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Research Article

Consistent Price Systems in Multiasset Markets

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Let X_t be any d-dimensional continuous process that takes values in an open connected domain $\mathcal{O}$ in $\mathbb{R}^d$. In this paper, we give equivalent formulations of the conditional full support (CFS) property of X_t in $\mathcal{O}$. We use them to show that the CFS property of X in $\mathcal{O}$ implies the existence of a martingale M under an equivalent probability measure such that M lies in the $\epsilon > 0$ neighborhood of X_t for any given ϵ under the supremum norm. The existence of such martingales, which are called consistent price systems (CPSs), has relevance with absence of arbitrage and hedging problems in markets with proportional transaction costs as discussed in the recent paper by Guasoni et al. (2008), where the CFS property is introduced and shown sufficient for CPSs for processes with certain state space. The current paper extends the results in the work of Guasoni et al. (2008), to processes with more general state space.

1. Introduction

We consider a financial market with d risky assets and a risk-free asset which is used as a numéraire and therefore assumed to be equal to one. We assume that the price processes of the d risky assets are given by an $\mathbb{R}^d$-valued process $Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^d)$, where $Y_t^i = e^{X_t^i}$, $1 \leq i \leq d$, and the d-dimensional process $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ and adapted to the filtration $\mathbb{F}$ that satisfies the usual assumptions. We assume that there are transaction costs in the market and they are fully proportional in the sense that each cost is equal to the actual dollar amount being traded beyond the riskless asset, multiplied by a fixed constant. In the presence of such transaction costs, it is reasonable to assume that purchases and sales do not overlap to avoid dissipation of wealth. In general, in markets with proportional transaction costs trading strategies $\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)$ are given by the difference of two processes $L_t = (L_t^1, L_t^2, \ldots, L_t^d)$ and $M_t = (M_t^1, M_t^2, \ldots, M_t^d)$ representing respectively the cumulative number of shares purchased and sold up to time t, namely, $\theta_t = L_t - M_t$. We are also required to start and end without any position in the risky assets to and this requirement corresponds to $\theta_0 = \theta_T = 0$.

For each such trading strategy $\theta_t = L_t - M_t$, the corresponding wealth process, after taking into account the incurred transaction costs, is given by

$$V_t(\theta) = \sum_{i=1}^{d} \int_0^t \theta_s^i dY_s^i - \epsilon \sum_{i=1}^{d} \int_0^t Y_s^i d\,\mathrm{Var}\left(\theta^i\right)_s, \tag{1.1}$$

where $\mathrm{Var}(\theta^i)_s = L_s^i + M_s^i$ is the total variation of θ^i in $[0, s]$ for each $1 \le i \le d$ and $\epsilon > 0$ is the proportion of the transaction costs. In our model (1.1), transaction costs between risky assets and cash are permitted and all transaction costs are charged to the cash account. Next, we introduce the class of trading strategies that we consider in this paper.

Definition 1.1. An admissible trading strategy is a predictable $\mathbb{R}^d$-valued process $\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)$ of finite variation with $\theta_0 = \theta_T = 0$ such that the corresponding wealth process $V_t(\theta)$ satisfies $V_t(\theta) \ge -C$ for some deterministic $C > 0$ and for all $t \in [0, T]$.

In the next definition, we state the absence of arbitrage condition for the market.

Definition 1.2. We say that the market $(1, Y_t^1, Y_t^2, \ldots, Y_t^d)$ does not admit arbitrage with ϵ-sized transaction costs if there is no admissible trading strategy $\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)$ such that the corresponding value process $V_t(\theta)$ satisfies

$$P(V_T(\theta) > 0) > 0, \qquad P(V_T(\theta) \ge 0) = 1. \tag{1.2}$$

The absence of arbitrage condition excludes trading strategies that enables the investors to have nonnegative payoff with the possibility of positive payoff with zero initial investment. The purpose of this note is to study the sufficient conditions on $(Y_t^1, Y_t^2, \ldots, Y_t^d)$ that ensure absence of arbitrage in the market $(1, Y_t^1, Y_t^2, \ldots, Y_t^d)$. It is clear that if the stock price process $Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^d)$ is a martingale under a measure Q that is equivalent to the original measure P, then the model (1.1) does not admit arbitrage. This can easily be seen from the fundamental theorem of asset pricing (see [1]) that states that martingale price processes do not admit arbitrage in frictionless markets (i.e., $\epsilon = 0$). In the absence of such martingale measure for Y, the existence of a process $\tilde{Y}_t = (\tilde{Y}_t^1, \tilde{Y}_t^2, \ldots, \tilde{Y}_t^d)$ which is a martingale under an equivalent measure Q and which has the following property:

$$\left|Y_t^i - \tilde{Y}_t^i\right| \le \epsilon Y_t^i, \quad \text{for } i = 1, 2, \ldots, d, \quad \forall t \in [0, T], \tag{1.3}$$

also implies absence of arbitrage for the model (1.1). To see this simple fact, observe the following:

$$\begin{aligned} V_T(\theta) &= \sum_{i=1}^{d} \int_0^T \theta_s^i d\left(Y_s^i - \tilde{Y}_s^i\right) + \sum_{i=1}^{d} \int_0^T \theta_s^i d\tilde{Y}_s^i - \epsilon \sum_{i=1}^{d} \int_0^T Y_s^i d\,\mathrm{Var}\left(\theta^i\right)_s \\ &= \sum_{i=1}^{d} \left[\int_0^T \left(Y_s^i - \tilde{Y}_s^i\right) d\theta_s^i - \epsilon \int_0^T Y_s^i d\,\mathrm{Var}\left(\theta^i\right)_s\right] + \sum_{i=1}^{d} \int_0^T \theta_s^i d\tilde{Y}_s^i. \end{aligned} \tag{1.4}$$

Note that because of (1.3), we have $\sum_{i=1}^{d}[\int_0^T (Y_s^i - \widetilde{Y}_s^i)d\theta_s^i - \epsilon \int_0^T Y_s^i d\,\mathrm{Var}(\theta^i)_s] \leq 0$ a.s. This implies that

$$V_T(\theta) \leq \sum_{i=1}^{d} \int_0^T \theta_s^i d\widetilde{Y}_s^i. \tag{1.5}$$

The financial interpretation of (1.5) is that trading at price process $\widetilde{Y}_t$ without transaction costs is always at least as profitable as trading at price process Y_t with transaction costs. The martingale property of $\widetilde{Y}_t$ implies that trading on $\widetilde{Y}_t$ is arbitrage free, and therefore trading on Y with transaction costs is also arbitrage-free.

The process $\widetilde{Y}_t$ is called consistent price systems (CPSs) for the price process Y. The origin of CPSs is due to [2] and the name consistent price system first appeared in [3]. In the following, we write down the formal definition of CPSs.

Definition 1.3. Let $\epsilon > 0$. We say that $\widetilde{Y}_t = (\widetilde{Y}_t^1, \widetilde{Y}_t^2, \ldots, \widetilde{Y}_t^d)$ is an ϵ-consistent price system for $Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^d)$, if there exists a measure $Q \sim P$ such that $\widetilde{Y}_t$ is a martingale under Q, and

$$\frac{1}{1+\epsilon} \leq \frac{\widetilde{Y}_t^i}{Y_t^i} \leq 1+\epsilon, \quad \text{for } i = 1, 2, \ldots, d, \quad \forall t \in [0, T]. \tag{1.6}$$

The existence of such pricing functions is a central question in markets with proportional transaction costs and their existence was extensively studied in the past literature. For example, the papers [4, 5] studied CPSs for semimartingale models and the papers [6–11] studied CPSs for non-semi-martingale models. Other papers that studied similar problems include [4, 8, 12–17]. Particularly, the recent paper [10] introduced a general condition, conditional full support (CFS), for price processes and showed that if a continuous process $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ with state space $\mathbb{R}^d$ has the CFS property, then the exponential process $Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^d)$ admits ϵ-CPS for any $\epsilon > 0$. The proof of this result is based on a clever approximation of Y by a discrete process which is called random walk with retirement (see [10]). In this paper, we consider continuous processes X_t with general state space $\mathcal{O}$, where $\mathcal{O}$ is any connected open set in $\mathbb{R}^d$. Unlike the original paper [10], where the random walk with retirement is constructed by using geometric grids, in this paper we choose to work on arithmetic grid. As a consequence, we show that if the process X_t with the state space $\mathcal{O}$ has the corresponding CFS property, then for any given $\epsilon > 0$ there exists a martingale $M_t = (M_t^1, M_t^2, \ldots, M_t^d)$, under an equivalent change of measure, such that

$$\left| M_t^i - X_t^i \right| \leq \epsilon \quad \text{for any } i = 1, 2, \ldots, d \text{ and any } t \in [0, T]. \tag{1.7}$$

By an abuse of language we call such M a ϵ-consistent price system for the process X. To achieve this goal, we first provide a few of equivalent formulations of the CFS property. We use these equivalent formulations in the proof of our result. The advantage is that with our approach the proofs become more transparent and also it enables us to state some stronger results than the original paper. For example, our Lemma 2.10 is a stronger result than the corresponding result in [10] that states that the CFS property is equivalent to the so-called strong CFS property which is stated in terms of stopping times.

Our main result in this paper is Theorem 2.6 which states that the CFS property of X in any open connected domain $\mathcal{O}$ implies the existence of CPSs. To prove this result, we first prove Lemmas 2.7, 2.8, 2.9, 2.10, and 2.11. In Lemma 2.7, we show that the CFS property implies the necessary properties of a random walk with retirement (see [10] for the formal definition of random walk with retirement). In Lemma 2.8, we prove that our approximating discrete time process is a martingale under an equivalent martingale measure. The proof of this Lemma gives an alternative and elementary proof for the corresponding result in the paper [10]. In Lemma 2.9, we prove that the approximating discrete time process is in fact a uniformly integrable martingale. The proof of this lemma is standard and similar to the corresponding proofs of the papers [10, 11]. In Lemma 2.10, we show the equivalence of the f-stickiness with the weak f-stickiness for each given f. In Lemma 2.11, we show that the CFS property is equivalent to the seemingly weaker linear stickiness property.

2. Main Results

Let $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$, $t \in [0,T]$ be a d-dimensional continuous process that takes values in an open connected domain $\mathcal{O} \subset \mathbb{R}^d$. For simplicity of our discussion, we assume that $0 \in \mathcal{O}$. We also assume that the process X_t is defined on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the usual assumptions in this space. Let $C([u,v],\mathcal{O})$ denote the set of continuous functions f defined on the interval $[u,v]$ and with values in $\mathcal{O}$ and, for any $x \in \mathbb{R}^d$, let $C_x([u,v],\mathcal{O})$ denote the set of functions in $C([u,v],\mathcal{O})$ with $f(u) = x$.

Definition 2.1. An adapted continuous process X_t satisfies the CFS property in $\mathcal{O}$, if for any $t \in [0,T)$ and for almost all $\omega \in \Omega$,

$$\operatorname{Supp}\Big(\operatorname{Law}\Big[(X_s)_{s\in[t,T]} \mid \mathcal{F}_t(\omega)\Big]\Big) = C_{X_t(\omega)}([t,T],\mathcal{O}), \quad \text{a.s.} \tag{2.1}$$

The CFS condition requires that, at any given time, the conditional law of the future of the process, given the past, must have the largest possible support. An equivalent formulation of this property is given in the following definition.

Definition 2.2. Let X_t be an adapted continuous process that takes values in an open and connected domain $\mathcal{O} \subset \mathbb{R}^d$. We say that X_t is linear sticky if for any $\alpha \in \mathbb{R}^d$, $\epsilon > 0$, and any deterministic $0 \le s \le \theta \le T$,

$$P\left(\left\{\sup_{t\in[s,\theta]} |X_t - X_s - \alpha(t-s)| < \epsilon\right\} \mid \mathcal{F}_s\right) > 0, \quad \text{a.s.} \tag{2.2}$$

on the set $\{X_s \in \bigcap_{t\in[0,\theta-s]}(\mathcal{O} - \alpha t\}$.

The equivalence of the CFS and the linear stickiness properties will be established in Lemmas 2.10 and 2.11. We also need the following definition.

Definition 2.3. Let X_t be an adapted continuous process that takes values in an open and connected domain $\mathcal{O} \subset \mathbb{R}^d$.

(a) We say that X_t is f-sticky for $f \in C_0([0,T],\mathbb{R}^d)$ if

$$P\left(\left\{\sup_{t\in[\tau,T]}\left|X_t - X_\tau - f(t-\tau)\right| < \epsilon\right\} \mid \mathcal{F}_\tau\right) > 0, \quad \text{a.s.} \tag{2.3}$$

on the set $\{X_\tau \in \bigcap_{t\in[0,T-\tau]}(\mathcal{O} - f(t))\}$ for any $\epsilon > 0$ and any stopping time τ.

(b) We say that X_t is weak f-sticky for $f \in C_0([0,T],\mathbb{R}^d)$ if

$$P\left(\left\{\sup_{t\in[s,T]}\left|X_t - X_s - f(t-s)\right| < \epsilon\right\} \mid \mathcal{F}_s\right) > 0, \text{ a.s.} \tag{2.4}$$

on the set $\{X_s \in \bigcap_{t\in[0,T-s]}(\mathcal{O} - f(t))\}$ for any $\epsilon > 0$ and any deterministic time $s \in [0,T]$.

Remark 2.4. It is clear that the CFS property of X in $\mathcal{O}$ is equivalent to the weak f-stickiness of X for all $f \in C_0([0,T],\mathcal{O})$. The linear stickiness of X_t is seemingly weaker condition than the weak f-stickiness of X_t for all $f \in C_0([0,T],\mathcal{O})$. However, this is not the case and in Lemma 2.11 we will show that linear stickiness is equivalent to weak f-stickiness of X_t for all $f \in C_0([0,T],\mathbb{R}^d)$. This, in turn, implies that the linear stickiness property is equivalent to the CFS property.

Remark 2.5. When a process X_t is 0-sticky as in (b) in Definition 2.3, we say that X_t is jointly sticky and this property was studied in the recent paper [14]. The f-stickiness roughly means that starting from any stopping time τ on, the process X_t has paths that are as close as one wants to the path $f(t) + X_\tau$. As it was shown in [14], the f-stickiness holds for any $f \in C_0([0,T],\mathbb{R}^d)$ for the process $(B_t^{H_1}, B_t^{H_2}, \ldots, B_t^{H_d})$, where $B_t^{H_1}, B_t^{H_2}, \ldots, B_t^{H_d}$ are independent fractional Brownian motions with respective Hurst parameters $H_1, H_2, \ldots, H_d \in (0,1)$. From [10], the f-stickiness also holds for any continuous Markov process with the full support property in $C_0([0,T],\mathbb{R}^d)$ for any $f \in C_0([0,T],\mathbb{R}^d)$.

The following is the main result of this paper. This result is an extension of the main result in [10] to processes with more general state space. We use [10] as a road map in the proof of this result.

Theorem 2.6. *Let $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ be a continuous process that takes values in a connected domain $\mathcal{O}$ in $\mathbb{R}^d$. If X_t is linear sticky, then X_t admits CPSs for all $\epsilon > 0$.*

To show this result, one fix any $\epsilon > 0$ and define the following increasing sequence of stopping times associated with the process X:

$$\tau_0 = 0, \ \tau_{n+1} = \inf_{t\geq\tau_n}\{|X_t - X_{\tau_n}| \geq \epsilon_{n+1}\} \wedge T, \quad \forall n \geq 0, \tag{2.5}$$

with $\epsilon_{n+1} := \epsilon \wedge d(X_{\tau_n}, \partial\mathcal{O})/2$. One should mention that the paper [10] defined the corresponding stopping times in a slightly different way, see the proof of Theorem 1.2 in [10].

In addition, for each $n \geq 1$ we define

$$\Delta_n = \begin{cases} X_{\tau_n} - X_{\tau_{n-1}} & \text{when } \tau_n < T, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Let $\mathcal{G}_n = \mathcal{F}_{\tau_n}$ for every $n \geq 0$. Note that ϵ_n is bounded and $\mathcal{G}_{n-1}$ measurable.

In the following, we use the notation $\mathrm{Supp}(\Delta_n \mid \mathcal{G}_{n-1})$ to denote the smallest closed set of $\mathbb{R}^d$ that contains the values of the random variable $E[\Delta_n \mid \mathcal{G}_{n-1}]$ with probability one. We use $B_r(x)$ to denote the open ball in $\mathbb{R}^d$ with center x and radius r. When the center is 0, we simply write B_r. We first prove the following lemma.

Lemma 2.7. *If X_t is f-sticky in $\mathcal{O}$ for all $f \in C_0([0,T])$, then the process $\{\Delta_n\}$ in* (2.6) *satisfies the following three properties:*

$$\begin{aligned} &\text{(i)} \quad P(\Delta_m = 0, \forall m \geq n \mid \Delta_n = 0) = 1; \\ &\text{(ii)} \quad \mathrm{Supp}\,(\Delta_n \mid \mathcal{G}_{n-1}) = 0 \cup \partial B_{\epsilon_n} \textit{ almost surely on } \{\Delta_{n-1} \neq 0\}, \\ &\text{(iii)} \quad P(\Delta_m \neq 0, \forall\ m \geq 1) = 0. \end{aligned} \tag{2.7}$$

Proof. Property (i) is obvious since $\{\Delta_n = 0\} = \{\tau_n = T\}$ and τ_n is increasing. Property (iii) follows from the fact that almost surely each path of X_t is contained in a compact set of $\mathcal{O}$ and therefore $\min_n \epsilon_n(\omega) > 0$ almost surely $\omega \in \Omega$. To prove property (ii), let us assume that $P(\tau_{n-1} < T) > 0$ and let G_{n-1} be any $\mathcal{G}_{n-1}$ measurable set such that $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$. Then, it is clear that there exist $T' < T$, $y \in \mathcal{O}$ and $\zeta > 0$ such that $\zeta < \epsilon \wedge d(y, \partial\mathcal{O})/4$ and $P(G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in B_\zeta(y)\}) > 0$, where $d(y, \partial\mathcal{O})$ is the distance of y with the boundary $\partial\mathcal{O}$ of $\mathcal{O}$. It is also clear that on the set $G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in B_\zeta(y)\}$ we have $\epsilon \wedge 3\zeta/2 \leq \epsilon_n < 2\zeta$.

First we show that $P(\Delta_n = 0 \mid \mathcal{G}_{n-1}) > 0$ a.s. on $\{\Delta_{n-1} \neq 0\}$. To see this, define the following stopping time:

$$\tau = \begin{cases} \tau_{n-1} & \text{on } G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in B_\zeta(y)\}, \\ T & \text{otherwise.} \end{cases} \tag{2.8}$$

The 0-stickiness of X implies that

$$P\left(\sup_{t \in [\tau, T]} |X_t - X_\tau| < \zeta, \tau < T\right) > 0. \tag{2.9}$$

But $\{\sup_{t\in[\tau,T]} |X_t - X_\tau| < \zeta, \tau < T\} \subset \{\Delta_n = 0\}$ and since G_{n-1} was an arbitrary $\mathcal{G}_{n-1}$ measurable set with $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$, we have

$$P(\Delta_n = 0 \mid \mathcal{G}_{n-1}) > 0 \text{ a.s. on} \{\Delta_{n-1} \neq 0\}. \tag{2.10}$$

Next we show that $\partial B_{\epsilon_n(\omega)} \subset \mathrm{Supp}(\Delta_n \mid \mathcal{G}_{n-1})(\omega)$ almost surely on $\{\Delta_{n-1} \neq 0\}$. Too see this, take any $x \in \partial B_1$, $0 < \epsilon' < \zeta$ and define

$$f(t) = \begin{cases} \dfrac{6\zeta t}{(T-T')}x & \text{if } 0 \le t \le \dfrac{T-T'}{2}, \\ 3\zeta x & \text{otherwise,} \end{cases} \tag{2.11}$$

and note that

$$P\left(\tau < T, X_\tau \in \bigcap_{t\in[\tau,T]} (\mathcal{O} - f(t-\tau))\right) > 0. \tag{2.12}$$

By f-stickiness of X, we obtain

$$P\left(\sup_{t\in[\tau,T]} \left|X_t - X_\tau - f(t-\tau)\right| < \epsilon', \tau < T\right) > 0, \tag{2.13}$$

or equivalently $P(A \cap G_{n-1}) > 0$, where

$$\begin{aligned} A = &\left\{\sup_{t\in[\tau_{n-1},T]} \left|X_t - X_{\tau_{n-1}} - f(t-\tau_{n-1})\right| < \epsilon'\right\} \\ &\cap \left\{\tau_{n-1} < T'\right\} \\ &\cap \left\{X_{\tau_{n-1}} \in \bigcap_{t\in[\tau_{n-1},T]} (\mathcal{O} - f(t-\tau_{n-1}))\right\}. \end{aligned} \tag{2.14}$$

We claim that $A \subset \{\Delta_n \subset B_{2\epsilon'}(\epsilon_n x)\}$. Indeed, if $\omega \in A$, we get

$$\begin{aligned} &\left|X_{\tau_{n-1}+(T-T')/2}(\omega) - X_{\tau_{n-1}}(\omega)\right| \\ &\quad \ge \left|f\left(\frac{T-T'}{2}\right)\right| - \left|X_{\tau_{n-1}+(T-T')/2}(\omega) - X_{\tau_{n-1}}(\omega) - f\left(\frac{T-T'}{2}\right)\right| \\ &\quad \ge 3\zeta - \epsilon' > \epsilon_n(\omega). \end{aligned} \tag{2.15}$$

Hence $\{\tau_n < T\}$ on A. Also, for $\omega \in A$ we have

$$\begin{aligned} 0 = d(X_{\tau_n} - X_{\tau_{n-1}}, \partial B_{\epsilon_n}) &\ge d\big(f(\tau_n - \tau_{n-1}), \partial B_{\epsilon_n}\big) - \left|X_{\tau_n} - X_{\tau_{n-1}} - f(t-\tau_{n-1})\right| \\ &> \left|f(\tau_n - \tau_{n-1}) - \epsilon_n x\right| - \epsilon', \\ |X_{\tau_n} - X_{\tau_{n-1}} - \epsilon_n x| &\le \left|f(\tau_n - \tau_{n-1}) - \epsilon_n x\right| + \left|X_{\tau_n} - X_{\tau_{n-1}} - f(t-\tau_{n-1})\right| \\ &< \left|f(\tau_n - \tau_{n-1}) - \epsilon_n x\right| + \epsilon' < 2\epsilon'. \end{aligned} \tag{2.16}$$

So for all $\omega \in A$, $d(\Delta_n, \epsilon_n x) < 2\epsilon'$. Since this is true for any small ϵ', $x \in B_1$ and any arbitrary $\mathcal{G}_{n-1}$ measurable set with $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$, we conclude that $\partial B_{\epsilon_n(\omega)} \subset \mathrm{Supp}(\Delta_n \mid \mathcal{G}_{n-1})(\omega)$ almost surely on $\{\Delta_{n-1} \neq 0\}$. Note that the other direction $\partial B_{\epsilon_n(\omega)} \cup \{0\} \supset \mathrm{Supp}(\Delta_n \mid \mathcal{G}_{n-1})(\omega)$ is clear from the definition of Δ_n. □

Now, define ϵ_n, Δ_n, $n \geq 0$ as above and let $M_n = X_0 + \sum_{i=1}^{n} \Delta_i$, $n \geq 0$. The $\mathbb{R}^d$-valued process $M_n =: (M_n^1, M_n^2, \ldots, M_n^d)$ will be used to construct CPSs for X_t. Next, we prove a lemma that shows that all of M_n^i, $1 \leq i \leq d$ are in fact uniformly integrable martingales under an equivalent change of measure. The proof of this lemma uses Lemma 3.1 of [10] as a road map (see also Proposition 2.2.14 of [18]).

Lemma 2.8. *There exists a measure Q equivalent to P under which the $\mathbb{R}^d$-valued discrete process $\{(M_n, \mathcal{G}_n)\}_{n=0}^{+\infty}$ is a martingale.*

Proof. For any $n \geq 0$, let μ_n be the regular conditional probability of Δ_n with respect to $\mathcal{G}_{n-1}$ and let $\Omega_n = \{\omega \in \Omega \mid \mathrm{Supp}(\Delta_n \mid \mathcal{G}_{n-1})(\omega) = 0 \cup \partial B_{\epsilon_n}\}$. Let k be any strictly increasing convex function defined on $\mathbb{R}$ with values in $(0, +\infty)$ such that $k(t) = t$ for every $t \geq 1$. Define $G_n : \Omega_n \times \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$G_n(\omega, \alpha) = \int_{\mathbb{R}^d} k(\alpha \cdot x) x d\mu_n(\omega, \cdot). \tag{2.17}$$

Obviously for each n, $G_n(\cdot, a)$ is $\mathcal{G}_{n-1}$ measurable and convex with respect to α. As a consequence, for any fixed $\omega \in \Omega_n$, $\mathrm{Im}(G_n(\omega, \cdot))$, the image of the function $G_n(\omega, \cdot)$ is convex. We first prove that for every $n \geq 1$ and $\omega \in \Omega_n$:

$$\lim_{|\alpha| \to \infty} G_n(\omega, \alpha) \cdot \frac{\alpha}{|\alpha|} = +\infty. \tag{2.18}$$

By the way of contrary, assume that this is not true, for some $n \geq 1$ and $\omega \in \Omega_n$. Then, there exists a sequence $(\alpha_m)_{m \geq 1}$ with $|\alpha_m| \to \infty$ such that $G_n(\omega, \alpha_m) \cdot (\alpha_m / |\alpha_m|)$ is bounded above. We can assume that $(\alpha_m / |\alpha_m|)$ converges to some α (this is a bounded sequence and therefore has a convergent subsequence). We have

$$\begin{aligned} G_n(\omega, \alpha_m) \cdot \frac{\alpha_m}{|\alpha_m|} &\geq \int_{\alpha_m \cdot x \leq 0} k(\alpha_m \cdot x) \frac{\alpha_m \cdot x}{|\alpha_m|} d\mu_n(\omega, \cdot) \\ &\quad + \int_{(\alpha_m \cdot x)/|\alpha_m| > \epsilon_n(\omega)/4} k(\alpha_m \cdot x) \frac{\alpha_m \cdot x}{|\alpha_m|} d\mu_n(\omega, \cdot) \\ &\geq -k(0) + \int_{2\alpha \cdot x > \epsilon_n(\omega)} \frac{(\alpha_m \cdot x)^2}{|\alpha_m|} d\mu_n(\omega, \cdot), \end{aligned} \tag{2.19}$$

for big enough m. Therefore, we can conclude that $\int_{2\alpha \cdot x > \epsilon_n(\omega)} ((\alpha_m \cdot x)^2 / |\alpha_m|) \cdot (1/|a_m|) d\mu_n(\omega, \cdot)$ converges to 0 as $|a_m| \to +\infty$, which will imply after passing to the limit that $\mu_n(\omega, \{2\alpha \cdot x > \epsilon_n\})$ and this is a contradiction. From this it follows easily that $0 \in \overline{\mathrm{Im}(G_n(\omega, \cdot))}$. If $0 \notin \overline{\mathrm{Im}(G_n(\omega, \cdot))}$, then using the geometric form of Hahn Banach theorem, there exists

a unit vector $\beta \in \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} k(\alpha \cdot x)\beta x d\mu_n(\omega,\cdot) < 0$ for every $\alpha \in \mathbb{R}^d$. Therefore, $\limsup_{t\to\infty} \int_{\mathbb{R}^d} k(t\beta \cdot x)\beta x d\mu_n(\omega,\cdot) \leq 0$. But

$$\int_{\mathbb{R}^d} k(t\beta \cdot x)\beta x d\mu_n(\omega,\cdot) = G_n(\omega, t\beta) \cdot \frac{t\beta}{|t\beta|}, \tag{2.20}$$

and so it contradicts (2.18).

Next, we want to show that $\mathrm{Im}(G_n(\omega,\cdot))$ is closed. Let $a \in \overline{\mathrm{Im}(G_n(\omega,\cdot))}$, so there exists a sequence $(\alpha_m)_{m\geq 1}$ such that $G_n(\omega,\alpha_m) \to a$. But then $|\alpha_m|$ is unbounded, and therefore this contradicts (2.18). So based on the continuity of $G_n(\omega,\cdot)$, $a \in \mathrm{Im}(G_n(\omega,\cdot))$.

Therefore, we conclude that for any $n \geq 1$ and $\omega \in \Omega_n$, there exists an $\alpha_n(\omega) \in \mathbb{R}^d$, unique, as a consequence of the strict monotonicity of k, such that $G_n(\omega,\alpha_n(\omega)) = 0$. G_n being continuous with respect to α and $\mathcal{G}_{n-1}$ measurable with respect to ω, it follows that α_n is $\mathcal{G}_{n-1}$ measurable. We extend α_n with 1 outside Ω_n and define:

$$Z_n = \frac{k(\alpha_n \cdot \Delta_n)\, 1_{\{\Delta_n \neq 0\}}}{2E\left(k(\alpha_n \cdot \Delta_n)1_{\{\Delta_n \neq 0\}} \mid \mathcal{G}_{n-1}\right)} + \frac{1_{\{\Delta_n = 0\}}}{2P(\Delta_n = 0 \mid \mathcal{G}_{n-1})}. \tag{2.21}$$

It is easy to check that Z_n satisfies

$$\begin{aligned} E(Z_n \mid \mathcal{G}_{n-1}) &= 1, \\ E(Z_n \Delta_n \mid \mathcal{G}_{n-1}) &= 0. \end{aligned} \tag{2.22}$$

Let $L_n = \prod_{i=1}^n Z_i$ and $L = \lim_{n\to\infty} L_n$. Note that this limit exists almost surely since $L_{n+1} = L_n$ a.s. on $\{\Delta_n = 0\}$ and $\{\Delta_n = 0\} \nearrow \Omega$. From (2.22), we get

$$\begin{aligned} E(L_n \mid \mathcal{G}_{n-1}) &= L_{n-1} \\ E(L_n M_n \mid \mathcal{G}_{n-1}) &= L_{n-1} M_{n-1}, \end{aligned} \tag{2.23}$$

which shows that $(L_n)_{n\geq 1}$ and $(M_n L_n)_{n\geq 1}$ are martingales under P. We thus get $E(L_n) = E(Z_1) = 1$, and Fatou's lemma gives $E(L) \leq 1$. We will show that $E(L) = 1$. We have

$$\begin{aligned} E(L) &= E\left(\lim_{n\to\infty} L 1_{\{\Delta_n = 0\}}\right) = \lim_{n\to\infty} E\left(L 1_{\{\Delta_n = 0\}}\right) = \lim_{n\to\infty} E\left(L_n 1_{\{\Delta_n = 0\}}\right) \\ &= 1 - \lim_{n\to\infty} E\left(L_n 1_{\{\Delta_n \neq 0\}}\right) = 1 - \lim_{n\to\infty} E\left(E\left(L_n 1_{\{\Delta_n \neq 0\}} \mid \mathcal{G}_{n-1}\right)\right) \\ &= 1 - \frac{1}{2} \lim_{n\to\infty} E\left(L_{n-1} 1_{\{\Delta_{n-1} \neq 0\}}\right) = 1 - \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 1. \end{aligned} \tag{2.24}$$

Combining Fatou's lemma with the equation $E(L_n) = E(L) = 1$, we obtain $E(L \mid \mathcal{G}_n) = L_n$. Also,

$$\begin{aligned} E(M_n L \mid \mathcal{G}_{n-1}) &= E(E(M_n L \mid \mathcal{G}_n) \mid \mathcal{G}_{n-1}) = E(M_n L \mid \mathcal{G}_{n-1}) \\ &= M_{n-1} L_{n-1} = E(M_{n-1} L \mid \mathcal{G}_{n-1}). \end{aligned} \tag{2.25}$$

Hence, L is the density of a measure Q under which our discrete process M_n is a martingale. And since $L > 0$ ($L_n > 0$ for all n), Q is equivalent to P. □

Lemma 2.9. *Under the measure Q of Lemma 2.8 the process M_n^i is uniformly integrable for each $1 \le i \le d$. In particular, $E_Q(\sup_{n\ge 0}|M_n^i|) < \infty$ for each $i = 1, 2, \ldots, d$.*

Proof. For any $1 \le i \le d$, set $M_*^i = \sup_{n\ge 0}|M_n^i|$ and observe that on $\{\Delta_k \ne 0, \Delta_{k+1} = 0\}$ we have $M_*^i \le |X_0^i| + k\epsilon$. Observing that $Q(\Delta_k \ne 0) = Q(\Delta_k \ne 0 \mid \Delta_{k-1} \ne 0) \cdots Q(\Delta_1 \ne 0 \mid \Delta_0 \ne 0) Q(\Delta_0 \ne 0)$ and that $Q(\Delta_k \ne 0 \mid \Delta_{k-1} \ne 0) = 1/2$ we obtain the following:

$$E_Q\left(M_*^i\right) = \sum_{k=0}^{\infty} E_Q\left(M_*^i 1_{\{\Delta_k \ne 0\}\cap\{\Delta_{k+1}=0\}}\right) \le \sum_{k=0}^{\infty}\left(\left|X_0^i\right| + k\epsilon\right) Q(\{\Delta_k \ne 0, \Delta_{k+1} = 0\}) < \infty. \tag{2.26}$$

□

The two lemmas above uses the f-stickiness. The f-stickiness is seemingly stronger condition than the weak f-stickiness since it involves stopping times. However, the next Lemma 2.10 shows that, in fact, these two conditions are equivalent.

Lemma 2.10. *Let X_t be an adapted continuous process with state space $\mathcal{O}$ and $f \in C_0([0,T], \mathbb{R}^d)$. Then, X_t is weak f-sticky if and only if it is f-sticky.*

Proof. Let us show first that for any $f \in C_0([0,T], \mathbb{R}^d)$ weak f-stickiness implies f-stickiness. Suppose for a contradiction that X_t is weak f-sticky but not f-sticky. Then there exists a stopping time τ with $P(\tau < T) > 0$, and an $\epsilon > 0$ such that

$$\begin{gathered} P\left(\tau < T, X_\tau \in \bigcap_{t\in[0,T-\tau]} (\mathcal{O} - f(t))\right) > 0, \\ P\left(\sup_{t\in[\tau,T]} \left|X_t - X_\tau - f(t-\tau)\right| < \epsilon, \tau < T\right) = 0. \end{gathered} \tag{2.27}$$

Since $f \in C_0([0,T], \mathbb{R}^d)$, there exists a $\delta > 0$ such that for all $t, s \in [0,T]$, $|t - s| < \delta$ implies $|f(t) - f(s)| < \epsilon/3$. In addition, we can find $t_1, t_2 \in [0,T)$, $0 < t_2 - t_1 < \delta$, and $0 < \zeta \le \epsilon/3$ such that

$$P\left(t_1 \le \tau < t_2, X_\tau \in \bigcap_{t\in[\tau,T]} (\mathcal{O}_\zeta - f(t-\tau))\right) > 0, \tag{2.28}$$

where $\mathcal{O}_\zeta = \{x \in \mathcal{O} / d(x, \partial\mathcal{O}) > \zeta\}$.

For each $q \in I = \mathbb{Q} \cap [t_1, t_2)$, let $A_q := A \cap \{t_1 \le \tau < q\} \cap \{\sup_{t\in[\tau,q]}|X_t - X_\tau| < \zeta\}$, where

$$A := \left\{t_1 \le \tau < t_2, X_\tau \in \bigcap_{t\in[\tau,T]} (\mathcal{O}_\zeta - f(t-\tau))\right\}. \tag{2.29}$$

Since $P(A) > 0$ and $A = \bigcup_{q\in I} A_q$, there exists a $q^* \in I$ such that $P(A_{q^*}) > 0$. Note that $A_{q^*} \in \mathcal{F}_{q^*}$ and $A_{q^*} \subset \bigcap_{t\in[0,T-q^*]}(\mathcal{O} - f(t))$. Hence, since X_t is weak f-sticky, we obtain

$$P\left(A_{q^*} \cap \left\{\sup_{t\in[q^*,T]} \left|X_t - X_{q^*} - f(t-q^*)\right| < \frac{\epsilon}{3}\right\}\right) > 0. \tag{2.30}$$

Let $C_{q^*} = A_{q^*} \cap \{\sup_{t\in[q^*,T]}|X_t - X_{q^*} - f(t-q^*)| < \epsilon/3\}$. Then we claim that

$$C_{q^*} \subset \left\{\sup_{t\in[\tau,T]} \left|X_t - X_\tau - f(t-\tau)\right| < \epsilon\right\} \cap \{\tau < T\}, \tag{2.31}$$

which contradicts (2.27). Indeed, if $\omega \in C_{q^*}$, then for $t \in [\tau, q^*]$ we have

$$\left|X_t - X_\tau - f(t-\tau)\right| < |X_t - X_\tau| + \left|f(t-\tau)\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon, \tag{2.32}$$

by the definition of A_{q^*} and the choice of δ. We will show also that $|X_t - X_\tau - f(t-\tau)| < \epsilon$ on C_{q^*} whenever $t \in [q^*, T]$:

$$\begin{aligned}\left|X_t - X_\tau - f(t-\tau)\right| &\le \left|X_{q^*} - X_\tau - f(t-\tau) + f(t-q^*)\right| + \left|X_t - X_{q^*} - f(t-q^*)\right| \\ &\le \left|X_\tau - X_{q^*}\right| + \left|f(t-q^*) - f(t-\tau)\right| + \left|X_t - X_{q^*} - f(t-q^*)\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned} \tag{2.33}$$

Thus, weak f-stickiness implies f-stickiness. Since the opposite direction is obvious, the proposition is proved. □

Lemma 2.11. *Let X_t be a continuous adapted process with state space $\mathcal{O}$. Then X_t is linear sticky if and only if X_t is f-sticky for all $f \in C_0([0,T],\mathbb{R}^d)$.*

Proof. We only need to show that linear stickiness implies the weak f-stickiness for each $f \in C_0([0,T],\mathbb{R}^d)$. Fix any $f \in C_0([0,T],\mathbb{R}^d)$, $s \in [0,T]$, $\epsilon_0 > 0$. We need to show that

$$P\left(\left\{\sup_{t\in[s,T]} \left|X_t - X_s - f(t-s)\right| < \epsilon_0\right\} \mid \mathcal{F}_s\right) > 0, \quad \text{a.s.} \tag{2.34}$$

on the set $B =: \{X_s \in \bigcap_{t\in[0,T-s]}(\mathcal{O} - f(t))\}$. To do this, for any $A \in \mathcal{F}_s$ with $P(A \cap B) > 0$, we need to show that

$$P\left(A \cap B \cap \left\{\sup_{t\in[s,T]} \left|X_t - X_s - f(t-s)\right| < \epsilon_0\right\}\right) > 0. \tag{2.35}$$

Define $Z(\omega) = \inf_{r\in[0,T-s]} d(X_s(\omega) + f(r), \partial\mathcal{O})$ for any $\omega \in A \cap B$. From the definition of B, it is clear that $Z > 0$ a.s. on $A \cap B$. Let $h > 0$ be a constant such that the set $B_0 = \{Z \geq h\}$ has positive probability. Note that $B_0 \in \mathcal{F}_s$ and $B_0 \subset A \cap B$. In the following, we show that

$$P\left(B_0 \cap \left\{\sup_{t\in[s,T]} \left|X_t - X_s - f(t-s)\right| < \epsilon_0\right\}\right) > 0. \tag{2.36}$$

Let $\epsilon = \min(\epsilon_0, h)$ and set $t_0 = 0$, and define

$$t_k = \inf\left\{t \geq t_{k-1} : \left|f(t) - f(t_{k-1})\right| \geq \frac{\epsilon}{4}\right\} \wedge (T-s), \tag{2.37}$$

for $k \geq 1$. Let N be the smallest positive integer such that $t_N = T - s$. For each $k \geq 1$, define $g(t)$ on $[t_{k-1}, t_k]$ to be equal to the linear function that connects the two points $f(t_{k-1})$ and $f(t_k)$. We can assume that

$$g(t) = f(t_{k-1}) + \alpha_{k-1} t, \quad \text{on } [t_{k-1}, t_k], \tag{2.38}$$

for some constant vector $\alpha_{k-1} \in \mathbb{R}^d$ for each $k \geq 1$. It is clear that

$$\sup_{t\in[0,T-s]} \left|f(t) - g(t)\right| \leq \frac{\epsilon}{2}. \tag{2.39}$$

Because of (2.39), to show (2.36) we only need to show

$$P\left(B_0 \cap \left\{\sup_{t\in[s,T]} \left|X_t - X_s - g(t-s)\right| < \frac{\epsilon}{2}\right\}\right) > 0. \tag{2.40}$$

For each $k = 0, 1, 2, \ldots, N-1$, let

$$B_{k+1} = B_k \cap \left\{\sup_{t\in[s+t_k, s+t_{k+1}]} \left|X_t - X_s - g(t-s)\right| < \frac{\epsilon}{2^{N-k}}\right\}. \tag{2.41}$$

Note that B_N is contained in the event in (2.40). Therefore, it is sufficient to prove that B_N has positive probability. When $t \in [s + t_k, s + t_{k+1}]$, we have $g(t-s) = f(t_k) + \alpha_k(t-s) = f(t_k) + \alpha_k t_k + \alpha_k[t - (s + t_k)] = g([(s + t_k) - s]) + \alpha_k[t - (s + t_k)]$. Therefore, we have the following relation:

$$\begin{aligned}
&\left\{\sup_{t\in[s+t_k, s+t_{k+1}]} \left|X_t - X_s - g(t-s)\right| < \frac{\epsilon}{2^{N-k}}\right\} \\
&\quad \supset \left\{X_{s+t_k} - X_s - g([(s+t_k) - s]) < \frac{\epsilon}{2^{N-k+1}}\right\} \\
&\quad \cap \left\{\sup_{t\in[s+t_k, s+t_{k+1}]} \left|X_t - X_{s+t_k} - \alpha_k[t - (s+t_k)]\right| < \frac{\epsilon}{2^{N-k+1}}\right\}.
\end{aligned} \tag{2.42}$$

By the definition of B_k and the above relation, it is easy see that

$$B_{k+1} \supset B_k \cap \left\{ \sup_{t\in[s+t_k,s+t_{k+1}]} |X_t - X_{s+t_k} - \alpha_k[t-(s+t_k)]| < \frac{\epsilon}{2^{N-k+1}} \right\}. \tag{2.43}$$

On B_k we have

$$\begin{gathered} d(X_{s+t_k},\partial\mathcal{O}) \geq d\big(X_s + g((s+t_k)-s),\partial\mathcal{O}\big) - d\big(X_s + g((s+t_k)-s), X_{s+t_k}\big) > \frac{\epsilon}{2} - \frac{\epsilon}{2^{N-k+1}} \geq \frac{\epsilon}{4}, \\ |X_{s+t_k} - [X_{s+t_k} + \alpha_k(t-(s+t_k))]| = |\alpha_k(t-(s+t_k))| = \big|g(t)-g(t_k)\big| \leq \frac{\epsilon}{4}, \end{gathered} \tag{2.44}$$

for each $k = 0,1,\ldots,N-1$ and for all $t \in [s+t_k, s+t_{k+1}]$. From this, we conclude that $B_k \subset \{X_{s+t_k} \in \bigcap_{t\in[s+t_k,s+t_{k+1}]}(\mathcal{O} - \alpha_k[t-(s+t_k)]\}$. Now, from the linear stickiness and the fact that $P(B_0) > 0$ we conclude $P(B_N) > 0$. This completes the proof. □

Proof of Theorem 2.6. By Lemmas 2.8 and 2.9, there exists an equivalent probability measure $Q \sim P$ such that $(M_n^i, \mathcal{G}_n)_{n\geq 0}$ is a uniformly integrable martingale for each $1 \leq i \leq d$. Let $M_\infty^i = \lim_{n\to\infty} M_n^i$. For each $t \in [0,T]$, set $\widetilde{M}_t^i = E_Q[M_\infty^i \mid \mathcal{F}_t]$. Observe that $\widetilde{M}_{\tau_n}^i = E_Q[M_\infty^i \mid \mathcal{F}_{\tau_n}] = M_n^i$, and $\widetilde{M}_t^i = E_Q[\widetilde{M}_{\tau_n}^i \mid \mathcal{F}_t]$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$ for all $n \geq 0$. Thus the following equation holds:

$$\left(\widetilde{M}_t^i - X_t^i\right)1_{\{\tau_{n-1}\leq t\leq\tau_n\}} = E_Q\left[\left(M_n^i - X_t^i\right)1_{\{\tau_{n-1}\leq t\leq\tau_n\}} \mid \mathcal{F}_t\right], \quad n \geq 1. \tag{2.45}$$

We write $M_n^i - X_t^i = (M_n^i - X_{\tau_n}^i) + (X_{\tau_{n-1}}^i - X_t^i) + (X_{\tau_n}^i - X_{\tau_{n-1}}^i)$. Note that each of $M_n^i - X_{\tau_n}^i$, $X_{\tau_{n-1}}^i - X_t^i$, and $X_{\tau_n}^i - X_{\tau_{n-1}}^i$ takes values in $(-\epsilon,\epsilon)$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$. Therefore, we have $-3\epsilon \leq |\widetilde{M}_t^i - X_t^i| \leq 3\epsilon$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$. Since $\bigcup_{n=1}^\infty \{\tau_{n-1} \leq t \leq \tau_n\} = \Omega$, we conclude that

$$-3\epsilon \leq \left|\widetilde{M}_t^i - X_t^i\right| \leq 3\epsilon. \tag{2.46}$$

Since $\epsilon > 0$ is arbitrary, the claim follows. □

Example 2.12. Let $B_t^{H_1}, B_t^{H_2}, \ldots, B_t^{H_d}$ be a sequence of independent fractional Brownian motions with respective Hurst parameters $H_1, H_2, \ldots, H_d \in (0,1)$. Let $f_i : \mathbb{R} \to (a_i, b_i)$ be a homeomorphism for each $i = 1,2,\ldots,d$, where (a_i, b_i) is an open interval in the real line. Then the new process $(f_1(B_t^{H_1}), f_2(B_t^{H_2}), \ldots, f_d(B_t^{H_d}))$ admits CPSs for each $\epsilon > 0$. This can be easily seen from the CFS property of the process $(B_t^{H_1}, B_t^{H_2}, \ldots, B_t^{H_d})$ which was shown in [14] and the fact that the map $f(x) = (f_1(x), f_2(x), \ldots, f_d(x))$ is a homomorphism from $\mathbb{R}^d$ to $(a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)$.

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