_1}^2 & X_{\xi_1 \xi_1}^3 \\ X_{\xi_1}^1 & X_{\xi_1}^2 & X_{\xi_1}^3 \\ X_{\xi_2}^1 & X_{\xi_2}^2 & X_{\xi_2}^3 \end{vmatrix}, \\ D' &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{\xi_1 \xi_2}^1 & X_{\xi_1 \xi_2}^2 & X_{\xi_1 \xi_2}^3 \\ X_{\xi_1}^1 & X_{\xi_1}^2 & X_{\xi_1}^3 \\ X_{\xi_2}^1 & X_{\xi_2}^2 & X_{\xi_2}^3 \end{vmatrix}, \\ D'' &= \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} X_{\xi_2 \xi_2}^1 & X_{\xi_2 \xi_2}^2 & X_{\xi_2 \xi_2}^3 \\ X_{\xi_1}^1 & X_{\xi_1}^2 & X_{\xi_1}^3 \\ X_{\xi_2}^1 & X_{\xi_2}^2 & X_{\xi_2}^3 \end{vmatrix}. \end{aligned} \quad (2.22c)$$

Finally we remark that one could easily generalize all the above written results to the general case of the initial ambient space R^D being a new general Riemann Manifold $(M, G_{\mu\nu}(X^\gamma))$. In this case, the “induced” surface metric tensor $h_{ab}(X(\xi)) = \partial_a X^\mu(\xi) \partial_b X_\nu(\xi)$ will be replaced by the new “induced” tensor:

$$h_{x\beta}(X(\xi), [G_{\mu\nu}]) = \partial_a X^\mu(\xi) G_{\mu\nu}(X^\gamma(\xi)) \partial_b X^\nu(\xi) \quad (2.23)$$

on the exposed formulae. The full classical and quantum theory of the surfaces σ -models can be found in [1].

3. Path Integral Quantization on Polyakov's Theory of Surface (or How to Quantize 2D Massless Scalar Fields in the Presence of 2D Quantum Gravity)

After exposing some basic concepts of the classical surface theory in Section 2, we now pass to the vital problem of quantization on the formalism of Feynman-Wiener path integrals.

Following R. P. Feynman in his theory of path integration sum over histories, a mathematical meaning for the continuous sum over (now), euclidean quantum (random) surfaces for (1.2), should be given by following path integrals for the Nambu-Goto case and of A. M. Polyakov, respectively, (see (2.1)–(2.6)):

$$G_{\text{NG}}(C) = \int_{X_\mu(\xi)|_{\xi \in \partial\Omega} = C_\mu} d\mu[X^\mu(\xi)] \exp\left\{-\frac{1}{2\pi\alpha'} A(S_c)\right\}, \quad (3.1a)$$

$$G_P(C) = \int_{\substack{X_\mu(\xi)|_{\xi \in \partial\Omega} = C_\mu \\ g_{ab}(\xi)|_{\xi \in \partial\Omega} = 0}} d\bar{\mu}[X^\mu(\xi), g_{ab}(\xi)] \exp\left\{-\frac{1}{2\pi\alpha'} \bar{A}(S_c)\right\} \\ \times \exp\left\{-\mu_0^2 \int_{\Omega} \sqrt{g}(\xi) d^2\xi\right\}. \quad (3.1b)$$

Here α' and μ_0^2 are constants to be identified with certain physical parameters of the theory. Note that α' has dimension of inverse if mass square and $[\mu_0^2] = [\alpha']^{-1}$. It is an important remark that the Feynman-Wiener path measures $d\mu[X^\mu(\xi)]$ and $d\bar{\mu}[X^\mu(\xi), g_{ab}(\xi)]$ must be defined in such a way in order to be fully invariant under the action of the local diffeomorphism group of both descriptions (see (2.2) and (2.7)).

We aim now to evaluate explicitly the A. M. Polyakov's integral equation (3.1b). As a first step one recall that we have imposed the "fluctuating" zero Dirichlet boundary condition for the intrinsic metric field $g_{ab}(\xi)(g_{ab}(\xi)|_{\partial\Omega} = 0)$. Those results lead us to the effective action for the surface dynamics weight function:

$$\bar{A}(S_c) = \frac{1}{2} \int_{\Omega} (\sqrt{g} X^\mu (-\Delta_g) X_\mu)(\xi) d^2\xi. \quad (3.1c)$$

Since at the classical level, the metric field decouples (one can choose it in the form $g_{ab} = e^\varphi \delta_{ab}$) and considers the usual classical plus quantum decomposition $X_\mu(\xi) = X_\mu^{\text{CL}}(\xi) + \sqrt{(\pi\alpha')} X_\mu^q(\xi)$, where the quantum correction $X_\mu^q(\xi)$ is such that $X_\mu^q(\xi)|_{\partial\Omega} = 0$.

After these preliminaries considerations one is led to evaluate the following covariant path integral (after disregarding classical field contributions to the path integral):

$$I[g_{ab}] = \int d\mu[X^\mu(\xi)] \exp\left\{-\frac{1}{2} \int_{\Omega} (\sqrt{g} X^\mu (-\Delta_g) X_\mu)(\xi) d^2\xi\right\}. \quad (3.1d)$$

It is worth recalling that there is a classical term (a functional depending on the loop boundary C) coming from the partial integration to arrive at ((3.1a), (3.1b), (3.1c), and (3.1d)) for the case $g_{ab} = \delta_{ab}$, which clearly does not satisfy the quantum-fluctuating nature of these nonconstant $g_{ab}(\xi)$:

$$\int_{\Omega} \left(\partial_a X_{\text{el}}^{\mu} \partial_a X_{\mu, \text{el}} \right) (\xi) d^2 \xi = \int_{\partial \Omega} \overbrace{\left(C^{\mu} \frac{\partial}{\partial n} C_{\mu} \right)}^{\text{classical contribution}} (\sigma) d\sigma. \quad (3.2)$$

The Feynman-Wiener measure $d\mu[X^{\mu}(\xi)]$ on (3.1d) must be chosen in order to be invariant under (2.7). The only local candidate is given by the “weighted” product Feynman measure [1]:

$$d\mu[X^{\mu}(\xi)] := \prod_{\xi \in \Omega} \left(\sqrt[4]{g(\xi)} dX^{\mu}(\xi) \right). \quad (3.3)$$

As a consequence the path integral equations ((3.1a), (3.1b), (3.1c), and (3.1d)) turns out to be Gaussian and metric dependent. It yields the immediate result:

$$I[g_{ab}] = \det^{-D/2}(-\Delta_g). \quad (3.4)$$

We thus have reduced the “explicitly” evaluation of the Polyakov’s path integral to the functional integration of the above written function over all fluctuating metric fields:

$$I = \int d\mu[g_{ab}] \det^{-D/2}(-\Delta_g) e^{-\mu_0^2 \int_{\Omega} \sqrt{g} d^2 \xi}. \quad (3.5)$$

By using the theory of invariant integration on Riemann Manifolds of ours [1], one can insure automatically the preservation of the diffeomorphism group by defining the above mentioned metric $d\mu[g_{ab}]$ as the (functional) element of volume of the following functional metric (so-called De Witt metric) with $(c \neq -1/2)$:

$$\|\delta g_{ab}\|^2 = \int_{\Omega} \left[\sqrt{g} \left\{ (\delta g_{ab}) (g^{aa'} g^{bb'} + c g^{ab} g^{a'b'}) (\delta g_{a'b'}) \right\} \right] (\xi) d^2 \xi. \quad (3.6)$$

Let us then choose the conformal gauge fixing to evaluate the above metric path integral successfully [1], but already considering the metric positivity:

$$\bar{g}_{ab}(\xi) = e^{\varphi(\xi)} \delta_{ab}. \quad (3.7)$$

As a result the functional infinitesimal displacements $\delta g_{ab}(\xi)$ can be written in the following general form on the functional manifold of the metric $[g_{ab}(\xi)]$ (on the formal infinite-dimensional functional tangent bundle):

$$\delta g_{ab}(\xi) = \delta \varphi(\xi) \bar{g}_{ab}(\xi) + (\nabla_a \xi_b + \nabla_b \xi_a)(\xi). \quad (3.8)$$

Let us rewrite the functional De-Witt metric in the form

$$\|\delta g_{ab}\|^2 = \int_{\Omega} (\delta g_{ab}(\xi)) \gamma^{(ab,a'b')} [g_{cd}(\xi)] (\delta g_{a'b'}(\xi)), \quad (3.9a)$$

$$\gamma^{(ab,a'b')} [g_{cd}(\xi)] = \left(\sqrt{g} (g^{aa'} g^{bb'} + c g^{ab} g^{a'b'}) \right) (\xi). \quad (3.9b)$$

Since there is only three independent components of the metric tensor $g_{ab}(\xi)$ in two dimensions ($g_{12} = g_{21}$), the metric coefficients $\{\bar{\gamma}_{ij}\}_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}}$ are effectively a 3×3 matrix on the functional manifold of the intrinsic 2D metric fields:

$$\begin{aligned} \|\delta g_{ab}\|^2 &= \int_{\Omega} \left\{ (\delta g_{11}, \delta g_{12}, \delta g_{22}) [\bar{\gamma}] \begin{pmatrix} \delta g_{11} \\ \delta g_{12} \\ \delta g_{22} \end{pmatrix} \right\} (\xi) d^2 \xi \\ &= \int \{ (\delta g_{11}) (\bar{\gamma}_{11}) (\delta g_{11}) + (\delta g_{12}) (\bar{\gamma}_{22}) (\delta g_{22}) \\ &\quad + (\delta g_{22}) (\bar{\gamma}_{33}) (\delta g_{22}) + (\delta g_{11}) (\bar{\gamma}_{12} + \bar{\gamma}_{21}) (\delta g_{12}) \\ &\quad + (\delta g_{11}) (\bar{\gamma}_{13} + \bar{\gamma}_{31}) (\delta g_{22}) + (\delta g_{12}) (\bar{\gamma}_{23} + \bar{\gamma}_{32}) (\delta g_{22}) \} (\xi) \times d^2 \xi. \end{aligned} \quad (3.10)$$

Here the explicitly expressions of the effective $[\bar{\gamma}]$ -matrix are related to those of (3.9b) by the results

$$\begin{aligned} \bar{\gamma}_{11} &= \gamma^{(11,11)} = e^{\varphi} (e^{-\varphi} \delta^{11} e^{-\varphi} \delta^{11} + c e^{-\varphi} \delta^{11} e^{-\varphi} \delta^{11}) = (1+c) e^{-\varphi}; \\ \bar{\gamma}_{12} &= \gamma^{(11,12)} = 0; \quad \bar{\gamma}_{21} = \gamma^{(12,11)} = 0; \quad \bar{\gamma}_{13} = \gamma^{(11,22)} = c e^{-\varphi}; \\ \bar{\gamma}_{31} &= \gamma^{(22,11)} = c e^{-\varphi}; \quad \bar{\gamma}_{23} = \gamma^{(12,22)}; \quad \bar{\gamma}_{33} = \gamma^{(22,12)} = (1+c) e^{-\varphi}. \end{aligned} \quad (3.11)$$

As a consequence

$$\det [\gamma^{(ab,a'b')}] = \det [\bar{\gamma}_{i\delta}] = \frac{(1+c)^2}{e^{3\varphi}} - \frac{c^2}{e^{3\varphi}} = (1+c) e^{-3\varphi}. \quad (3.12)$$

So the De Witt metric on 2D is nonsingular only if $(C \neq -1/2)$.

Let us substitute the general functional displacement equation (3.8) on (3.9a)-(3.9b).

Firstly, we get (one could choose from the beginning on (3.9b))

$$\begin{aligned} \|\delta g_{ab}\|^2 &= \int_{\Omega} d^2 \xi e^{\varphi(\xi)} \left\{ (\delta \varphi g_b^a + \nabla^a \mathcal{E}_b + \nabla_b \mathcal{E}^a) \cdot (\delta \varphi g_a^b) + \nabla^b \mathcal{E}_a + \nabla_a \mathcal{E}^b \right. \\ &\quad \left. + 4c (\nabla_c \mathcal{E}^c + \delta \varphi)^2 \right\}. \end{aligned} \quad (3.13)$$

Let us analyze those terms involving the conformal factor.

By noting that $\nabla_c g_{ab} = \nabla_c g^{ab} = 0$ and $(\nabla_a \xi_b) g^{aa'} g^{bb'} (\nabla_{b'} \xi_{a'}) = \nabla^{a'} \xi_b \nabla^b \xi_{a'}$, added with $\nabla_a \xi^a = 0$; $\nabla_{a'} (g^{bb'} \xi_{b'}) = g^{bb'} \nabla_{a'} \xi_{b'} = \nabla_{a'} \xi^b$, one obtains the chain of results written below

$$\begin{aligned}
\|\delta g_{ab}\|^2 &= \int_{\Omega} d^2 \xi e^{\varphi(\xi)} \left\{ (\delta \varphi g_b^a + \nabla^a \xi_b + \nabla_b \xi^a) \cdot (\delta \varphi g_a^b) \right. \\
&\quad \left. + \nabla^b \xi_a + \nabla_a \xi^b + 4c(\nabla_c \xi^c + \delta \varphi)^2 \right\}(\xi) \\
&= \int_{\Omega} d^2 \xi e^{\varphi(\xi)} \left\{ (\nabla^a \xi_b + \nabla_b \xi^a) (\nabla^b \xi_a + \nabla_a \xi^b) \right. \\
&\quad \left. - 2(\nabla_d \xi^d)^2 + 2(1+2c)(\nabla_d \xi^d + \delta \varphi)^2 \right\}(\xi) \\
&= 2(1+2c) \int_{\Omega} d^2 \xi e^{\varphi(\xi)} (\nabla_d \xi^d + \delta \varphi)^2(\xi) \\
&\quad - 2 \int_{\Omega} d^2 \xi e^{\varphi(\xi)} \left\{ \xi^a (\nabla_d \nabla^d + [\nabla^b, \nabla_a]) \xi_b \right\},
\end{aligned} \tag{3.14}$$

where we have used the covariant by parts, integration rule ($g|_{\partial\Omega} = 0$):

$$\int_{\Omega} d^2 \xi \sqrt{g} f (\nabla_d h) = - \int_{\Omega} d^2 \xi \sqrt{g} (\nabla_d f) h. \tag{3.15}$$

As a consequence we have the full result:

$$\begin{aligned}
\|\delta g_{ab}\|^2 &= (1+2c) \int_{\Omega} d^2 \xi \sqrt{g(\xi)} (\delta \mathcal{U}(\xi))^2 \\
&\quad + \int_{\Omega} d^2 \xi \sqrt{g(\xi)} \left| (\xi_1(\xi), \xi_2(\xi)) [\hat{\mathcal{L}}] \begin{pmatrix} \xi_1(\xi) \\ \xi_2(\xi) \end{pmatrix} \right|,
\end{aligned} \tag{3.16}$$

where

$$\delta \mathcal{U}(\xi) = \delta \varphi(\xi) + \overbrace{(\nabla^c \xi_c)}^{=0}(\xi) = \delta \varphi(\xi) \tag{3.17}$$

and the Faddeev-Popov operator acting on the two-dimensional vector field generators of the local group of coordinate transformations is explicitly given by

$$(\hat{\mathcal{L}} \xi)_a = - \left((\nabla^d \nabla_d \delta_a^b + [\nabla^b, \nabla_a]) \xi_b \right)(\xi). \tag{3.18}$$

As a consequence the functional volume element is written as

$$d\mu[g_{ab}(\xi)] = d^{\text{cov}}[\varphi(\xi)] d^{\text{cov}}[\xi_1(\xi)] d^{\text{cov}}[\xi_2(\xi)] \det^{1/2}[\mathcal{L}]. \tag{3.19}$$

Here

$$\begin{aligned}
 d^{\text{cov}}(\varphi(\xi)) &= \prod_{(\xi \in \Omega)} e^{(\varphi(\xi)/2)} \delta\varphi(\xi) = \prod_{\xi \in \Omega} \delta\left(2e^{(\varphi(\xi)/2)}\right) = D^F\left[e^{(\varphi(\xi)/2)}\right] \\
 d^{\text{cov}}[\mathcal{E}_a(\xi)] &= \prod_{\xi \in \Omega} d\mathcal{E}_a(\xi) = D^F[\mathcal{E}_a(\xi)].
 \end{aligned} \tag{3.20}$$

Just for pedagogical purpose let us evaluate in details the relevant term for the Fadeev-Popov operator in the above formulae.

We have, for instance, the following sample calculation:

$$\begin{aligned}
 &\int_{\Omega} d^2\xi \sqrt{g} (\nabla_a \mathcal{E}_b + \nabla_b \mathcal{E}_a) g^{aa'} g^{bb'} (\nabla_{a'} \mathcal{E}_{b'} + \nabla_{b'} \mathcal{E}_a) \\
 &= \left(\int d^2\xi \left\{ \sqrt{g} (\nabla_a \mathcal{E}_b) g^{aa'} g^{bb'} (\nabla_{a'} \mathcal{E}_{b'}) \right\}(\xi) \right) + \text{similar terms} \\
 &= \int d^2\xi \sqrt{g} \left[\overbrace{\left(\nabla^a \mathcal{E}_b, \nabla_a \mathcal{E}^b \right)}^{g^{bb'} \nabla_{a'} \mathcal{E}_{b'} = \nabla_{a'} \left(g^{bb'} \mathcal{E}_{b'} \right) = \nabla_{a'} \mathcal{E}^b} + \nabla_a \mathcal{E}^b \nabla_b \mathcal{E}^a + \nabla_b \mathcal{E}^a \nabla_a \mathcal{E}^b + \nabla_b \mathcal{E}^a \nabla^b \mathcal{E}_a \right](\xi) \\
 &= \int d^2\xi \sqrt{g} \left\{ \mathcal{E}_b (-\nabla^a \nabla_a) \mathcal{E}^b - \mathcal{E}^b (\nabla_a \nabla_b) \mathcal{E}^a - \mathcal{E}^a (\nabla_b \nabla_a) \mathcal{E}^b - \mathcal{E}^a (\nabla_b \nabla^b) \mathcal{E}_a \right\} \\
 &= - \int d^2\xi \sqrt{g} \left\{ 2\mathcal{E}_b (-\nabla_a \nabla_a) \mathcal{E}^b - \mathcal{E}^b \overbrace{(\nabla_a \nabla_b + \nabla_a \nabla_b) \mathcal{E}^a}^{2\nabla_a \nabla_b = 2\nabla_b \nabla_a + 2[\nabla, \nabla^b]_-} \right\} \\
 &= -2 \int d^2\xi \sqrt{g} \mathcal{E}_a \left[(\nabla^c \nabla_c) \delta_{ab} + \overbrace{[\nabla_b, \nabla_a]_-}^{(\nabla_a \mathcal{E}^a = 0)} \right] \mathcal{E}_b.
 \end{aligned} \tag{3.21}$$

As an exercise to our diligent readers, the form of the above elliptic operator in the conformal gauge is given by (action on two-component objects)

$$\begin{aligned}
 \Delta_{(g=e^{\varphi} \delta_{ab})} &= e^{-\varphi(\xi)} \partial_{\bar{z}} \partial_z, \\
 \hat{\mathcal{L}}_{(g=e^{\varphi} \delta_{ab})} &= e^{-2\varphi(\xi)} \partial_{\bar{z}} \left(e^{\varphi(\xi)} \partial_z \right),
 \end{aligned} \tag{3.22}$$

where we have introduced the complex plane notation and the above displayed operators now acting on complex functions (isomorphic to two-component objects = vector fields) [8]:

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \bar{z}} &= \frac{\partial}{\partial \xi_1} + i \frac{\partial}{\partial \xi_2}.\end{aligned}\quad (i = \sqrt{-1}), \quad (3.23)$$

In [3], we present the explicit evaluation of elliptic operators of the form (for j being a positive integer) considered below:

$$\mathcal{L}_j = -e^{-(j+1)\varphi(\xi)} \partial_z \left(e^{j\varphi(\xi)} \partial_{\bar{z}} \right). \quad (3.24)$$

The result is

$$\det_F(\mathcal{L}_j) = \exp \left\{ -\frac{1+6j(j+1)}{12\pi} \int_{\Omega} d^2\xi \frac{1}{2} (\partial_a \varphi)^2(\xi) - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi\varepsilon} \int_{\Omega} d^2\xi e^{\varphi(\xi)} \right\}. \quad (3.25)$$

Let us point out the validity of the relationship below:

$$\begin{aligned}\det_F^{-D/2}(\Delta_{g=e^{\varphi}\delta_{ab}}) &= \det_F^{-D/4}(\mathcal{L}_{j=0}), \\ \det_F^{+1/2}(\hat{L}_{g=e^{\varphi}\delta_{ab}}) &= \det_F^{+1/2}(\mathcal{L}_{j=1}).\end{aligned} \quad (3.26)$$

As our final result of this covariant Polyakov path integral, we get the 2D-induced quantum gravity Liouville model:

$$I = \int D^F \left[e^{(\varphi/2)(\xi)} \right] \exp \left\{ -\frac{(26-D)}{48\pi} \int_{\Omega} \frac{1}{2} (\partial_a \varphi)^2(\xi) d^2\xi \right\} \times \left\{ e^{-\mu_{\text{ren}}^2} \int_{\Omega} e^{\varphi(\xi)} d^2\xi \right\}. \quad (3.27)$$

Here $\mu_r^2 = \mu_0^2 + \lim_{\varepsilon \rightarrow 0^+} ((2-D)/8\pi\varepsilon)$.

Note that the quantum measure on the above-displayed Liouville model is not the usual flat Feynman measure $\prod_{\xi \in \Omega} d\varphi(\xi) = D^F[\varphi(\xi)]$ as initially supposed by A. M. Polyakov.

As a consequence one should rewrite it in terms of the canonical Goldstone boson field $\bar{\rho}(\xi) = e^{(\varphi/2)(\xi)}$ which leads us to a kind of “noncompact” σ -model with a “renormalized” mass term:

$$I = \int D^F [\bar{\rho}(\xi)] \exp \left\{ -\frac{26-D}{12\pi} \int_{\Omega} \frac{(\partial_a \bar{\rho})^2}{\bar{\rho}^2} d^2\xi - \mu_r^2 \int_{\Omega} \bar{\rho}^2(\xi) d^2\xi \right\} \quad (3.28)$$

and signaling the dynamical breaking of the conformal group [1].

Sometimes one can formally consider the variable change in the path measure of (3.27):

$$\begin{aligned}
D^F \left[e^{(\varphi/2)(\xi)} \right] &= \overbrace{\det_F \left(e^{\varphi/2} \delta_{ab} \delta(\xi - \xi') \right)}^{\text{Functional Jacobian}} D^F [\varphi(\xi)] \\
&= \left(\lim_{\varepsilon \rightarrow 0^+} \exp \text{Tr} \left[\ell g \left(e^{\varphi/2} \delta_{ab} \right) e^{-\varepsilon \Delta} g = e^\varphi \right] \right) \times D^F [\varphi(\xi)] \\
&= \lim_{\varepsilon \rightarrow 0^+} \exp \left\{ \int_{\Omega} d^2 \xi e^{\varphi(\xi)} \left(\frac{\varphi(\xi)}{2} \right) \left[\frac{1}{4\pi\varepsilon} + \frac{1}{12\pi} R(\varphi) \right] \right\} \times D^F [\varphi(\xi)] \\
&= \lim_{\varepsilon \rightarrow 0^+} \left\{ \exp \left[\frac{1}{8\pi\varepsilon} \int_{\Omega} e^{\varphi(\xi)} \varphi(\xi) d^2 \xi \right] \times \exp \left[-\frac{1}{24\pi} \int_{\Omega} (\varphi(-\Delta)\varphi)(\xi) d^2 \xi \right] \right\} \\
&\quad \times D^F [\varphi(\xi)]
\end{aligned} \tag{3.29}$$

which would lead to the same expression (3.28) but with the “reduced” anomaly conformal coefficient 26D to 25D, if one could disregard the infinite piece on (3.29) and if the “Fujikawa-like” evaluation of the functional jacobian could be done more invariant. At this point it appears that the above 2D-Liouville-Polyakov path integral only makes sense at the classical level which is formally equivalent to evaluate all the observables in the $\varphi(\xi)$ -theory at the limit of $D \rightarrow -\infty$ [1]. One can see this after considering the rescale (see Appendix C for a complete analysis of this study):

$$\begin{aligned}
\varphi &\longrightarrow \sqrt{\frac{48\pi}{26-D}} \varphi, \\
D^F \left[e^{\varphi/2} \right] &\longrightarrow D^F \left[e^{\sqrt{(48\pi/(26-D))}\varphi} \right] \xrightarrow{D \rightarrow -\infty} D^F [0] = \text{classical}.
\end{aligned} \tag{3.30}$$

However, we should point out that all there questions on two-dimensional quantum gravity as a well-defined problem still are not well-understood since its inception in 1981 [1].

Finally it is very important on applications to the Dual model theory for strong interactions (off-shell Scattering Amplitudes) to have a covariant regularized form for the formal Green function of the Beltrami-Laplace operator in the conformal gauge $g_{ab}(\xi) = e^{\varphi(\xi)} \delta_{ab}$. It is well known from Hadamard theory of parametrix singular solutions that one has the following covariant behavior for the Beltrami-Laplace operator for small distances:

$$\begin{aligned}
\left(\Delta_{g=e^\varphi \delta_{ab}}^{-1} \right) (\xi, \xi')^{\varepsilon=(\xi-\xi') \rightarrow 0} &\sim \frac{1}{2\pi\varepsilon} - \frac{1}{2\pi} \ell g \left(e^{\varphi(\xi)} |\xi - \xi'| \right) \\
&\sim \frac{1}{2\pi} \left(\frac{1}{\varepsilon} \right) - \frac{1}{2\pi} \varphi(\xi) - \frac{1}{2\pi} \ell g |\xi - \xi'|.
\end{aligned} \tag{3.31}$$

This lead us to consider the regularized from of the Green function used on dual models based on the Polyakov's path integral:

$$\left(\xi \left| \Delta_{g=\varepsilon^q \delta_{ab}}^{-1} e^{-\mathcal{L} \Delta_{g=\varepsilon^q \delta_{ab}}} \right| \xi' \right) = \begin{cases} -\frac{1}{2\pi} \ell g |\xi - \xi'| & \xi \neq \xi' \\ -\frac{1}{4\pi} \ell_g \left(\frac{1}{\varepsilon} \right) + \frac{1}{4\pi} \varphi(\xi) & \xi = \xi'. \end{cases} \quad (3.32)$$

Note that we have used the following formulae to write explicitly the functional trace of the general two-dimensional (strongly) elliptic operators [3]:

$$A = - \left(g_{11} \frac{\partial^2}{\partial \xi_1^2} + g_{22} \frac{\partial^2}{\partial \xi_2^2} \right) \mathbf{1}_{2 \times 2} - (A_1) \frac{\partial}{\partial x_1} - (A_2) \frac{\partial}{\partial x_2} - A_0 \quad (3.33a)$$

$$\begin{aligned} \lim_{t \rightarrow 0^+} \text{tr}(e^{-tA}) &\sim \frac{1}{2\pi t} \left(\int d^2 \xi \sqrt{g_{11} g_{22}} \right) \\ &+ \left\{ \frac{1}{2\pi} \left(\int d^2 \xi \sqrt{g_{11} g_{22}} \left(-\frac{1}{6} R(g) \right) \right) \right\} \\ &\times \int d^2 \xi \left(-\frac{1}{2} \frac{1}{\sqrt{g_{11} g_{22}}} \right) \text{Tr}_{\text{matrix}} \left[\left(\frac{\partial}{\partial x_1} (\sqrt{g_{11} g_{22}} A_1) + \frac{\partial}{\partial x_2} (\sqrt{g_{11} g_{22}} A_2) \right) \right] \\ &+ \int d^2 \xi \left(-\frac{1}{4} \text{Tr} \left[\frac{(A_1)^2}{g_{11}} + \frac{(A_2)^2}{g_{22}} \right] + A_0 \right) \Big\} + O(t). \end{aligned} \quad (3.33b)$$

4. Path-Integral Quantization of the Nambu-Goto Theory of Random Surfaces

In order to give a path integral meaning for the symbolic Feynman continuum sum over surfaces' histories (3.1a), we start rewriting it in the following Polyakov's form, but in the presence of a constraint [1]:

$$\begin{aligned} G_{\text{NG}}(C) &= \int d\bar{\mu} [x^\mu(\xi), g_{ab}(\xi)] \exp \left\{ -\frac{1}{2\pi\alpha'} \int_{\Omega} d^2 \xi \left(\sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \right) (\xi) \right\} \\ &\times \delta_{\text{cov}}^{(F)} (g_{ab} - \partial_a X^\mu \partial_b X_\mu). \end{aligned} \quad (4.1)$$

Note that our constraint is rewritten as a covariant (diffeomorphism) invariant delta functional:

$$\delta_{\text{cov}}^{(F)}(g_{ab} - \partial_a X^\mu \partial_b X_\mu) = \int D^{\text{cov}}[\lambda_{ab}] e^{i \int_{\Omega} d^2 \xi (\sqrt{g} \lambda_{ab} (\partial^a X^\mu \partial^b X_\mu - g^{ab}))(\xi)}, \quad (4.2)$$

where the covariant Feynman-Wiener measure on (4.2) is the element of volume of the functional metric:

$$ds^2 = \|\delta \lambda_{ab}\|^2 = \int \left(\sqrt{g} \delta \lambda_{ab} (g^{aa'} g^{bb'}) \delta \lambda_{a'b} \right) (\xi) d^2 \xi. \quad (4.3)$$

After introducing again the complex-euclidean light-cone coordinates on the domain Ω :

$$\begin{aligned} \partial_+ &= \frac{(\partial_{\xi_1} - i \partial_{\xi_2})}{2} = \partial_z, \\ \partial_- &= \frac{(\partial_{\xi_1} + i \partial_{\xi_2})}{2} = \partial_{\bar{z}} \end{aligned} \quad (4.4)$$

and imposing the surface conformal gauge $g_{ab}(\xi) = \rho(\xi) \delta_{ab}$, which means that $g_{11} = g_{22}$, $g_{12} = g_{21} = 0$, or equivalently using light-cone coordinates again,

$$\begin{aligned} \partial_+ X^\mu \partial_+ X_\mu &= \partial_- X^\mu \partial_- X_\mu & (\iff \partial_{\xi_1} X^\mu \partial_{\xi_2} X_\mu = g_{12} = g_{21} = 0), \\ \sqrt{g} &= \partial_+ X^\mu \partial_- X_\mu & (g_{11} = \partial_+ X^\mu \partial_- X_\mu = g_{22} \neq 0). \end{aligned} \quad (4.5)$$

Now by taking into account the Jacobian relationship between the covariant functional measures and the associated pure Feynman-Wiener-Kac path measures, one obtains the following results.

A careful discussion presented in [1]-chapter 12 leads us to the following result on the surface light-cone gauge (4.5):

$$\begin{aligned} G_{\text{NG}}(C) &= \int D^F[\bar{\rho}(\xi)] D^F[X^\mu(\xi)] \\ &\times \exp \left\{ -\frac{(26-D)}{12\pi} \int_{\Omega} d^2 \xi \left[\frac{1}{2} \left(\frac{\partial_a \bar{\rho}}{\bar{\rho}} \right)^2 \right] + \lim_{\varepsilon \rightarrow 0} \frac{(2-D)}{2\pi\varepsilon} \int_{\Omega} d^2 \xi \bar{\rho}^2 \right\} \\ &\times \exp \left\{ -\frac{1}{2\pi\alpha'} \int_{\Omega} d^2 \xi (\partial_a X^\mu)^2(\xi) \right\} \\ &\times \delta^{(F)} \left[(\bar{\rho})^2(\xi) \delta_{ab} - \overbrace{h_{ab}(X^\mu(\xi))}^{=\partial_+ X^\mu \partial_- X_\mu} \right]. \end{aligned} \quad (4.6)$$

Note the usual noncovariant function “flat” constraint relating the surface vector position to the auxiliary metric field $g_{ab}(\xi)$ on the conformal gauge.

After realizing the immediate $\bar{\rho}(\xi)$ functional integral, one gets the final result, after a renormalization of the Regge slope constant ($\partial_+ = \partial_z$, $\partial_- = \partial_{\bar{z}}$). (It is worth to remark that at a Feynman diagrammatic perturbative level, the measure “tad-pole” factor in (4.7), $\exp\{((- \delta^{(2)}(0))/4) \int_{\Omega} \ell g(h(X^\mu(z, \bar{z}))) \times dz d\bar{z}\}$, can be assigned to $\exp\{0\} = 1$ in the dimensional regularization scheme):

$$\begin{aligned}
 G_N(C) = & \int \left[\prod_{(\mu, \xi, \xi')} dX^\mu(z, \bar{z}) \sqrt{h(X^\mu(z, \bar{z}))} \right] \\
 & \times \delta^{(F)}(\partial_+ X^\mu \partial_+ X_\mu) \delta^{(F)}(\partial_- X^\mu \partial_- X_\mu) \\
 & \exp \left\{ -\frac{1}{2\pi\alpha'_{\text{ren}}} \int_{\Omega} dz d\bar{z} (\partial_+ X^\mu)(\partial_- X_\mu)(z, \bar{z}) \right\} \\
 & \exp \left\{ -\frac{(26-D)}{48\pi} \int_{\Omega} dz d\bar{z} \left[\frac{(1/2)\partial_+(\partial_+ X^\mu \partial_- X_\mu)}{(\partial_+ X^\mu \partial_- X_\mu)} \times \frac{\partial_-(\partial_+ X^\mu \partial_- X_\mu)}{(\partial_+ X^\mu \partial_- X_\mu)} \right] (z, \bar{z}) \right\}.
 \end{aligned} \tag{4.7}$$

Now we realize that one must choose $D = 26$ or introduce intrinsic Majorana fermions living on those Nambu-Goto random surfaces in order to vanish the nonrenormalizable Liouville surface term. The numbers of such Neutral Majorana 2D fields m must be such that $26 = 4 + m$ on $D = 4$. Note that complex pairing such neutral Majorana fermion one gets eleven complex fermion Dirac fields living on the random surface.

It is worth (all the reader attention that in this framework of Nambu-Goto path integrals on surface light-cone gauge, the vertices are given by object:

$$\begin{aligned}
 V = & \left\{ \sqrt{h} h^{[a_1, a_2]} \dots h^{a_{2m-1}, a_{2n}} \right\} (X^\mu(z, \bar{z})) \\
 & \times \left\{ \partial_{a_1} X^\mu \dots \partial_{a_{2n}} X^{\mu_{2m}} \cdot \overbrace{\mathcal{E}_{\mu_1 \dots \mu_{2m}}}^{\text{spin polarization tensors}} \cdot e^{iK_\mu X^\mu} \right\} (z, \bar{z}),
 \end{aligned} \tag{4.8}$$

which are free from tachyons, and so forth.

A complete study of the spectrum of such Nambu-Goto strings with those tachyons-free “geometrical” vertices is still missing in literature and left to our diligent readers.

Similar analysis can be straightforwardly implemented for the extrinsic surface path-integral weight functional equations (2.15)-(2.16).

Appendices

A. On the Formal Evaluation of the Euclidean Dirac Functional Determinant on Two Dimensions

Let us start by this appendix by considering the euclidean Dirac partition functional on a two-dimensional space time:

$$\begin{aligned} Z[A_\mu] &=: \frac{1}{Z[A_\mu = 0]} \times \left\{ \int D\psi D\bar{\psi} e^{-\int (\bar{\psi} \not{D}(A) \psi) d^2\xi} \right\} \\ &=: \det(\not{D}(A)) = \det^{1/2} \left((\not{D}(A))^2 \right), \end{aligned} \quad (\text{A.1})$$

where the (nonself adjoint) euclidean Dirac operator is explicitly given by (including the massive case!)

$$\not{D}(A) = \gamma_\mu (\partial_\mu + g A_\mu) + im. \quad (\text{A.2})$$

The two-dimensional euclidean matrixes on the Dirac operator (A.2) satisfy the relationship below:

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \\ \gamma_\mu \gamma_5 &= i\epsilon_{\mu\nu} \gamma_\nu, \gamma^5 = i\gamma_0 \gamma_1, \\ \epsilon_{01} &= -\epsilon_{10} = 1. \end{aligned} \quad (\text{A.3})$$

Let us note the more suitable redefinition of the euclidean Dirac operator:

$$\tilde{\not{D}}(A) = i\not{D}(A) = e^{i\gamma_5 \not{D}} \overbrace{(i\gamma_\mu \partial_\mu + ig\gamma_5 \not{A}_\mu)}^{=i\gamma_\mu \partial_\mu + ig\gamma_5 \not{A}_\mu} e^{ig\gamma_5 \not{D}}, \quad (\text{A.4})$$

where the background abelian gauge field configurations are chosen on the landau gauge $\partial_\mu A_\mu = 0$:

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \varphi. \quad (\text{A.5})$$

Let us note that (with a undetermined infinite phase!)

$$\begin{aligned} \det_F(\tilde{\not{D}}(A)) &= \det(\not{D}(A)), \\ \left| \det_F(\tilde{\not{D}}(A)) \right|^2 &= \left| \det(\not{D}(A)) \right|^2. \end{aligned} \quad (\text{A.6})$$

Since we have the result

$$\begin{aligned} \left(\tilde{\mathcal{D}}(A)\right)^2 &= -\partial^2 - 2g A_\mu(\xi)\partial_\mu - g^2 A_\mu^2(\xi) \\ &\quad - \frac{g}{4} \overbrace{\sigma^{\mu\nu}}^{[\gamma^\mu, \gamma^\nu]} \cdot F_{\mu\nu}(A(\xi)) - g(\partial_\mu A_\mu)(\xi) \end{aligned} \quad (\text{A.7})$$

thus

$$\begin{aligned} \lim_{t \rightarrow 0^+} \left\langle \left| \left\{ \exp\left(-t\left(\tilde{\mathcal{D}}(A)\right)^2\right) \right\} \right| \xi \right\rangle \\ \sim \frac{1}{4\pi t} \mathbf{1}_{2 \times 2} + \frac{1}{4\pi} \left[g^2 A_\mu^2 + \frac{g}{4} \sigma^{\mu\nu} F_{\mu\nu} + g \partial_\mu A_\mu - \frac{1}{4} (2g A_\mu)^2 - \frac{1}{2} \partial_\mu (2g A_\mu) \right], \end{aligned} \quad (\text{A.8})$$

one can straightforwardly apply the Romanov-Schwartz theorem [1] to write the following (formal) differential equation for the functional determinant under analysis:

$$\begin{aligned} 2 \frac{d}{d\sigma} \ell_g \det(\mathcal{D}(A^\sigma)) \\ = \lim_{\varepsilon \rightarrow 0^+} 4 \left\{ \int d^2 \xi \text{Tr}_{\text{Dirac}} \left(i g \gamma^5 \phi \right) \left[\frac{1}{4\pi \varepsilon} + \frac{g}{16\pi} \sigma \overbrace{\sigma^{\mu\nu}}{=[\gamma^\mu, \gamma^\nu]} F_{\mu\nu}(A) \right] \right\}. \end{aligned} \quad (\text{A.9})$$

Here

$$A^{(\sigma)} = \sigma \xi_{\mu\nu} \partial_\gamma \varphi. \quad (\text{A.10})$$

Since $\text{Tr}_{\text{Dirac}}(\gamma_5) = 0$, one has the immediate result

$$\left(\frac{\det \mathcal{D}(A)}{\det \mathcal{D}} \right) = e^{-(g^2/2\pi) \int (\partial\phi)^2 d^2 \xi} = e^{-(g^2/2\pi) \int A_\mu^2(\xi) d^2 \xi} \times \mathcal{D}^{(F)}(\partial_\mu A_\mu). \quad (\text{A.11})$$

In the general nongauge fixed case, one has the result

$$\begin{aligned} \frac{\det(\mathcal{D}(A))}{\det(\mathcal{D})} &= \int D^F[\omega(x)] e^{-(g^2/2\pi) \int (A_\mu + \partial_\mu \omega)^2 d^2 \xi} \\ &= e^{-(g^2/2\pi) \int d^2 \xi [A_\mu (\delta_{\mu\nu} - ((\partial_\mu \partial_\nu)/(-\partial^2))) A_\nu]}(\xi). \end{aligned} \quad (\text{A.12})$$

Let us show in detail the above calculation:

$$\begin{aligned} & \int_0^1 \left(\ell_g \det \left(\not{D} \left(A^{(\sigma)} \right) \right)^2 \right) d\sigma \\ &= \lim_{\varepsilon \rightarrow 0^+} 4 \left\{ \int d^2 \xi \operatorname{tr} \left(i g \gamma^5 \varphi \left[\frac{1}{4\pi\varepsilon} + \frac{g}{16} \sigma^{\mu\nu} F_{\mu\nu}(A) \right] \right) \right\} \times \left(\int_0^1 d\sigma \right). \end{aligned} \quad (\text{A.13})$$

So

$$\begin{aligned} \ell_g \left(\frac{\det(\not{D}(A))}{\det(\not{D})} \right) &= \frac{g^2}{16\pi} \left\{ \int d^2 \xi \operatorname{Tr}_{\text{Dirac}} \left(i \gamma^5 (-2i \mathcal{E}_{\mu\nu}) \gamma_5 \phi \times F_{\mu\nu}(A) \right) \right\} \\ &= \frac{g^2}{4\pi} \int d^2 \xi \phi \mathcal{E}_{\mu\nu} F_{\mu\nu}(A) \\ &= \frac{g^2}{2\pi} \int d^2 \xi \phi \partial^2 \phi = -\frac{g^2}{2\pi} \int d^2 \xi (\partial \phi)^2 = -\frac{g^2}{2\pi} \int d^2 \xi (A_\mu)^2. \end{aligned} \quad (\text{A.14})$$

It is important to remark that if one had used the following self-adjoint Dirac euclidean operator,

$$\widehat{\not{D}}(A) = e^{e\gamma_5 \phi} (i\gamma_\mu \partial_\mu) e^{g\gamma_5 \phi} \quad (\text{A.15})$$

one would obtain a “Tachyon” (imaginary mass) term for the dynamically generated mass term for the abelian gauge field:

$$\frac{\det_F \widehat{\not{D}}(A)}{\det \not{D}} = \exp \left\{ + \frac{g^2}{2\pi} \int d^2 \xi (A_\mu)^2 \right\}. \quad (\text{A.16})$$

At this point it is worth to recall the Seeley asymptotic expansion

$$\lim_{t \rightarrow 0^+} \langle \xi | \exp(-tA) | \xi \rangle = \frac{1}{4\pi t} \mathbf{1}_{2 \times 2} + \frac{1}{4\pi} \left[V_0 - \frac{V_\mu^2}{4} - \frac{\partial_\mu V_\mu}{2} \right] \quad (\text{A.17})$$

for a formal second order operator of the form on R^2 :

$$A = -\partial^2 - V_\mu \partial_\mu - V_0. \quad (\text{A.18})$$

B. On Atiyah-Singer Index Theorem in the Framework of Feynman

B.1. Pseudoclassical Path Integrals: Author's Original Remarks

One of the most celebrated (pure) mathematical theorems on modern geometry and topology of compact orientable manifolds on R^N is the so-called index theorem [4]. By the other side, it

is widely known that the techniques to obtain such results are extremely intricate and based on the most difficult aspects of the theory of elliptic operators on closed compact orientable N -dimensional manifolds (see [5], Chapter 11). Our aim in this section is to somewhat simplify a very important index theorem on the Laplacian operator acting on forms [6].

As in the last reference [6], one can argue that the index of the Laplacian operator Δ_f acting on differential forms on a N -dimensional Riemannian compact orientable manifold $\mathcal{M}$ is given by the object called supertrace which by its turn can be written as a Grassmannian (supersymmetric) euclidean path integral [4]:

$$\begin{aligned} \text{index}(\Delta_f) =: & \lim_{\hbar \rightarrow 0} \left\{ \int_{X^\mu(0)=X^\mu(1)=x^\mu} \left(\overbrace{\prod_{\substack{0 \leq s \leq 1 \\ 1 \leq n \leq N}} \det^{1/2}[G_{\mu\nu}(X^\mu(s))] dX^\mu(s)}^{\equiv D_{\text{cov}}^F[X^\mu(s)]} \right) \right\} \\ & \times \int \left(\overbrace{\prod_{\substack{0 \leq s \leq 1 \\ \alpha=1,2 \\ 1 \leq i \leq 2N}} \det^{1/2}[\sqrt{G}(X^\mu(s))] d\xi_{(\alpha)}^i(s) d\xi_{(\alpha)}^{*i}(\alpha)}^{\equiv D_{\text{cov}}^F[\xi_{(\alpha)}^i(\xi)]} \right) \\ & \times \exp \left\{ -\frac{1}{\hbar} \int_0^1 ds \left[\frac{1}{2} G_{\mu\nu}(X^\alpha(s)) \frac{dX^\mu}{ds}(s) \frac{dX^\mu}{ds}(s) + \frac{i}{2} G_{\mu\nu}(X^\alpha(s)) \left(\xi_{(\alpha)}^i(s) \frac{d}{ds} \xi_{(\alpha)}^i(s) \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{8} R_{ijkl}(X^\alpha(s)) \left(\xi_{(1)}^i(s) \xi_{(2)}^j(s) \right) \left(\xi_{(1)}^k(s) \xi_{(2)}^\ell(s) \right) \right] \right\}. \end{aligned} \quad (\text{B.1})$$

An important remark to be done now originally due to ourselves [7] is about the “bosonic” behavior of the Grassmannian composite products $\xi_{(1)}^i(s) \xi_{(2)}^j = \sigma^{ij}(s)$ as can be easily verified from the computation below ($[a, b]_- = ab - ba$; $[a, b]_+ = ab + ba$):

$$\begin{aligned} \left[\xi_{(1)}^i \xi_{(2)}^j, \xi_{(1)}^p \xi_{(2)}^q \right]_- &= \xi_{(1)}^i \left[\xi_{(1)}^p \xi_{(2)}^q, \xi_{(2)}^j \right]_+ - \left[\xi_{(1)}^p \xi_{(2)}^q, \xi_{(1)}^i \right]_+ \xi_{(2)}^j \\ &= \left\{ \xi_{(j)}^i \left(\xi_{(1)}^p \xi_{(2)}^q \xi_{(2)}^j \right) - \xi_{(1)}^p \xi_{(2)}^q \xi_{(1)}^i \xi_{(2)}^j \right\} \\ &\quad + \left\{ \xi_{(1)}^i \xi_{(2)}^j \xi_{(1)}^p \xi_{(2)}^q - \xi_{(1)}^i \xi_{(1)}^p \xi_{(2)}^q \xi_{(2)}^j \right\} \\ &= 0 + 0 = 0. \end{aligned} \quad (\text{B.2})$$

On the basis of the above made remark (B.2) one can rewrite the Grassmannian path integral equation (B.1) in the terms of “composite bosonic operators” [8]:

$$\begin{aligned} \text{index}(\Delta_f) = & \lim_{t \rightarrow 0} \int_{X^\mu(0)=X^\mu(1)=\bar{X}^\mu} D_{\text{cov}}^F[X^\mu(s)] \int_{\xi_{\alpha}^i(0)=-\xi_{\alpha}^i(1)} D^F[\xi_{\alpha}^i(s)] \\ & \times \int_{\sigma_{ik}(0)=\sigma_{ik}(1)} \overbrace{D_{\text{cov}}^F[\sigma_{ik}(s)]}^{\text{Bosonic Variable}} \int_{\lambda_{ik}(0)=\lambda_{ik}(1)} \overbrace{D^F[\lambda_{ik}(s)]}^{\text{Bosonic Variable}} \end{aligned}$$

$$\begin{aligned}
& \times \exp \left\{ \frac{i}{h} \int_0^1 ds \lambda^{ik} \left(\xi_{(1)}^i(s) \xi_{(2)}^k(s) - \sigma_{ik}(s) \right) \right\} \\
& \times \exp \left\{ -\frac{1}{4h} \int_0^1 \left(\frac{1}{2} R_{ijk\ell}(X^\alpha(s)) \sigma_{(s)}^{ik} \sigma_{(s)}^{j\ell} \right) \right\}.
\end{aligned} \tag{B.3}$$

After integrating the Grassmannian variables $\{\xi_{(1)}^i(s), \xi_{(2)}^i(s)\}_{1 \leq i \leq N}$, one gets the below written functional determinant as outcome (prove it with the definition $\theta(0) = 0$, here $\theta(s)$ denotes the heaviside step function-distribution):

$$\det_{\text{cov}} \left[G_{ij}(X^\alpha(s)) \frac{d}{ds} \delta_{ij} + \lambda_{ik}(s) \right] = 1. \tag{B.4}$$

Now the σ_{ik} functional integral is Gaussian and yields the result

$$\exp \left\{ -\frac{1}{2h} \int_0^1 ds \int_0^1 ds' (\lambda_{ik}(s)) \left[\frac{1}{2} R_{pqrs}(X^\alpha(s)) \right]_{(ik) \cdot (j\ell)}^{-1} \delta(s-s') (\lambda_{j\ell}(s')) \right\}. \tag{B.5}$$

At the classical limit $\hbar \rightarrow 0$, the closed periodic quantum trajectories $X^\mu(0) = X^\mu(s) = \bar{x}^\mu$; all reduce to their initial common point $\bar{x}^\mu$. As a consequence one gets that at $\hbar \rightarrow 0$.

$$\begin{aligned}
X^\mu(s) & \longrightarrow \bar{x}^\mu, \\
D_{\text{cov}}^F(X^\mu(s)) & \longrightarrow \int_{\mathcal{M}} d^N(\bar{x}) \cdot \det^{1/2}[G_{\mu\nu}(\bar{x}^\mu)], \\
\int D_{\text{cov}}^F(\lambda_{pq}(s)) \exp \left\{ -\frac{1}{4 \text{tr}} \int_0^1 ds \int_0^1 ds' \left(\lambda_{\underbrace{ik}_A}(s) \right) [R_{(A,B)}(X^x(s))]^{-1} \left(\lambda_{\underbrace{j\ell}_B}(s') \right) \right\} \\
& \longrightarrow \det^{1/2}[R_{AB}(\bar{x}^\mu)] =: P_f(R_{AB}(\bar{x}^\mu)).
\end{aligned} \tag{B.6}$$

By grouping together (B.6), one gets the index of the Laplacian operator acting on differential forms defined on a compact orientable N -dimensional which is universally proportional to the famous Chern index of the endowed riemannian structure of M . Namely (see [8] for topological details),

$$\text{index}(\Delta_f) = \xi_{\text{Chern}}(\mathcal{M}) = \bar{e} \left\{ \int_{\mathcal{M}} d^N(\bar{x}) \det^{1/2}[G_{\mu\nu}(\bar{x}^\mu)] \times \overbrace{P_f(R_{AB}(\bar{x}^\mu))}^{\text{Pfaffinian form of the curvature two-form}} \right\}. \tag{B.7}$$

The universal constant is adjusted by defining as unity the Chern induces or the Euler-Poincaré characteristic) of spheres S^{N-1} with the usual R^N embedding metric (exercise to our

differential geometric oriented readers!). Now it is well known that $\chi_{\text{Chem}}(\mathcal{M})$ are integer numbers.

In two dimensions, (B.7) reduces to the famous Gauss theorem for Riemann surfaces and its connections to the Euler topological surface genera:

$$\text{index}(\Delta_f) = \chi_{\text{Chem}}(\Sigma) = \int_{\Sigma} (\sqrt{g}R(g))(\bar{x}^\mu) d^2\bar{x} = 2\pi(2-2g), \quad (\text{B.8})$$

where g is the number of “holes” of Σ .

C. A Perturbative Analysis of the Liouville Quantum

C.1. Field Model on the Botelho Sigma Model Parametrization

We start this appendix by considering the Liouville quantum field model partition functional in terms of the correct $\bar{p}(z)$ -field parametrization equation (3.28) of the text; called by is the σ -model Liouville field theory:

$$Z = \int D^F [\bar{p}(z)] \exp \left\{ -\frac{(26-D)}{12\pi} \int_{\Omega} \left(\frac{1}{2} \left(\frac{\partial_a \bar{p}}{\bar{p}} \right)^2 \right) (\xi) d^2\xi - \mu_R^2 \int_{\Omega} \bar{p}^2(z) d^2\xi \right\}. \quad (\text{C.1})$$

In order to define (C.1) perturbatively in the number of “inverse dimension” $1/D$ by a loop expansion, we consider the usual hypothesis of the field decomposition:

$$\bar{p}(\xi) = p_0(\xi) + \overbrace{\left(\frac{12\pi}{26-D} \right)^{1/2}}^{:=\varepsilon(\text{the parameter loop expansion})} h(\xi). \quad (\text{C.2})$$

Here $p_0(\xi)$ is the classical background field configuration associated to the vanishing of the parameter loop expansion, that is, the well-known $D \rightarrow -\infty$ limit of the path-integral equation (C.1) [7].

After that step has been taken, one has the result (with the new cosmological constant $\mu = (\sqrt{2})\mu_R \cdot \varepsilon$) holding true for all orders of the loop expansion:

$$\begin{aligned} Z = & \int D^F [4] \exp \left\{ -\frac{1}{\varepsilon^2} \int_{\Omega} \frac{1}{2} \left(\frac{\partial p_0}{p_0} \right)^2 \left[\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot \varepsilon^{n-1} \frac{h^{n-1}}{p_0^{n-1}} \right] \right\} \\ & \times \exp \left\{ -\frac{1}{\varepsilon} \int_{\Omega} \frac{1}{2} \left(\frac{\partial p_0 \partial h}{p_0^2} \right) \left[\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot \varepsilon^{n-1} \frac{h^{n-1}}{p_0^{n-1}} \right] \right\} \\ & \times \exp \left\{ -\frac{1}{2} \int_{\Omega} \left(\frac{\partial h}{p_0^2} \right)^2 \left[\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot \varepsilon^{n-1} \frac{h^{n-1}}{p_0^{n-1}} \right] \right\} \\ & \times \exp \left\{ -\frac{\mu^2}{\varepsilon^2} \int_{\Omega} \frac{1}{2} (p_0^2 + \varepsilon^2 h^2 + 2\varepsilon h p_0) \right\}. \end{aligned} \quad (\text{C.3})$$

Clearly at a perturbation level around the one-loop expansion Gaussian path integral, one can see that all the infinite set of perturbative vertices (higher terms ε^p , with $p \geq 2$) renders the theory nonnormalizable, which by its turn is an already expected outcome from the result of nonperturbative renormalizability of quantum gravity theories in general.

Note that we are indeed considering the Liouville-Model as the two-dimensional version of quantum gravity.

Just for completeness, we write the full expression of the one-loop Gaussian path integral associated to our proposed $(1/D)$ expansion (or correctly the ε expansion with $\varepsilon = \sqrt{(12/(26-D))}$; $D < 26$):

$$\begin{aligned}
 Z^{\text{one-loop}} \sim & \exp \left(\overbrace{-\frac{1}{\varepsilon^2} \int_{\Omega} \frac{1}{2} \left[\left(\frac{\partial p_0}{p_0} \right)^2 + \mu^2 (p_0)^2 \right] (\xi) d^2 \xi}^{\text{classical term}} \right) \\
 & \times \left\{ \int D^F[h] \exp \left(-\frac{1}{2} \int_{\Omega} \left[\left(\frac{\partial h}{p_0} \right)^2 + \mu^2 h^2 \right] (\xi) d^2 \xi \right) \right. \\
 & \quad \times \exp \left(+ \int_{\Omega} \frac{\partial p_0}{p_0^3} \cdot h^2 \right) \\
 & \quad \times \exp \left(+ \frac{1}{\varepsilon} \int_{\Omega} \frac{(\partial p_0)^2}{p_0^3} \cdot h \right) \\
 & \quad \times \exp \left(-\frac{3}{2} \int_{\Omega} \left(\frac{\partial p_0}{p_0} \right)^2 \cdot \frac{h^2}{p_0^2} \right) \\
 & \quad \left. \times \exp \left(-\frac{\mu^2}{\varepsilon} \int h p_0 \right) \right\}. \tag{C.4}
 \end{aligned}$$

Note again that one should choose the Liouville quantum fluctuations in such a way that the constraints between the field h and p_0 displayed below should vanish and leading thus to a massive scalar field h (however, interacting with the background field configuration p_0 as one can see from (C.4)):

$$\int \frac{(\partial p_0)^2}{p_0^3} \cdot h = 0 = \int h p_0. \tag{C.5}$$

As a general conclusion on this sigma model for two-dimensional quantum gravity, one can see that at least for applications on string theory for Q.C.D., one necessarily must set $\varepsilon = 0$ on the theory effective path integral averages like scattering amplitudes, or equivalently, one must take into account the theory already at the classical fix point $p(\xi) = p_0(\xi)$ theory path integral (the classical $D \rightarrow -\infty$ Liouville field configuration, see [1] for details in the fermionic string path integral).

However, it is worth to remark and keep on mind that the full Liouville path integral although being defined for $\varepsilon = 0$ must be defined for all topological surface genera (the

old dual model unitarization) by taking into full account the quantization of the conformal surface metrical structures' Teichmüller parameters degrees of freedom which makes the theory highly nontrivial [9].

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