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*Research Article*

# The Theory for $J$-Hermitian Subspaces in a Product Space

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This paper is concerned with the theory for $J$-Hermitian subspaces. The defect index of a $J$-Hermitian subspace is defined, and a formula for the defect index is established; the result that every $J$-Hermitian subspace has a $J$-self-adjoint subspace extension is obtained; all the $J$-self-adjoint subspace extensions of a $J$-Hermitian subspace are characterized. This theory will provide a fundamental basis for characterizations of $J$-self-adjoint extensions for linear nonsymmetric expressions on general time scales in terms of boundary conditions, including both differential and difference cases.

## 1. Introduction

The spectral theory for differential and difference has been investigated extensively. In general, under certain definiteness conditions, a formally symmetric differential expression can generate a minimal operator which is symmetric, that is, a densely defined Hermitian operator, in a related Hilbert space and its adjoint, is the corresponding maximal operator (see, e.g., [1–3]). There are many results on self-adjoint extensions of the minimal operators since self-adjoint extension problems are fundamental in the study of spectral theory for differential expressions [2–6]. However, for some formally symmetric differential expressions, their minimal operators may be nondensely defined, or their maximal operators may be multivalued (e.g., [7, Example 2.2]). Further, for a formally symmetric difference expression, even a second-order one, its minimal operator is nondensely defined, and its maximal operator is multivalued in the related Hilbert space in general [8]. Therefore, the classical von Neumann self-adjoint extension theory and the Glazman-Krein-Naimark (GKN) theory for symmetric operators are not applicable in these cases.

The appropriate framework is linear subspaces (linear relations in the terminology of [7, 9, 10]) in a Hilbert space to study the linear differential or difference expressions for which

the corresponding operators are nondensely defined or multivalued. Lesch and Malamud studied formally symmetric Hamiltonian systems in the framework of linear subspaces [7]. Coddington studied self-adjoint extensions of Hermitian subspaces in a product space [11]. He had extended the von Neumann self-adjoint extension theory for symmetric operators to Hermitian subspaces. By applying the relevant results in [11], Shi established the GKN theory for Hermitian subspaces [12]. For more results about nondensely defined Hermitian operators or Hermitian subspaces, we refer to [13–15].

The study of spectral problems involving linear differential and difference expressions with complex-valued coefficients is becoming a well-established area of analysis, and many results have been obtained [1, 16–22]. Such expressions are not formally symmetric in general, and hence, the spectral theory of self-adjoint subspaces is not applicable. To study such problems, Glazman introduced a concept of $J$-symmetric operators in [23], where $J$ is a conjugation operator given in Section 2. The minimal operators generated by certain differential and difference expressions with complex-valued coefficients are $J$-symmetric operators in the related Hilbert spaces (e.g., [18, 24, 25]). $J$-self-adjoint extension problems are also fundamental in the spectral theory for such expressions. Many results have been obtained on $J$-self-adjoint extensions [24–27]. Knowles gave a complete solution to the problem of describing all the $J$-self-adjoint extensions of any given $J$-symmetric operator provided the regularity field of this operator is nonempty [26]. Given a differential or difference expression, it is in practice difficult however to determine whether the appropriate $J$-symmetric operator has empty or nonempty regularity field. Therefore, Race established the theory for $J$-self-adjoint extensions of $J$-symmetric operators without the restrictions on the regularity fields [24]. However, the appropriate framework is also linear subspaces in a Hilbert space to study the linear nonsymmetric differential or difference expressions for which the corresponding minimal operators are nondensely defined, or the corresponding maximal operators are multivalued. So, the $J$-self-adjoint extension theory mentioned the above needs to be extended to linear subspace when we consider the nonsymmetric Hamiltonian systems which induce the nondensely defined or multivalued operators.

In this present paper, the concept of the defect indices of $J$-Hermitian subspaces is given and a formula for the defect indices is obtained. Further, the result that every $J$-Hermitian subspace has a $J$-self-adjoint subspace extension is given, and the characterizations for all the $J$-self-adjoint subspace extensions of a $J$-Hermitian subspace are established, which can be regarded as the GKN theorem for $J$-Hermitian subspaces.

*Remark 1.1.* We will apply the results obtained in the present paper to characterizations of $J$-self-adjoint extensions for linear Hamiltonian difference systems in terms of boundary conditions in the near future.

The rest of this present paper is organized as follows. In Section 2, some basic concepts and fundamental results about linear subspaces are introduced. In Section 3, the defect index of a $J$-Hermitian subspace is defined, and a formula for the defect index is given. Section 4 pays attention to the existence of $J$-self-adjoint subspace extensions and the GKN theorem for $J$-Hermitian subspace.

## 2. Preliminaries

In this section, we introduce some basic concepts and give some fundamental results about linear subspaces in a product space.

Let $X$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. The norm $\|\cdot\|$ is defined by $\|f\| = \langle f, f \rangle^{1/2}$ for $f \in X$. Let $X^2$ be the product space $X \times X$. By definition, the elements of $X^2$ consist of all possible ordered pairs $(x, f)$ with $x \in X$ and $f \in X$, and for arbitrary two elements $(x, f)$, $(y, g) \in X^2$ and $\alpha \in C$,

$$\alpha(x, f) = (\alpha x, \alpha f), \qquad (x, f) + (y, g) = (x + y, f + g). \tag{2.1}$$

The null element of $X^2$ is $(0, 0)$. The inner product in $X^2$ is defined by

$$\langle (x, f), (y, g) \rangle^* = \langle x, y \rangle + \langle f, g \rangle, \quad (x, f), (y, g) \in X^2, \tag{2.2}$$

and $\|\cdot\|^*$ denotes the induced norm.

Let $T$ be a linear subspace in $X^2$ which is called to be a linear relation in [7, 9, 10]. For brevity, a linear subspace is only called a subspace. For subspace $T$ in $X^2$, we shall use the following definitions and notations:

$$\begin{gathered} D(T) = \{x \in X : (x, f) \in T \text{ for some } f \in X\}, \\ R(T) = \{f \in X : (x, f) \in T \text{ for some } x \in X\}, \\ T(x) = \{f \in X : (x, f) \in T\}, \\ \ker(T) = \{x \in X : (x, 0) \in T\}, \\ T^{-1} = \{(f, x) : (x, f) \in T\}, \\ T - \lambda = \{(x, f - \lambda x) : (x, f) \in T\}. \end{gathered} \tag{2.3}$$

Clearly, $T(0) = \{0\}$ if and only if $T$ can determine a unique linear operator from $D(T)$ into $X$ whose graph is $T$, and $T^{-1}$ is closed if and only if $T$ is closed. Since the graph of a linear operator in $X$ is a subspace in $X^2$ and a linear operator is identified with its graph, the concept of subspaces in $X^2$ generalizes that of linear operators in $X$.

*Definition 2.1* (see [11]). Let $T$ be a subspace in $X^2$.

(1) Its adjoint, $T^*$, is defined by

$$T^* = \left\{ (y, g) \in X^2 : \ \langle f, y \rangle = \langle x, g \rangle \ \forall (x, f) \in T \right\}. \tag{2.4}$$

(2) $T$ is said to be a Hermitian subspace if $T \subset T^*$.

(3) $T$ is said to be a self-adjoint subspace if $T = T^*$.

**Lemma 2.2** (see [11]). *Let $T$ be a subspace in $X^2$, then $T^*$ is a closed subspace in $X^2$, $T^* = (\overline{T})^*$, and $T^{**} = \overline{T}$, where $\overline{T}$ is the closure of $T$.*

*Definition 2.3.* An operator $J$ defined on $X$ is said to be a conjugation operator if for all $x, y \in X$,

$$\langle Jx, Jy \rangle = \langle y, x \rangle, \qquad J^2 x = x. \tag{2.5}$$

It can be verified that $J$ is a conjugate linear, that is, $J(x+y) = Jx+Jy$ and $J(\lambda x) = \overline{\lambda}Jx$ for $x, y \in X$ and $\lambda \in C$, and norm-preserving bijection on $X$ satisfying

$$\langle Jx, y\rangle = \langle Jy, x\rangle \quad \forall x, y \in X. \tag{2.6}$$

For example, the complex conjugation $x \mapsto \overline{x}$ in any $L^2$ space is a conjugation operator on $L^2$.

*Definition 2.4.* Let $T$ be a subspace in $X^2$, and let $J$ be a conjugation operator.

(1) Its $J$-adjoint, $T_J^*$, is defined by

$$T_J^* = \left\{(y, g) \in X^2 : \langle f, Jy\rangle = \langle x, Jg\rangle \forall (x, f) \in T\right\}. \tag{2.7}$$

(2) $T$ is said to be a $J$-Hermitian subspace if $T \subset T_J^*$.

(3) $T$ is said to be a $J$-self-adjoint subspace if $T = T_J^*$.

*Remark 2.5.* (1) It can be easily verified that $T_J^*$ is a closed subspace. Consequently, a $J$-self-adjoint subspace $T$ is a closed subspace since $T = T_J^*$. In addition, $S_J^* \subset T_J^*$ if $T \subset S$.

(2) From the definition, we have that $\langle f, Jy\rangle = \langle x, Jg\rangle$ for all $(x, f) \in T$ and $(y, g) \in T_J^*$, and that $T$ is a $J$-Hermitian subspace if and only if

$$\langle f, Jy\rangle = \langle x, Jg\rangle \quad \forall (x, f), (y, g) \in T. \tag{2.8}$$

(3) The concepts of $J$-Hermitian and $J$-self-adjoint subspaces generalize those of $J$-symmetric and $J$-self-adjoint operators, respectively (see, e.g., [1, 24] for the concepts of $J$-symmetric and $J$-self-adjoint operators).

**Lemma 2.6.** *Let $T$ be a subspace in $X^2$, then*

(1) $T^* = \{(Jy, Jg) : (y, g) \in T_J^*\}$,

(2) $T_J^* = \{(Jy, Jg) : (y, g) \in T^*\}$.

*Proof.* Result (2) follows from result (1) and the second relation of (2.5). So, one needs only to prove result (1). Set $D_1 = \{(Jy, Jg) : (y, g) \in T_J^*\}$. Let $(y, g) \in T^*$, then $\langle f, y\rangle = \langle x, g\rangle$ for all $(x, f) \in T$. So, the second relation of (2.5) yields that $\langle f, J(Jy)\rangle = \langle x, J(Jg)\rangle$ for all $(x, f) \in T$. Hence, $(Jy, Jg) \in T_J^*$. Then $(y, g) = (J^2y, J^2g) \in D_1$, which implies that $T^* \subset D_1$. Conversely, let $(y, g) \in D_1$, then there exists $(\tilde{y}, \tilde{g}) \in T_J^*$ such that $(y, g) = (J\tilde{y}, J\tilde{g})$. It follows from $(\tilde{y}, \tilde{g}) \in T_J^*$ that $\langle f, J\tilde{y}\rangle = \langle x, J\tilde{g}\rangle$ for all $(x, f) \in T$, which implies that $(J\tilde{y}, J\tilde{g}) \in T^*$, that is, $(y, g) \in T^*$. So, $D_1 \subset T^*$. Consequently, $T^* = D_1$, and result (1) holds. □

*Remark 2.7.* Let $T$ be a subspace in $X^2$, then from Lemmas 2.2 and 2.6, and the closedness of $T_J^*$, one has that $T_J^* = (\overline{T})_J^*$, and $\overline{T}$ is $J$-Hermitian if $T$ is $J$-Hermitian.

**Lemma 2.8.** *Let $T$ be a closed $J$-Hermitian subspace, then $(y, g) \in T$ if and only if $(y, g) \in T_J^*$ and $\langle f, Jy\rangle = \langle x, Jg\rangle$ for all $(x, f) \in T_J^*$.*

*Proof.* Let $T$ be a closed $J$-Hermitian subspace. Clearly, the necessity holds by (2) of Remark 2.5. Now, consider the sufficiency. Suppose that $(y,g) \in T_J^*$ and $\langle f, Jy\rangle = \langle x, Jg\rangle$ for all $(x,f) \in T_J^*$, then we get from (2.6) that $\langle y, Jf\rangle = \langle g, Jx\rangle$ for all $(x,f) \in T_J^*$. This, together with (1) of Lemma 2.6, implies that $\langle y, \tilde{f}\rangle = \langle g, \tilde{x}\rangle$ for all $(\tilde{x}, \tilde{f}) \in T^*$. So, $(y,g) \in T^{**}$, and hence, $(y,g) \in T$ by Lemma 2.2. So, the sufficiency holds. □

**Lemma 2.9.** *Let $T$ be a closed $J$-Hermitian subspace in $X^2$, then*

(1) $(T_J^*)^* = \{(Jy, Jg) : (y,g) \in T\}$,

(2) $T = \{(Jy, Jg) : (y,g) \in (T_J^*)^*\}$.

*Proof.* Since result (1) and the second relation of (2.5) imply that result (2) holds, it suffices to prove result (1). Set $D = \{(Jy, Jg) : (y,g) \in T\}$. Let $(y,g) \in D$, then there exists $(\tilde{y}, \tilde{g}) \in T$ such that $(y,g) = (J\tilde{y}, J\tilde{g})$. By Lemma 2.8, $(\tilde{y}, \tilde{g}) \in T$ implies that $\langle f, J\tilde{y}\rangle = \langle x, J\tilde{g}\rangle$ for all $(x,f) \in T_J^*$. Then $(J\tilde{y}, J\tilde{g}) \in (T_J^*)^*$, that is, $(y,g) \in (T_J^*)^*$. So, $D \subset (T_J^*)^*$. Conversely, let $(y,g) \in (T_J^*)^*$, then $\langle f, y\rangle = \langle x, g\rangle$ for all $(x,f) \in T_J^*$, that is,

$$\langle f, J(Jy)\rangle = \langle x, J(Jg)\rangle \quad \forall (x,f) \in T_J^*. \tag{2.9}$$

Clearly, (2.9) holds for all $(x,f) \in T$ since $T \subset T_J^*$. So, $(Jy, Jg) \in T_J^*$. This, together with (2.9) and Lemma 2.8, implies that $(Jy, Jg) \in T$. So, $(J(Jy), J(Jg)) \in D$, that is, $(y,g) \in D$. Then, $(T_J^*)^* \subset D$ and then $(T_J^*)^* = D$. □

*Remark 2.10.* Since $T_J^* = (\overline{T})_J^*$ by Remark 2.7, Lemma 2.9 yields that $(T_J^*)^* = \{(Jy, Jg) : (y,g) \in \overline{T}\}$ and $\overline{T} = \{(Jy, Jg) : (y,g) \in (T_J^*)^*\}$ for a $J$-Hermitian subspace $T$ which may not be closed.

## 3. Defect Index of a $J$-Hermitian Subspace

In this section, the definition of the defect index of a $J$-Hermitian subspace is introduced, and a formula for the defect index is obtained.

Let $T$ be a closed $J$-Hermitian subspace. It has been known that $T_J^*$ is a closed subspace by (1) of Remark 2.5. Then the closedness of $T$ and $T_J^*$ and $T \subset T_J^*$ gives that

$$T_J^* = T \oplus \mathcal{T}, \tag{3.1}$$

where $\mathcal{T}$ denotes the orthogonal complement of $T$ in $T_J^*$, that is, $\mathcal{T} = T_J^* \ominus T$. Now, let $S$ be a closed $J$-Hermitian subspace extension of $T$, that is, $T \subset S$ and $S$ is $J$-Hermitian. Then, it follows from the closedness of $S$ and $T$ that there exists a unique subspace $K_{S,T} = S \ominus T$ such that

$$S = T \oplus K_{S,T}. \tag{3.2}$$

Clearly, $S \subset T_J^*$ since $S_J^* \subset T_J^*$ and $S \subset S_J^*$. Then by (3.1) and (3.2), $K_{S,T}$ can be expressed as

$$\begin{aligned} &K_{S,T} \\ &= \{(y,g) \in \mathcal{T} : \text{there exist } (y_1,g_1) \in T, (y_2,g_2) \in S, \\ &\quad \text{such that } (y_2,g_2) = (y_1,g_1) + (y,g)\}. \end{aligned} \tag{3.3}$$

Further, we have the following result.

**Theorem 3.1.** *Let $T$ be a closed $J$-Hermitian subspace, and let $S$ be a $J$-self-adjoint subspace extension (briefly, $J$-SSE) of $T$, that is, $T \subset S$ and $S$ is $J$-self-adjoint, then*

$$\dim \frac{T_J^*}{S} = \dim \frac{S}{T}. \tag{3.4}$$

*Proof.* If $T$ is a $J$-self-adjoint subspace, then $T = T_J^*$ and $T$ is the only $J$-SSE of itself. So, (3.4) holds. Now, suppose that $T$ is $J$-Hermitian but not $J$-self-adjoint, that is, $T \subset T_J^*$ and $T \neq T_J^*$. It follows that (3.2) holds with $K_{S,T} \neq \{0\}$. Let $\dim S/T = m$, then (3.2) yields that $\dim K_{S,T} = m$. In the case of $m < \infty$, let $\{(x_j, f_j)\}_{j=1}^m$ be a basis of $K_{S,T}$, then

$$S = T \oplus \operatorname{span}\{(x_1,f_1),(x_2,f_2),\ldots,(x_m,f_m)\}. \tag{3.5}$$

Define

$$T_j = T \oplus \operatorname{span}\{(x_1,f_1),(x_2,f_2),\ldots,(x_j,f_j)\}, \quad j = 1,2,\ldots,m. \tag{3.6}$$

Clearly, $T \neq T_j \neq T_{j+1}$ for $j = 1,2,\ldots,m-1$ and

$$S = S_J^* = (T_m)_J^* \subset (T_{m-1})_J^* \subset \cdots \subset (T_2)_J^* \subset (T_1)_J^* \subset T_J^*, \tag{3.7}$$

since $T \subset T_1 \subset T_2 \subset \cdots \subset T_m = S$ and $S$ is $J$-self-adjoint. Further, $T_j$ $(j = 1,2,\ldots,m)$ is a closed subspace since $T$ is closed. It holds that

$$T^* \neq T_j^* \neq T_{j+1}^*, \quad j = 1,2,\ldots,m-1. \tag{3.8}$$

Otherwise, for example, suppose that $T_1^* = T_2^*$, then by Lemma 2.2, we have $T_1 = T_1^{**} = T_2^{**} = T_2$ since $T_1$ and $T_2$ are closed. It contradicts $T_1 \neq T_2$. So, (3.8) holds. It follows from (3.8) and (2) of Lemma 2.6 that

$$T_J^* \neq (T_j)_J^* \neq (T_{j+1})_J^*, \quad j = 1,2,\ldots,m-1. \tag{3.9}$$

We get from (3.7) and (3.9) that $\dim T_J^*/S \geq m = \dim S/T$.

In the case of $m = +\infty$, we have the linear span of an infinite set in (3.5). So we can construct an infinite sequence of subspaces of the form (3.6) which satisfies the relations like those in (3.7) and (3.9). So, we have $\dim T_J^*/S = +\infty = \dim S/T$.

Next, we prove that $\dim S/T \geq \dim T_J^*/S$. Since $T_J^*$ and $S$ are closed subspaces, there exists uniquely a closed subspace $K_{T_J^*,S} = T_J^* \ominus S$ such that $T_J^* = S \oplus K_{T_J^*,S}$. Set $\dim T_J^*/S = m'$. Then $\dim K_{T_J^*,S} = m'$. If $m' < \infty$, let $\{(y_j, g_j)\}_{j=1}^{m'}$ be a basis of $K_{T_J^*,S}$, then

$$T_J^* = S \oplus \operatorname{span}\{(y_1, g_1), (y_2, g_2), \ldots, (y_{m'}, g_{m'})\}. \tag{3.10}$$

Define

$$S_j = S \oplus \operatorname{span}\{(y_1, g_1), (y_2, g_2), \ldots, (y_j, g_j)\}, \quad j = 1, 2, \ldots, m'. \tag{3.11}$$

Clearly, it holds that $S_J^* = S \neq S_j \neq S_{j+1}$ for $j = 1, 2, \ldots, m' - 1$ and

$$\left(T_J^*\right)^* = S_{m'}^* \subset S_{m'-1}^* \subset \cdots \subset S_2^* \subset S_1^* \subset S^* = \left(S_J^*\right)^*, \tag{3.12}$$

since $S \subset S_1 \subset S_2 \subset \cdots \subset S_{m'} = T_J^*$ and $S$ is $J$-self-adjoint. Further, $S_j$ $(j = 1, 2, \ldots, m')$ is a closed subspace since $S$ is closed. Similarly, it holds that

$$\left(S_J^*\right)^* \neq S_j^* \neq S_{j+1}^*, \quad j = 1, 2, \ldots, m' - 1. \tag{3.13}$$

We get from (3.12) and (3.13) that $\dim (S_J^*)^*/(T_J^*)^* \geq m'$. It can be verified that

$$\dim \frac{\left(S_J^*\right)^*}{\left(T_J^*\right)^*} = \dim \frac{S}{T} \tag{3.14}$$

by Lemma 2.9. So, $\dim S/T \geq m' = \dim T_J^*/S$. Further, it can be verified that $\dim S/T \geq \dim T_J^*/S$ also holds for $m' = +\infty$. Based on the above discussions, (3.4) holds. □

*Remark 3.2.* (1) From Theorem 3.1 and its proof, one has that if one of the two dimensions in (3.4) is finite, so is the other and they are equal, and if one of the two dimensions is infinite, so is the other. Here, there is no distinction between degrees of infinity.

(2) The case for $J$-symmetric operators was established in [24, Theorem 3.1].

*Remark 3.3.* Note that $T_J^* = (\overline{T})_J^*$. We get from Theorem 3.1 that if $S$ is a $J$-SSE of $T$, which may not be closed, then it holds that

$$\dim \frac{T_J^*}{\overline{T}} = 2 \dim \frac{S}{\overline{T}}. \tag{3.15}$$

Now, we give the concept of the defect index of a $J$-Hermitian subspace. The concept of the defect index of a $J$-symmetric operator in $X$ was given by [24, Definition 3.2].

*Definition 3.4.* Let $T$ be a $J$-Hermitian subspace, then $d(T) = (1/2) \dim T_J^*/\overline{T}$ is called to be the defect index of $T$.

*Remark 3.5.* (1) It will be proved that every $J$-Hermitian subspace has a $J$-SSE in Theorem 4.3 in Section 4. So, by (3.15) we have that the defect index of every $J$-Hermitian subspace is a nonnegative integer.

(2) Since $T_J^* = (\overline{T})_J^*$ by Remark 2.7 and every $J$-SSE is closed, we have that a $J$-symmetric subspace $T$ and its closure $\overline{T}$ have the same defect index and the same $J$-SSEs.

*Definition 3.6* (see [12]). Let $T$ be a subspace in $X^2$. The set

$$\Gamma(T) := \left\{\lambda \in \mathbf{C} : \text{there exists } c(\lambda) > 0 \text{ such that } \|f - \lambda x\| \geq c(\lambda)\|x\| \forall (x, f) \in T\right\} \tag{3.16}$$

is called to be the regularity field of $T$.

It is evident that $\Gamma(T) = \Gamma(\overline{T})$ for a subspace $T$.

**Lemma 3.7.** *Let $T$ be a subspace in $X$, then*

(1) $R(T - \lambda)^{\perp} = \ker(T^* - \overline{\lambda})$ *for each* $\lambda \in \mathbf{C}$,

(2) *for each* $\lambda \in \Gamma(T)$,

$$X = R\left(\overline{T} - \lambda\right) \oplus \ker\left(T^* - \overline{\lambda}\right) \text{ (orthogonal sum in X)}, \tag{3.17}$$

(3) $X = R(T_J^* - \lambda)$ *for each* $\lambda \in \Gamma(T)$.

*Proof.* (1) Let $\lambda \in \mathbf{C}$. It is clear that

$$\ker\left(T^* - \overline{\lambda}\right) = \left\{x \in X : \left(x, \overline{\lambda}x\right) \in T^*\right\}. \tag{3.18}$$

For every $x \in \ker(T^* - \overline{\lambda})$, we have from (3.18) that $(x, \overline{\lambda}x) \in T^*$. So, $\langle g, x\rangle = \langle y, \overline{\lambda}x\rangle$ for all $(y, g) \in T$, which implies that $\langle g - \lambda y, x\rangle = 0$. Therefore, $x \in R(T - \lambda)^{\perp}$, and then $\ker(T^* - \overline{\lambda}) \subset R(T - \lambda)^{\perp}$. Conversely, for every $x \in R(T - \lambda)^{\perp}$, we have that $\langle g - \lambda y, x\rangle = 0$ for all $(y, g) \in T$. It follows that $\langle g, x\rangle = \langle y, \overline{\lambda}x\rangle$ for all $(y, g) \in T$. So, $(x, \overline{\lambda}x) \in T^*$, and hence $x \in \ker(T^* - \overline{\lambda})$ by (3.18). So, $R(T - \lambda)^{\perp} \subset \ker(T^* - \overline{\lambda})$. Consequently, $R(T - \lambda)^{\perp} = \ker(T^* - \overline{\lambda})$, and result (1) is proved.

(2) By result (1) and Lemma 2.2, we have that $R(\overline{T} - \lambda)^{\perp} = \ker((\overline{T})^* - \overline{\lambda}) = \ker(T^* - \overline{\lambda})$. So, by the projection theorem, in order to prove (3.17), it suffices to show that $R(\overline{T} - \lambda)$ with $\lambda \in \Gamma(T)$ is closed in $X$. It is evident that

$$\left(\overline{T} - \lambda\right)^{-1} = \left\{(f - \lambda x, x) : (x, f) \in \overline{T}\right\}. \tag{3.19}$$

Let $\lambda \in \Gamma(T)$. Since $\Gamma(\overline{T}) = \Gamma(T)$, one has that $\lambda \in \Gamma(\overline{T})$, that is, there exists a constant $c(\lambda) > 0$ such that

$$\|f - \lambda x\| \geq c(\lambda)\|x\| \quad \forall (x, f) \in \overline{T}. \tag{3.20}$$

Then we get from (3.19) and (3.20) that $(\overline{T}-\lambda)^{-1}(0)=\{0\}$. So, $(\overline{T}-\lambda)^{-1}$ determines a linear operator from $R(\overline{T}-\lambda)$ to $X$. Further, the closedness of $\overline{T}-\lambda$ and (3.20) imply that this operator is a closed and bounded operator. So, its domain $R(\overline{T}-\lambda)$ is closed in $X$. Therefore, (3.17) holds, and result (2) is proved.

(3) Let $\lambda\in\Gamma(T)$. We first show that there exists a constant $M>0$ such that

$$\|x\|\le M\|f-\lambda x\|\quad\forall(x,f)\in T_J^*. \tag{3.21}$$

Assume the contrary, then there exists a sequence $\{f_k-\lambda x_k\}_{k=1}^{\infty}\subset R(T_J^*-\lambda)$ with $\|f_k-\lambda x_k\|=1$ $(k=1,2,\ldots)$ such that

$$\|x_k\|>k,\qquad k=1,2,\ldots. \tag{3.22}$$

Clearly, $(x_k,f_k)\in T_J^*$ and $(\overline{T})_J^*=T_J^*$ imply that $(x_k,f_k)\in(\overline{T})_J^*$. So, $\langle g,Jx_k\rangle=\langle y,Jf_k\rangle$ for all $(y,g)\in\overline{T}$ by (1) of Remark 2.5, which, together with $\langle\lambda y,Jx_k\rangle=\langle y,J(\lambda x_k)\rangle$, implies that for $k=1,2,\ldots$,

$$\langle g-\lambda y,Jx_k\rangle=\langle y,J(f_k-\lambda x_k)\rangle\quad\forall(y,g)\in\overline{T}. \tag{3.23}$$

Define $\phi_k(g-\lambda y)=\langle g-\lambda y,Jx_k\rangle$ for $(y,g)\in\overline{T}$. Then $\phi_k$, $k=1,2\ldots$, are linear functionals in $R(\overline{T}-\lambda)$. Since $\phi_k(g-\lambda y)=\langle y,J(f_k-\lambda x_k)\rangle$ by (3.23) and $\|J(f_k-\lambda x_k)\|=\|f_k-\lambda x_k\|=1$, we have that $\{\phi_k(g-\lambda y)\}_{k=1}^{\infty}$ is bounded for any given $g-\lambda y\in R(\overline{T}-\lambda)$. Note that $R(\overline{T}-\lambda)$ with $\lambda\in\Gamma(T)$ is closed by the proof of result (2), and hence it is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. So, by the resonance theorem, $\{\|\phi_k\|\}_{k=1}^{\infty}$ is bounded, that is, $\{\|Jx_k\|\}_{k=1}^{\infty}$ is bounded. Since $\|Jx_k\|=\|x_k\|$, $k=1,2,\ldots$, we have a contradiction with (3.22). So, (3.21) holds.

Inserting (3.21) into $\|x\|+\|f\|\le\|f-\lambda x\|+(1+|\lambda|)\|x\|$, we get that

$$\|x\|+\|f\|\le[1+(1+|\lambda|)M]\|f-\lambda x\|. \tag{3.24}$$

It can be easily verified from the closedness of $T_J^*$ and (3.24) that $R(T_J^*-\lambda)$ is closed in $X$. So, result (1) implies that

$$X=R\left(T_J^*-\lambda\right)\oplus\ker\left(\left(T_J^*\right)^*-\overline{\lambda}\right). \tag{3.25}$$

By Remark 2.10, $\ker((T_J^*)^*-\overline{\lambda})=\{Jy:y\in\ker(\overline{T}-\lambda)\}$, while it can be verified that $\ker(\overline{T}-\lambda)=\{0\}$ for $\lambda\in\Gamma(T)=\Gamma(\overline{T})$. Therefore, (3.25) yields that result (3) holds. □

If $\Gamma(T)\neq\emptyset$, we have the following results which give a formula for the defect index of a $J$-Hermitian space.

**Theorem 3.8.** *Assume that $T$ is a $J$-Hermitian subspace with $\Gamma(T) \neq \emptyset$. Let $\lambda \in \Gamma(T)$, then*

$$T_J^* = \overline{T} + \mathcal{N}, \tag{3.26}$$

*where*

$$\mathcal{N} = \left\{ (y, g) \in T_J^* : g - \lambda y \in \ker\left(T^* - \overline{\lambda}\right) \right\}, \tag{3.27}$$

$$d(T) = \dim R(T - \lambda)^{\perp} = \dim \ker\left(T^* - \overline{\lambda}\right). \tag{3.28}$$

*Proof.* We first prove (3.26). Since $T$ is $J$-Hermitian, one has that $\overline{T}$ is $J$-Hermitian, and hence $\overline{T} \subset (\overline{T})_J^* = T_J^*$. Clearly, $\mathcal{N} \subset T_J^*$. So, $\overline{T} + \mathcal{N} \subset T_J^*$. On the other hand, let $(x, f) \in T_J^*$. It follows from (3.17) and $\lambda \in \Gamma(T)$ that there exist $(y, g) \in \overline{T}$ and $w \in \ker(T^* - \overline{\lambda})$ such that $f - \lambda x = g - \lambda y + w$, that is, $(f - g) - \lambda(x - y) = w$. Let $(\tilde{x}, \tilde{f}) = (x - y, f - g)$, then $(x, f) = (y, g) + (\tilde{x}, \tilde{f})$ and $(\tilde{x}, \tilde{f}) \in \mathcal{N}$. So, $T_J^* \subset \overline{T} + \mathcal{N}$, and consequently, $T_J^* = \overline{T} + \mathcal{N}$.

Now, let $(u, h) \in \overline{T} \cap \mathcal{N}$, then

$$h - \lambda u \in R\left(\overline{T} - \lambda\right), \qquad h - \lambda u \in \ker\left(T^* - \overline{\lambda}\right). \tag{3.29}$$

Consequently, $h - \lambda u = 0$ by $R(\overline{T} - \lambda)^{\perp} = \ker(T^* - \overline{\lambda})$, which can be obtained from (3.17). Since $\lambda \in \Gamma(T) = \Gamma(\overline{T})$, one has by Definition 3.6 that there exists a constant $c(\lambda) > 0$ such that $0 = \|h - \lambda u\| \geq c(\lambda)\|u\|$. It follows that $u = 0$, which, together with $\|h - \lambda u\| = 0$, implies that $h = 0$. Then, $(u, h) = (0, 0)$ if $(u, h) \in \overline{T} \cap \mathcal{N}$. So, (3.26) holds.

Next, we prove (3.28). Let $\lambda \in \Gamma(T)$. Set $U_1 = \{(x, \lambda x) \in T_J^*\}$, then $U_1$ is closed since $T_J^*$ is closed. Let $U_2 = \mathcal{N}/U_1$. We will show that

$$\dim U_2 = \dim \ker\left(T^* - \overline{\lambda}\right). \tag{3.30}$$

Let $\{\varphi_j\}_{j=1}^m \subset \ker(T^* - \overline{\lambda})$ be linearly independent, then, by (3) of Lemma 3.7, there exists $(u_j, h_j) \in T_J^*$ such that $\varphi_j = h_j - \lambda u_j$, $1 \leq j \leq m$. It follows from $\varphi_j \in \ker(T^* - \overline{\lambda})$ that $(u_j, h_j) \in U_2$ for $1 \leq j \leq m$. In addition, if

$$\sum_{j=1}^{m} c_j (u_j, h_j) \in U_1, \quad c_j \in \mathbf{C}, \tag{3.31}$$

then $\sum_{j=1}^m c_j(h_j - \lambda u_j) = 0$, that is, $\sum_{j=1}^m c_j \varphi_j = 0$. So, $c_j = 0$ for $1 \leq j \leq m$, and hence $\{(u_j, h_j)\}_{j=1}^m \subset U_2$ is linearly independent $(\operatorname{mod} U_1)$. Conversely, let $\{(u_j, h_j)\}_{j=1}^m \subset U_2$ be linearly independent $(\operatorname{mod} U_1)$, and let $\varphi_j = h_j - \lambda u_j$, then $\varphi_j \in \ker(T^* - \overline{\lambda})$. If $\sum_{j=1}^m c_j \varphi_j = 0$, then $\sum_{j=1}^m c_j(u_j, h_j) \in U_1$. So, $c_j = 0$ $(1 \leq j \leq m)$, and hence the set $\{\varphi_j\}_{j=1}^m$ is linearly

independent. Based on the above discussions, (3.30) holds. On the other hand, it is evident that

$$\dim\left\{\left(x,\overline{\lambda}x\right)\in T^*\right\}=\dim\ \ker\left(T^*-\overline{\lambda}\right). \tag{3.32}$$

Further, by Lemma 2.6, we have that

$$\dim U_1=\dim\left\{\left(x,\overline{\lambda}x\right)\in T^*\right\}. \tag{3.33}$$

It follows from (3.30)–(3.33) that $\dim U_1=\dim U_2$, and hence (3.26) implies that

$$d(T)=\frac{1}{2}\dim\mathcal{N}=\dim\ \ker\left(T^*-\overline{\lambda}\right). \tag{3.34}$$

So, (3.28) holds by (1) of Lemma 3.7. □

*Remark 3.9.* From Theorem 3.8, one has the following result: for a $J$-Hermitian subspace $T$, $\dim R(T-\lambda)^{\perp}$ and $\dim\ \ker(T^*-\overline{\lambda})$ are constants in $\Gamma(T)$ which are equal to the defect index of $T$. This result extends [24, Theorem 5.7] for $J$-symmetric operators to $J$-Hermitian subspaces. Similarly, there is no distinction between degrees of infinity.

## 4. $J$-Self-Adjoint Subspace Extensions of a $J$-Hermitian Subspace

In this section, we consider the existence of $J$-SSEs of a $J$-Hermitian space and the characterizations of all the $J$-SSEs.

Define the form [:] as

$$[(x,f):(y,g)]=\langle f,Jy\rangle-\langle x,Jg\rangle,\quad (x,f),\ (y,g)\in T_J^*. \tag{4.1}$$

Then, $\langle f,Jy\rangle=\langle x,Jg\rangle$ if and only if $[(x,f):(y,g)]=0$. Further, for all $Y_j=(x_j,f_j)\in T_J^*$ $(j=1,2,3)$ and $\mu\in\mathbf{C}$,

$$\begin{gathered}[Y_3:Y_1+Y_2]=[Y_3:Y_1]+[Y_3:Y_2],\qquad [Y_1+Y_2:Y_3]=[Y_1:Y_3]+[Y_2:Y_3],\\ [\mu Y_1:Y_2]=[Y_1:\mu Y_2]=\mu[Y_1:Y_2],\qquad [Y_1:Y_2]=-[Y_2:Y_1].\end{gathered} \tag{4.2}$$

Since the closure $\overline{T}$ of a $J$-Hermitian subspace $T$ is also a $J$-Hermitian subspace by Remark 2.7, and $T$ and $\overline{T}$ have the same defect indices and the same $J$-SSEs by (2) of Remark 3.5, we shall assume that $T$ is closed in the rest of this section. Let $T$ be a closed subspace in $X^2$, and let $K\subset\mathcal{T}$ be a subspace, where $\mathcal{T}$ is given in (3.1). Let $K_J^*|_{\mathcal{T}}$ be a restriction of $K_J^*$ to $\mathcal{T}$, that is,

$$K_J^*\Big|_{\mathcal{T}}=\{(y,g)\in\mathcal{T}:[(x,f):(y,g)]=0\forall(x,f)\in K\}, \tag{4.3}$$

then $K$ is called to be $J$-Hermitian in $\mathcal{T}$ if $K\subset K_J^*|_{\mathcal{T}}$, and $K$ is called to be $J$-self-adjoint in $\mathcal{T}$ if $K=K_J^*|_{\mathcal{T}}$.

*Remark 4.1.* From the definition, we have that $[(x,f):(y,g)]=0$ for all $(x,f)\in K$ and $(y,g)\in K_J^*|_{\mathcal{T}}$, and $K$ is $J$-Hermitian in $\mathcal{T}$ if and only if $[(x,f):(y,g)]=0$ for all $(x,f)$, $(y,g)\in K$.

**Lemma 4.2.** *Let $T$ be a closed $J$-Hermitian subspace, and let $K\subset\mathcal{T}$ be a subspace. Assume that $S=T\oplus K$, then*

(1) *$S$ is $J$-Hermitian if and only if $K$ is $J$-Hermitian in $\mathcal{T}$,*

(2) *$S$ is $J$-self-adjoint if and only if $K$ is $J$-self-adjoint in $\mathcal{T}$.*

*Proof.* (1) Suppose that $S$ is $J$-Hermitian. It can be easily verified that $K$ is $J$-Hermitian in $\mathcal{T}$ by (2) of Remark 2.5, $K\subset S$, and Remark 4.1. So, the necessity holds. We now prove the sufficiency. Suppose that $K$ is $J$-Hermitian in $\mathcal{T}$. For all $(x,f)$, $(y,g)\in S$, we get from $S=T\oplus K$ that

$$\begin{aligned}(x,f)&=(x_1,f_1)+(x_2,f_2),\quad (x_1,f_1)\in T,\ (x_2,f_2)\in K,\\(y,g)&=(y_1,g_1)+(y_2,g_2),\quad (y_1,g_1)\in T,\ (y_2,g_2)\in K.\end{aligned}\tag{4.4}$$

Since $T$ is $J$-Hermitian and $K\subset K_J^*|_{\mathcal{T}}\subset T_J^*$, it can be obtained from (2) of Remark 2.5 and Remark 4.1 that

$$\begin{aligned}[(x,f):(y,g)]&=[(x_1,f_1):(y_1,g_1)]+[(x_1,f_1):(y_2,g_2)]\\&\quad+[(x_2,f_2):(y_1,g_1)]+[(x_2,f_2):(y_2,g_2)]=0.\end{aligned}\tag{4.5}$$

So, $S$ is $J$-Hermitian. The sufficiency holds, and result (1) is proved.

(2) First, consider the necessity. Suppose that $S$ is $J$-self-adjoint, then $K$ is $J$-Hermitian in $\mathcal{T}$, that is, $K\subset K_J^*|_{\mathcal{T}}$, by result (1). It suffices to show that $K_J^*|_{\mathcal{T}}\subset K$. For any given $(x,f)\in S$, there exist $(x_1,f_1)\in T$ and $(x_2,f_2)\in K$ such that the first relation of (4.4) holds. Let $(y,g)\in K_J^*|_{\mathcal{T}}$, then $[(x_2,f_2):(y,g)]=0$ by Remark 4.1. Note that $(y,g)\in T_J^*$ by $K_J^*|_{\mathcal{T}}\subset T_J^*$. Then $[(x_1,f_1):(y,g)]=0$ by (2) of Remark 2.5. Therefore, the first relation of (4.4) yields that

$$[(x,f):(y,g)]=[(x_1,f_1):(y,g)]+[(x_2,f_2):(y,g)]=0\quad\forall(x,f)\in S.\tag{4.6}$$

So, $(y,g)\in S_J^*$. Therefore, $S=S_J^*$ yields that $(y,g)\in S$, which, together with $(y,g)\in K_J^*|_{\mathcal{T}}$, $K_J^*|_{\mathcal{T}}\cap T=\{0\}$, and $S=T\oplus K$, implies that $(y,g)\in K$. Hence, $K_J^*|_{\mathcal{T}}\subset K$, and hence $K=K_J^*|_{\mathcal{T}}$. The necessity holds.

Next, consider the sufficiency. Suppose that $K$ is $J$-self-adjoint in $\mathcal{T}$. By result (1), one has that $S\subset S_J^*$. It suffices to show that $S_J^*\subset S$. Let $(y,g)\in S_J^*$, then $(y,g)\in T_J^*$ since $S_J^*\subset T_J^*$ by $T\subset S$. It follows from (3.1) that there exist $(y_1,g_1)\in T$ and $(y_2,g_2)\in\mathcal{T}$ such that

$$(y,g)=(y_1,g_1)+(y_2,g_2).\tag{4.7}$$

We claim that $(y_2,g_2)\in K$. In fact, since $(y,g)\in S_J^*$, we have

$$[(x,f):(y,g)]=0\quad\forall(x,f)\in S.\tag{4.8}$$

Inserting the first relation of (4.4) and (4.7) into (4.8) and using (2) of Remark 2.5 and Remark 4.1, we get that $[(x_2, f_2) : (y_2, g_2)] = 0$ for all $(x_2, f_2) \in K$. Then, $(y_2, g_2) \in K_J^*|_{\mathcal{T}} = K$. It follows from $(y_2, g_2) \in K$, $S = T \oplus K$, and (4.7) that $(y, g) \in S$. So, $S_J^* \subset S$, and hence $S = S_J^*$. The sufficiency holds. □

Now, we give the following result on the existence of $J$-SSEs.

**Theorem 4.3.** *Every closed $J$-Hermitian subspace has a $J$-SSE.*

*Proof.* Let $T$ be a closed $J$-Hermitian subspace. If $T$ is $J$-self-adjoint, then this conclusion holds. So, we assume that $T \neq T_J^*$. To prove that $T$ has a $J$-SSE, it suffices to prove that there exists a $J$-self-adjoint subspace $K$ in $\mathcal{T}$ by Lemma 4.2. The proof uses Zorn's lemma. Since $T \neq T_J^*$, one has that $\mathcal{T} \neq \{0\}$. Choose $0 \neq (x_0, f_0) \in \mathcal{T}$ and set $K_0 = \operatorname{span}\{(x_0, f_0)\}$. Then $K_0$ is $J$-Hermitian in $\mathcal{T}$ since $[(x_0, f_0) : (x_0, f_0)] = 0$. Let $\mathcal{K}$ be the set of all the $J$-Hermitian subspaces in $\mathcal{T}$ which contain $K_0$, then $\mathcal{K}$ is not empty since $K_0 \in \mathcal{K}$. Further, let $\mathcal{K}$ be ordered by extension, that is, $A < B$ if and only if $A \subset B$, and let $\mathcal{M} = \{K_\alpha\}$ be an arbitrary totally ordered subset of $\mathcal{K}$. Set $\widehat{K} = \bigcup_\alpha K_\alpha$. Then, it can be verified that $\widehat{K}$ is $J$-Hermitian in $\mathcal{T}$ by Remark 4.1 and the fact that all the elements of $\mathcal{M}$ are $J$-Hermitian in $\mathcal{T}$. So, $\widehat{K}$ is an upper bound of $\mathcal{M}$ in $\mathcal{K}$. Therefore, $\mathcal{K}$ has a maximal element by Zorn's lemma. This means that $K_0$ has a maximal $J$-Hermitian subspace extension, denoted by $K$, in $\mathcal{T}$. We now prove $K = K_J^*|_{\mathcal{T}}$. Suppose that $K \neq K_J^*|_{\mathcal{T}}$ on the contrary. Note that $K \subset K_J^*|_{\mathcal{T}}$. Choose $(\tilde{x}_0, \tilde{f}_0) \in K_J^*|_{\mathcal{T}}$ satisfying $(\tilde{x}_0, \tilde{f}_0) \notin K$, and set $\widetilde{K} = K + \operatorname{span}\{(\tilde{x}_0, \tilde{f}_0)\}$. It can be verified by Remark 4.1 and the fact that $K$ is $J$-Hermitian in $\mathcal{T}$ that $[(x, f) : (y, g)] = 0$ holds for all $(x, f), (y, g) \in \widetilde{K}$. Note that $\widetilde{K} \subset \mathcal{T}$. Then, $\widetilde{K}$ is $J$-Hermitian in $\mathcal{T}$, which contradicts the maximality of $K$. Hence, $K = K_J^*|_{\mathcal{T}}$. □

*Remark 4.4.* Since $T$ and its closure have the same $J$-SSEs, we have that every $J$-Hermitian subspace has a $J$-SSE. In addition, Theorem 4.3 extends the relevant result, for example, [1, Chapter III, Theorem 5.8], for $J$-symmetric operators to $J$-Hermitian subspaces.

The following result will give a characterization of all the $J$-SSEs.

**Theorem 4.5.** *Let $T$ be a closed $J$-Hermitian subspace. Assume that $d(T) = d < +\infty$, then a subspace $S$ is a $J$-SSE of $T$ if and only if $T \subset S \subset T_J^*$, and there exists $\{(x_j, f_j)\}_{j=1}^d \subset T_J^*$ such that*

(1) $(x_1, f_1), (x_2, f_2), \ldots, (x_d, f_d)$ *are linearly independent* (mod$T$),

(2) $[(x_s, f_s) : (x_j, f_j)] = 0$ *for* $s, j = 1, 2, \ldots, d$,

(3) $S = \{(y, g) \in T_J^* : [(y, g) : (x_j, f_j)] = 0,\ j = 1, 2, \ldots, d\}$.

*Proof.* First, consider the necessity. Suppose that $S$ is a $J$-SSE of $T$, then it holds that $T \subset S \subset T_J^*$ since $S_J^* \subset T_J^*$ and $S = S_J^*$. We also have that (3.2) holds and $K_{S,T}$ in (3.2) is $J$-self-adjoint in $\mathcal{T}$ by Lemma 4.2. Note that $\dim S/T = d$ by Theorem 3.1. Then $\dim K_{S,T} = d$, and let $\{(x_j, f_j)\}_{j=1}^d$ be a basis of $K_{S,T}$, then we get from (3.2) that (1) holds. In addition, since $K_{S,T}$ is $J$-self-adjoint in $\mathcal{T}$, one has that (2) holds by Remark 4.1. For convenience, set

$$D = \left\{(y, g) \in T_J^* : [(y, g) : (x_j, f_j)] = 0,\ j = 1, 2, \ldots, d\right\}. \tag{4.9}$$

Now, we prove $T \oplus K_{S,T} = D$, that is, $S = D$. Let $(y, g) \in T \oplus K_{S,T}$, then $(y, g) \in T_J^*$ by $S \subset T_J^*$, and there exist $(\tilde{y}, \tilde{g}) \in T$ and $c_j \in C$ such that

$$(y, g) = (\tilde{y}, \tilde{g}) + \sum_{j=1}^{d} c_j (x_j, f_j). \tag{4.10}$$

Inserting (4.10) into $[(y, g) : (x_s, f_s)]$ and using (2) of Remark 2.5 and (2) of this theorem, we get that $[(y, g) : (x_s, f_s)] = 0$ for $s = 1, 2, \ldots, d$. So, $(y, g) \in D$, and hence $T \oplus K_{S,T} \subset D$. Conversely, suppose that $(y, g) \in D$, then by (3.1), there exist $(y_1, g_1) \in T$ and $(y_2, g_2) \in \mathcal{T}$ such that (4.7) holds. The definition of $D$, (4.7), and (2) of Remark 2.5 implies that for $s = 1, 2, \ldots, d$,

$$\left[(y_2, g_2) : (x_s, f_s)\right] = \left[(y, g) : (x_s, f_s)\right] - \left[(y_1, g_1) : (x_s, f_s)\right] = 0. \tag{4.11}$$

So, $(y_2, g_2) \in (K_{S,T})_J^*|_{\mathcal{T}}$, which implies that $(y_2, g_2) \in K_{S,T}$ since $K_{S,T}$ is $J$-self-adjoint in $\mathcal{T}$. One has from (4.7) that $(y, g) \in T \oplus K_{S,T}$, and consequently, $D \subset T \oplus K_{S,T}$. Hence, $T \oplus K_{S,T} = D$, that is, $S = D$. The necessity holds.

Next, consider the sufficiency. Suppose that there exists $\{(x_j, f_j)\}_{j=1}^{d} \subset T_J^*$ such that conditions (1) and (2) hold and $S$ is given in condition (3). From (3.1), we have

$$(x_j, f_j) = (x_{j0}, f_{j0}) + \left(\tilde{x}_j, \tilde{f}_j\right), \quad (x_{j0}, f_{j0}) \in T, \quad \left(\tilde{x}_j, \tilde{f}_j\right) \in \mathcal{T}, \quad j = 1, 2, \ldots, d. \tag{4.12}$$

It can be easily verified that the set $\{(\tilde{x}_j, \tilde{f}_j)\}_{j=1}^{d}$ satisfies conditions (1) and (2). Let $\tilde{K} = \operatorname{span}\{(\tilde{x}_1, \tilde{f}_1), (\tilde{x}_2, \tilde{f}_2), \ldots, (\tilde{x}_d, \tilde{f}_d)\}$, then $\tilde{K}$ is $J$-Hermitian in $\mathcal{T}$ since $\{(\tilde{x}_j, \tilde{f}_j)\}_{j=1}^{d}$ satisfies condition (2) and $\tilde{K} \subset \mathcal{T}$. By the proof of Theorem 4.3, there exists $K_1 \subset \mathcal{T}$ such that $K_1$ is $J$-self-adjoint in $\mathcal{T}$ and $\tilde{K} \subset K_1$. Then, by Lemma 4.2, $T \oplus K_1$ is a $J$-SSE of $T$, which, together with Theorem 3.1, yields that

$$d = \frac{\dim(T \oplus K_1)}{T} = \dim K_1 \geq d. \tag{4.13}$$

Therefore, $\tilde{K} \subset K_1$ and $\dim \tilde{K} = d$ imply that $\tilde{K} = K_1$, and hence $\tilde{K}$ is $J$-self-adjoint in $\mathcal{T}$. With a similar argument to the proof of $T \oplus K_{S,T} = D$, we have

$$T \oplus \tilde{K} = \tilde{D}, \tag{4.14}$$

where

$$\tilde{D} = \left\{(y, g) \in T_J^* : \left[(y, g) : \left(\tilde{x}_j, \tilde{f}_j\right)\right] = 0, \ j = 1, 2, \ldots, d\right\}. \tag{4.15}$$

On the other hand, it can be easily verified that $\tilde{D} = S$. So, $S = T \oplus \tilde{K}$, and hence $S$ is $J$-self-adjoint by (2) of Lemma 4.2. The sufficiency holds. □

*Remark 4.6.* The case for $J$-symmetric operators is given by [24, 27]. Theorem 4.5 can be regarded as the GKN theorem for $J$-Hermitian subspaces, which will be used for characterizations of $J$-self-adjoint extensions for linear Hamiltonian difference systems in terms of boundary conditions.

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