Research Article

On Certain Subclasses of Meromorphic Functions with Positive and Fixed Second Coefficients Involving the Liu-Srivastava Linear Operator

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We introduce and study a subclass $\Sigma_P(\gamma, k, \lambda, c)$ of meromorphic univalent functions defined by certain linear operator involving the generalized hypergeometric function. We obtain coefficient estimates, extreme points, growth and distortion inequalities, radii of meromorphic starlikeness, and convexity for the class $\Sigma_P(\gamma, k, \lambda, c)$ by fixing the second coefficient. Further, it is shown that the class $\Sigma_P(\gamma, k, \lambda)$ is closed under convex linear combination.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the punctured unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \quad (1.2)$$

Let Σ_S , $\Sigma^*(\gamma)$ and $\Sigma_K(\gamma)$, $(0 \leq \gamma < 1)$ denote the subclasses of Σ that are meromorphically univalent, meromorphically starlike functions of order γ , and meromorphically convex

functions of order γ , respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if, f is of the form (1.1) and satisfies

$$-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \quad z \in \mathbb{U}, \quad (1.3)$$

similarly, $f \in \Sigma_K(\gamma)$, if and only if, f is of the form (1.1) and satisfies

$$-\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \gamma, \quad z \in \mathbb{U}, \quad (1.4)$$

and similar other classes of meromorphically univalent functions have been extensively studied by Aouf and Darwish [1], Aouf and Joshi [2], Mogra et al. [3], Uralegaddi and Ganigi [4], Uralegaddi and Somanatha[5], and Owa and Pascu [6].

Let $f, g \in \Sigma$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n. \quad (1.5)$$

Then the Hadamard product (or convolution) $f * g$ of the functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \quad (1.6)$$

Let Σ_p be the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (1.7)$$

that are analytic and univalent in $\mathbb{U}^*$.

For complex parameters $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m (\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m)$ the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!}, \quad (1.8)$$

$(l \leq m+1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U),$

where $\mathbb{N}$ denotes the set of all positive integers and $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1, & n=0; \theta \in \mathbb{C} \setminus \{0\}, \\ \theta(\theta+1)(\theta+2) \cdots (\theta+n-1), & n \in \mathbb{N}; \theta \in \mathbb{C}. \end{cases} \quad (1.9)$$

Corresponding to a function ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ defined by

$$\mathcal{F}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z^{-1} {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z). \quad (1.10)$$

Liu and Srivastava [7] considered a linear operator $\mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \Sigma \rightarrow \Sigma$ defined by the following Hadamard product (or convolution):

$$\begin{aligned} \mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z) &= \mathcal{F}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &= z^{-1} + \sum_{n=1}^{\infty} \left| \frac{(\alpha_1)_{n+1} \dots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \dots (\beta_m)_{n+1}} \right| \frac{a_n z^n}{(n+1)!}, \end{aligned} \quad (1.11)$$

where, $\alpha_i > 0$, ($i = 1, 2, \dots, l$), $\beta_j > 0$, ($j = 1, 2, \dots, m$), $l \leq m+1$; $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For notational simplicity, we use a shorter notations $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$ and

$$\Gamma_n(\alpha_1) = \frac{(\alpha_1)_{n+1} \dots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \dots (\beta_m)_{n+1}} \frac{1}{(n+1)!}, \quad (1.12)$$

unless otherwise stated in the sequel. We note that the linear operator $H_m^l[\alpha_1]$ was earlier defined for multivalent functions by Dziok and Srivastava [8] and was investigated by Liu and Srivastava [7].

Making use of the operator $\mathcal{H}_m^l[\alpha_1]$, now we consider a subclass of functions in Σ_P as follows.

Definition 1.1. For $0 \leq \gamma < 1$ and $k \geq 0$ and $0 \leq \lambda < 1/2$, let $\Sigma(\gamma, k, \lambda)$ denote a subclass of Σ consisting functions of the form (1.1) satisfying the condition that

$$\begin{aligned} & -\Re \left\{ \frac{z(\mathcal{H}[\alpha_1]f(z))' + \lambda z^2(\mathcal{H}[\alpha_1]f(z))''}{(1-\lambda)\mathcal{H}[\alpha_1]f(z) + \lambda z(\mathcal{H}[\alpha_1]f(z))'} + \gamma \right\} \\ & > k \left| \frac{z(\mathcal{H}[\alpha_1]f(z))' + \lambda z^2(\mathcal{H}[\alpha_1]f(z))''}{(1-\lambda)\mathcal{H}[\alpha_1]f(z) + \lambda z(\mathcal{H}[\alpha_1]f(z))'} + 1 \right|, \quad z \in \mathbb{U}^*, \end{aligned} \quad (1.13)$$

where $\mathcal{H}[\alpha_1]f(z)$ is given by (1.11). Furthermore, we say that a function $f \in \Sigma_P(\gamma, k, \lambda)$, whenever $f(z)$ is of the form (1.7).

We observe that, by specializing the parameters $l, m, \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m, k, \gamma$ and λ the class leads to various subclasses. As for illustrations, we present some examples for the cases.

Example 1.2. If $l = 2$ and $m = 1$ with $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$ and $f(z)$ of the form (1.7), then we obtain the new subclass $\mathcal{M}_P(\gamma, k, \lambda)$ defined by

$$-\Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} + \gamma \right\} > k \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} + 1 \right|, \quad z \in \mathbb{U}^*. \quad (1.14)$$

Example 1.3. For $l = 2$, $m = 1$, $\alpha_1 = \delta + 1$, $\beta_1 = \alpha_2 = 1$, and $f(z)$ of the form (1.7), then we get the new subclass $\mathfrak{D}_P^\delta(\gamma, k, \lambda)$ defined by

$$-\Re \left\{ \frac{z(\mathfrak{D}^\delta f(z))' + \lambda z^2(\mathfrak{D}^\delta f(z))''}{(1-\lambda)\mathfrak{D}^\delta f(z) + \lambda z(\mathfrak{D}^\delta f(z))'} + \gamma \right\} > k \left| \frac{z(\mathfrak{D}^\delta f(z))' + \lambda z^2(\mathfrak{D}^\delta f(z))''}{(1-\lambda)\mathfrak{D}^\delta f(z) + \lambda z(\mathfrak{D}^\delta f(z))'} + 1 \right|, \quad z \in \mathbb{U}^*, \quad (1.15)$$

where $\mathfrak{D}^\delta f(z) = 1/(z(1-z)^{\delta+1}) * f(z)$ ($\delta > -1$), is the differential operator which was introduced by Ganigi and Uralegaddi [9]. Also we note that the class $\mathfrak{D}_P^\delta(\gamma, k, \lambda)$ was introduced by Atshan and Kulkarni [10].

Example 1.4. For $l = 2$, $m = 1$, $\alpha_2 = 1$, and $f(z)$ of the form (1.7), then we obtain the new subclass $\mathcal{L}_P(\gamma, k, \lambda)$ defined by

$$\begin{aligned} & -\Re \left\{ \frac{z(\mathcal{L}[\alpha_1, \beta_1]f(z))' + \lambda z^2(\mathcal{L}[\alpha_1, \beta_1]f(z))''}{(1-\lambda)\mathcal{L}[\alpha_1, \beta_1]f(z) + \lambda z(\mathcal{L}[\alpha_1, \beta_1]f(z))'} + \gamma \right\} \\ & > k \left| \frac{z(\mathcal{L}[\alpha_1, \beta_1]f(z))' + \lambda z^2(\mathcal{L}[\alpha_1, \beta_1]f(z))''}{(1-\lambda)\mathcal{L}[\alpha_1, \beta_1]f(z) + \lambda z(\mathcal{L}[\alpha_1, \beta_1]f(z))'} + 1 \right|, \quad z \in \mathbb{U}^*, \end{aligned} \quad (1.16)$$

where the operator $\mathcal{L}[\alpha_1; \beta_1]f(z)$ was introduced and studied by Liu and Srivastava [11] (see also [12, 13]).

For the class $\Sigma_P(\gamma, k, \lambda)$, the following characterization was given by Magesh et al. [14].

Theorem 1.5. Let $f \in \Sigma_P$ be given by (1.7). Then $f \in \Sigma_P(\gamma, k, \lambda)$, if and only if,

$$\sum_{n=1}^{\infty} (1 + (n-1)\lambda) [n(k+1) + (k+\gamma)] \Gamma_n(\alpha_1) a_n \leq (1-\gamma)(1-2\lambda), \quad (1.17)$$

where $\Gamma_n(\alpha_1)$ is given by (1.12).

For a function defined by (1.7) and in the class $\Sigma_P(\gamma, k, \lambda)$, Theorem 1.5, immediately yields

$$\Gamma_1(\alpha_1) a_1 \leq \frac{(1-\gamma)(1-2\lambda)}{(2k+\gamma+1)}. \quad (1.18)$$

Hence we may take

$$\Gamma_1(\alpha_1) a_1 = \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}, \quad 0 < c < 1. \quad (1.19)$$

Motivated by Aouf and Darwish [1], Aouf and Joshi [2], Ghanim and Darus [12], and Uralegaddi [15], we now introduce the following class of functions and use the similar techniques to prove our results.

Let $\Sigma_P(\gamma, k, \lambda, c)$ be the subclass of $\Sigma_P(\gamma, k, \lambda)$ consisting functions in of the form

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \sum_{n=2}^{\infty} \Gamma_n(\alpha_1) a_n z^n, \quad 0 < c < 1. \quad (1.20)$$

In this paper, coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and convexity are discussed for the class $\Sigma_P(\gamma, k, \lambda)$ by fixing the second coefficient. Further, it is shown that the class $\Sigma_P(\gamma, k, \lambda)$ is closed under convex linear combination.

2. Main Results

In our first theorem, we now find out the coefficient inequality for the class $\Sigma_P(\gamma, k, \lambda, c)$.

Theorem 2.1. *Let the function $f(z)$ defined by (1.20). Then $f \in \Sigma_P(\gamma, k, \lambda, c)$, if and only if,*

$$\sum_{n=2}^{\infty} (1 + (n-1)\lambda) [n(k+1) + (k+\gamma)] \Gamma_n(\alpha_1) a_n \leq (1-\gamma)(1-2\lambda)(1-c). \quad (2.1)$$

The result is sharp.

Proof. By putting

$$\Gamma_1(\alpha_1) a_1 = \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}, \quad 0 < c < 1, \quad (2.2)$$

in (1.17), the result is easily derived. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1) + (k+\gamma)] \Gamma_n(\alpha_1)} z^n, \quad n \geq 2. \quad (2.3)$$

□

Corollary 2.2. *If the function f defined by (1.20) is in the class $\Sigma_P(\gamma, k, \lambda, c)$, then*

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1) + (k+\gamma)] \Gamma_n(\alpha_1)}, \quad n \geq 2. \quad (2.4)$$

The result is sharp for the function $f(z)$ given by (2.3).

Next we prove the following growth and distortion properties for the class $\Sigma_P(\gamma, k, \lambda, c)$.

Theorem 2.3. *If the function $f(z)$ defined by (1.20) is in the class $\Sigma_P(\gamma, k, \lambda, c)$ for $0 < |z| = r < 1$, then one has*

$$\begin{aligned} \frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r - \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r^2 &\leq |f(z)| \\ &\leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r^2. \end{aligned} \quad (2.5)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}z^2. \quad (2.6)$$

Proof. Since $\Sigma_P(\gamma, k, \lambda, c)$, Theorem 2.1 yields

$$\Gamma_n(\alpha_1)a_n \leq \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]}, \quad n \geq 2. \quad (2.7)$$

Thus, for $0 < |z| = r < 1$,

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}|z| + \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_n|z|^n \\ &\leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r + r^2 \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_n \\ &\leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r^2, \\ |f(z)| &\geq \frac{1}{|z|} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}|z| - \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_n|z|^n \\ &\geq \frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r - r^2 \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_n \\ &\geq \frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r - \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r^2. \end{aligned} \quad (2.8)$$

Thus the proof of the theorem is complete. $\square$

Theorem 2.4. If the function $f(z)$ defined by (1.20) is in the class $\Sigma_P(\gamma, k, \lambda, c)$ for $0 < |z| = r < 1$, then one has

$$\begin{aligned} \frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} - \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r \\ \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}r. \end{aligned} \quad (2.9)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)}z^2. \quad (2.10)$$

Proof. In view of Theorem 2.1, it follows that

$$n\Gamma_n(\alpha_1)a_n \leq \frac{n(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]}, \quad n \geq 2. \quad (2.11)$$

Thus, for $0 < |z| = r < 1$ and making use of (2.11), we obtain

$$\begin{aligned} |f'(z)| &\leq \left| \frac{-1}{z^2} \right| + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} + \sum_{n=2}^{\infty} n\Gamma_n(\alpha_1)a_n |z|^{n-1} \\ &\leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} + r \sum_{n=2}^{\infty} n\Gamma_n(\alpha_1)a_n \\ &\leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} + \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)} r, \\ |f'(z)| &\geq \left| \frac{-1}{z^2} \right| - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} - \sum_{n=2}^{\infty} n\Gamma_n(\alpha_1)a_n |z|^{n-1} \\ &\geq \frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} - r \sum_{n=2}^{\infty} n\Gamma_n(\alpha_1)a_n \\ &\geq \frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} - \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+\lambda)(3k+\gamma+2)} r. \end{aligned} \quad (2.12)$$

□

Hence the result follows.

Next, we will show that the class $\Sigma_P(\gamma, k, \lambda, c)$ is closed under convex linear combination.

Theorem 2.5. *If*

$$f_1(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} z, \quad (2.13)$$

and for $n \geq 2$

$$f_n(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)} z + \sum_{n=2}^{\infty} \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)} z^n. \quad (2.14)$$

Then $f \in \Sigma_P(\gamma, k, \lambda, c)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (2.15)$$

where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n \leq 1$.

Proof. From (2.13), (2.14), and (2.15), we have

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \sum_{n=2}^{\infty} \frac{(1-\gamma)(1-2\lambda)(1-c)\mu_n}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)}z^n. \quad (2.16)$$

Since

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(1-\gamma)(1-2\lambda)(1-c)\mu_n}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)} \frac{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)}{(1-\gamma)(1-2\lambda)(1-c)} \\ &= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1, \end{aligned} \quad (2.17)$$

it follows from Theorem 1.5 that the function $f \in \Sigma_P(\gamma, k, \lambda, c)$. Conversely, suppose that $f \in \Sigma_P(\gamma, k, \lambda, c)$. Since

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)}, \quad n \geq 2. \quad (2.18)$$

Setting

$$\begin{aligned} \mu_n &= \frac{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)}{(1-\gamma)(1-2\lambda)(1-c)} a_n, \\ \mu_1 &= 1 - \sum_{n=2}^{\infty} \mu_n. \end{aligned} \quad (2.19)$$

it follows that

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \quad (2.20)$$

This completes the proof of the theorem. $\square$

Theorem 2.6. *The class $\Sigma_P(\gamma, k, \lambda, c)$ is closed under linear combination.*

Proof. Suppose that the function f be given by (1.20), and let the function g be given by

$$g(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \sum_{n=2}^{\infty} |b_n|z^n, \quad n \geq 2. \quad (2.21)$$

Assuming that f and g are in the class $\Sigma_P(\gamma, k, \lambda, c)$, it is enough to prove that the function h defined by

$$h(z) = \mu f(z) + (1-\mu)g(z), \quad 0 \leq \mu \leq 1, \quad (2.22)$$

is also in the class $\Sigma_P(\gamma, k, \lambda, c)$. Since

$$h(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}z + \sum_{n=2}^{\infty} |a_n\mu + (1-\mu)b_n|z^n, \quad (2.23)$$

we observe that

$$\sum_{n=2}^{\infty} (1 + (n-1)\lambda) [n(k+1) + (k+\gamma)] \Gamma_n(\alpha_1) |a_n\mu + (1-\mu)b_n| \leq (1-\gamma)(1-2\lambda)(1-c), \quad (2.24)$$

with the aid of Theorem 2.1. Thus, $h \in \Sigma_P(\gamma, k, \lambda, c)$. $\square$

Next we determine the radii of meromorphically starlikeness and convexity of order δ for functions in the class $\Sigma_P(\gamma, k, \lambda, c)$.

Theorem 2.7. *Let the function $f(z)$ defined by (1.20) be in the class $\Sigma_P(\gamma, k, \lambda, c)$, then*

- (i) *f is meromorphically starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_1(\gamma, k, \lambda, c, \delta)$ where $r_1(\gamma, k, \lambda, c, \delta)$, is the largest value for which*

$$\begin{aligned} & \frac{(3-\delta)(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \frac{(n+2-\delta)(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]}r^{n+1} \\ & \leq (1-\delta), \quad n \geq 2, \end{aligned} \quad (2.25)$$

- (ii) *f is meromorphically convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2(\gamma, k, \lambda, c, \delta)$ where $r_2(\gamma, k, \lambda, c, \delta)$, is the largest value for which*

$$\frac{(3-\delta)(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \frac{n(n+2-\delta)(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]}r^{n+1} \leq (1-\delta), \quad n \geq 2. \quad (2.26)$$

Each of these results is sharp for function $f(z)$ given by (2.3).

Proof. It is enough to highlight that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \delta, \quad |z| < r_1. \quad (2.27)$$

Thus, we have

$$\begin{aligned} & \left| \frac{zf'(z)}{f(z)} + 1 \right| \\ &= \left| \frac{-1/z + ((V)(E)c/(Q))z + \sum_{n=2}^{\infty} n\Gamma_n(\alpha_1)a_nz^n + 1/z + ((V)(E)c/(Q))z + \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_nz^n}{1/z + ((V)(E)c/(Q))z + \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_nz^n} \right|, \end{aligned} \quad (2.28)$$

where (Q) denotes $(2k + \gamma + 1)$, V denotes $(1 - \gamma)$, and (E) denotes $(1 - 2\lambda)$.
Hence (2.28) holds true if

$$\begin{aligned} & \frac{2(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \sum_{n=2}^{\infty} (n+1)\Gamma_n(\alpha_1)a_nr^{n+1} \\ & \leq (1-\delta) \left[1 - \frac{(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 - \sum_{n=2}^{\infty} \Gamma_n(\alpha_1)a_nr^{n+1} \right], \end{aligned} \quad (2.29)$$

or,

$$\frac{(3-\delta)(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \sum_{n=2}^{\infty} (n+2-\delta)\Gamma_n(\alpha_1)a_nr^{n+1} \leq (1-\delta), \quad (2.30)$$

and it follows that, from (2.1), we may take

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)(1-c)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]\Gamma_n(\alpha_1)}\mu_n, \quad n \geq 2, \quad (2.31)$$

where $\mu_n \geq 0$ and $\sum_{n=2}^{\infty} \mu_n \leq 1$.

For each fixed r , we choose the positive integer $n_0 = n_0(r)$ for which

$$\frac{(n+2-\delta)}{(1+(n-1)\lambda)[n(k+1)+(k+\gamma)]}\Gamma_n(\alpha_1)r^{n+1} \quad (2.32)$$

is maximal. Then it follows that

$$\sum_{n=2}^{\infty} (n+2-\delta)\Gamma_n(\alpha_1)a_nr^{n+1} \leq \frac{(n_0+2-\delta)(1-\gamma)(1-2\lambda)(1-c)}{(1+(n_0-1)\lambda)[n_0(k+1)+(k+\gamma)]}r^{n_0+1}. \quad (2.33)$$

Then f is starlike of order δ in $0 < |z| < r_1(\gamma, k, \lambda, c, \delta)$ provided that

$$\frac{(3-\delta)(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \frac{(n_0+2-\delta)(1-\gamma)(1-2\lambda)(1-c)}{(1+(n_0-1)\lambda)[n_0(k+1)+(k+\gamma)]}r^{n_0+1} \leq (1-\delta). \quad (2.34)$$

We find the value $r_0 = r_0(k, c, \delta, n)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\delta)(1-\gamma)(1-2\lambda)c}{(2k+\gamma+1)}r^2 + \frac{(n_0+2-\delta)(1-\gamma)(1-2\lambda)(1-c)}{(1+(n_0-1)\lambda)[n_0(k+1)+(k+\gamma)]}r^{n_0+1} = (1-\delta). \quad (2.35)$$

It is the value for which the function $f(z)$ is starlike in $0 < |z| < r_0$. (ii) In a similar manner, we can prove our result providing the radius of meromorphic convexity of order δ ($0 \leq \delta < 1$) for functions in the class $\Sigma_P(\gamma, k, \lambda, c)$, so we skip the details of the proof of (ii). $\square$

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