The equilibrium conditions are

$$\overset{\bullet}{E}/E = r - \rho, \tag{1}$$

$$v_{jk} = \frac{\pi_{jk}}{r + \iota_{oj} + \iota_{rj} - \left(\overset{\bullet}{v}_{jk}/v_{jk}\right)}, \ jk \in J_n, \tag{2}$$

$$\pi_s = p_s - w_s a_s \left(1 - s_s\right), \tag{3}$$

$$\pi_j = \frac{E}{\lambda w_s} \left(\lambda w_s - w_s\right), \ j \in J_n, \tag{4}$$

$$v_i \leq w_n a_o \left(1 - s_n\right) + p_s, \text{ with equality for } \iota_i > 0, \ i = or, oo, \tag{5}$$

$$v_s \leq \frac{p_s - w_s a_s \left(1 - s_s\right)}{r}, \tag{6}$$

$$\sum_{i \in J_n} n_i = 1, \tag{7}$$

$$n_{jk} \left(\iota_{rj} + \iota_{oj}\right) = \iota_{jk} \left(n_{kr} + n_{ko}\right), \text{ for } j = r, o \text{ and } k = r, o, \tag{8}$$

$$L_s = a_s \sum_{k=r,o,s} \iota_{ok} n_{ok} + \frac{E}{\lambda w_s}, \tag{9}$$

$$L_n = a_n \sum_{k=r,o,s} \iota_{rk} n_{rk} + a_o \sum_{k=r,o,s} \iota_{ok} n_{ok}, \tag{10}$$

$$x_j \equiv x_{jr} = x_{jo} = x_{js} \text{ for } \ x = \iota, v, \pi \ \text{ and } j = r, o, \ \text{ and} \tag{11}$$

$$v_s = w_s a_s \left(1 - s_s\right). \tag{12}$$

First of all, plugging in the 6 conditions:

First of all, plugging in the following 6 conditions:

1. $\overset{\bullet}{v}_i/v_i = \overset{\bullet}{E}/E = r - \rho = 0$, $i \in J_n$,

2. $x_j \equiv x_{jr} = x_{jo} = x_{js}$ for $x = \iota, \upsilon, \pi$ and $j = r, o,$

3. $\pi_i = \frac{E}{\lambda w_s}(\lambda w_s - w_s) = E\left(1 - \frac{1}{\lambda}\right), \, i \in J_n,$

4. $n_o \equiv n_{or} + n_{oo} + n_{os},$

5. $n_r \equiv n_{rr} + n_{ro} + n_{rs},$

6. $p_s = (1+\rho) w_s a_s (1 - s_s),$

into equations (1) to (12), we get the following equilibrium conditions:

$$w_n a_n = \frac{Ee}{\iota_o + \iota_r + \rho}, \tag{13}$$

$$w_n a_o (1 - s_n) + (1+\rho) w_s a_s (1 - s_s) = \frac{Ee}{\iota_o + \iota_r + \rho}, \tag{14}$$

$$n_{jk}(\iota_r + \iota_o) = \iota_k n_k, \; j = r, \text{ and } \; k = r, o, \tag{15}$$

$$L_s = a_s \iota_o n_o + \frac{E}{\lambda w_s} \text{ and} \tag{16}$$

$$L_n = a_n \iota_r n_r + a_o \iota_o n_o. \tag{17}$$

Equations (13) to (14) give

$$\frac{Ee}{w_n} = a_n(\iota_o + \iota_r + \rho),$$

$$\frac{w_s}{w_n} = \frac{a_n - a_o(1 - s_n)}{(1+\rho) a_s (1 - s_s)} \text{ and}$$

$$\frac{Ee}{w_s} = \frac{(1+\rho) a_s (1 - s_s) a_n}{a_n - a_o(1 - s_n)} (\iota_o + \iota_r + \rho).$$

Equation (15) give

$$n_{jk} = (\iota_r + \iota_o)^{-1} \iota_j n_k, \; j = r, \text{ and } \; k = r, o,$$

which and $\sum_{i \in J_n} n_i = 1$, we get

$$\iota_o n_r = \iota_r n_o, \; n_r = \frac{\iota_r}{\iota_r + \iota_o} \text{ and } \; n_o = \frac{\iota_o}{\iota_r + \iota_o}.$$

Thus, we can solve all measures of firms,

$$n_{jk} = \frac{\iota_j \iota_j}{(\iota_r + \iota_o)^2}, \; j = r, \text{ and } \; k = r, o.$$

Therefore, the equilibrium conditions reduce to

$$L_s = a_s \iota_o n_o + \frac{\phi}{\lambda e}\left(\iota_o + \iota_r + \rho\right), \tag{18}$$

$$L_n = a_n \iota_r n_r + a_o \iota_o n_o, \tag{19}$$

$$n_r = \frac{\iota_r}{\iota_r + \iota_o} \quad \text{and} \quad n_o = \frac{\iota_o}{\iota_r + \iota_o}, \tag{20}$$

where $\phi = \frac{(1+\rho)a_s(1-s_s)a_n}{a_n - a_o(1-s_n)}$. Define η as the aggregate rate of innovation,

$$\eta \equiv \iota_o n_o + \iota_r n_r. \tag{21}$$

Then we get equations

$$a_s \iota_o n_o + \frac{\phi}{\lambda e}\left(\frac{\iota_o}{n_o} + \rho\right) - L_s = 0,$$
$$a_n \eta + \iota_o n_o \left(a_o - a_n\right) - L_n = 0 \quad \text{and}$$
$$\eta - 2\iota_o n_o + 2\iota_o - \frac{\iota_o}{n_o} = 0.$$

Total differentiation of the equations evaluating at $s_n = s_s = 0$ yields

$$A \begin{bmatrix} d\eta \\ d\iota_o \\ dn_o \end{bmatrix} = \begin{bmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & a_s \iota_o - \frac{\phi}{\lambda e}\frac{\iota_o}{n_o}\frac{1}{n_o} \\ a_n & n_o\left(a_o - a_n\right) & \iota_o\left(a_o - a_n\right) \\ 1 & -2n_o + 2 - \frac{1}{n_o} & -2\iota_o + \frac{\iota_o}{n_o}\frac{1}{n_o} \end{bmatrix} \begin{bmatrix} d\eta \\ d\iota_o \\ dn_o \end{bmatrix}$$
$$= \begin{bmatrix} -\left(\iota_o n_o + \frac{\phi\left(\frac{\iota_o}{n_o}+\rho\right)}{a_s \lambda e}\right) da_s + \frac{\phi\left(\frac{\iota_o}{n_o}+\rho\right)}{\lambda e} ds_s + \frac{a_o \phi\left(\frac{\iota_o}{n_o}+\rho\right)}{\lambda e\left(a_n - a_o\right)} ds_n + dL_s \\ dL_n \\ 0 \end{bmatrix},$$

and

$$\begin{aligned} |A| &= \frac{\iota_o}{n_o}\begin{vmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & a_s n_o - \frac{\phi}{\lambda e}\frac{1}{n_o} \\ a_n & n_o\left(a_o - a_n\right) & n_o\left(a_o - a_n\right) \\ 1 & -2n_o + 2 - \frac{1}{n_o} & -2n_o + \frac{1}{n_o} \end{vmatrix} = \frac{-2\iota_o}{n_o}\begin{vmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & \frac{\phi}{\lambda e}\frac{1}{n_o} \\ a_n & n_o\left(a_o - a_n\right) & 0 \\ 1 & -2n_o + 2 - \frac{1}{n_o} & 1 - \frac{1}{n_o} \end{vmatrix} \\ &= \frac{2a_n \iota_o}{n_o}\begin{vmatrix} a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & \frac{\phi}{\lambda e}\frac{1}{n_o} \\ -2n_o + 2 - \frac{1}{n_o} & 1 - \frac{1}{n_o} \end{vmatrix} + \frac{-2\iota_o}{n_o}\begin{vmatrix} a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & \frac{\phi}{\lambda e}\frac{1}{n_o} \\ n_o\left(a_o - a_n\right) & 0 \end{vmatrix} \\ &= \frac{2a_n \iota_o}{n_o}\left[a_s\left(n_o - 1\right) + \left(1 - \frac{1}{n_o} + \frac{a_o}{a_n}\right)\frac{\phi}{\lambda e}\right]. \end{aligned}$$

Let $X \equiv n_o - 1 + \left(1 - \frac{1}{n_o} + \frac{a_o}{a_n}\right)\frac{\phi}{\lambda e a_s}$. If $n_o = 1$ we get $X = \frac{(1+\rho)a_o}{(\lambda-1)(a_n-a_o)} > 0$. Thus, for some $n_o^* < 1$ we have $X > 0$. If $n_o \to 0$ we get $X \to -\infty$. Thus, for some $n_o^* > 1$ we have $X < 0$. Because that

$$\frac{dX}{dn_o} = 1 + \frac{1}{n_o^2}\frac{(1+\rho)\,a_n}{(\lambda-1)\,(a_n-a_o)} > 0,$$

there is a $0 < n_o^* < 1$ such that if $n_o = n_o^*$ we get $X = 0$. Thus, if $n_o > n_o^*$ then $|A| > 0$ and if $n_o < n_o^*$ then $|A| < 0$.

Given $X = 0$ or

$$n_o^2 + \left[\frac{(1+\rho)\,(a_n+a_o)}{(\lambda-1)\,(a_n-a_o)} - 1\right] n_o - \frac{(1+\rho)\,a_n}{(\lambda-1)\,(a_n-a_o)} = 0, \tag{22}$$

if we assume that $\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} > 1$ then there are one positive root and one negative root for n_o, and in terms of absolute values the positive root is smaller than the negative one. Thus, we have

$$n_o^* \equiv \frac{-\left[\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} - 1\right] + \sqrt{\left[\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} - 1\right]^2 + 4\frac{(1+\rho)a_n}{(\lambda-1)(a_n-a_o)}}}{2} > 0;$$

if we assume that $\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} < 1$ then there are one positive root and one negative root for n_o, and in terms of absolute values the positive root is larger than the negative root. Thus, we have

$$n_o^* \equiv \frac{-\left[\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} - 1\right] + \sqrt{\left[\frac{(1+\rho)(a_n+a_o)}{(\lambda-1)(a_n-a_o)} - 1\right]^2 + 4\frac{(1+\rho)a_n}{(\lambda-1)(a_n-a_o)}}}{2} > 0.$$

If $n_o^* = 0$, the LHS of (22) is less than zero; if $n_o^* = 1$, the LHS of (22) is equal to a_o. Thus, we get $n_o^* \in (0,1)$.

We derive some comparative statics as follows:

$$\frac{\partial \eta}{\partial L_s} = \frac{1}{|A|}\begin{vmatrix} 1 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & a_s \iota_o - \frac{\phi}{\lambda e}\frac{\iota_o}{n_o}\frac{1}{n_o} \\ 0 & n_o\,(a_o - a_n) & \iota_o\,(a_o - a_n) \\ 0 & -2n_o + 2 - \frac{1}{n_o} & -2\iota_o + \frac{\iota_o}{n_o}\frac{1}{n_o} \end{vmatrix} = -\frac{2}{|A|}\frac{\iota_o}{n_o}\left[\left(1 - \frac{1}{n_o}\right) n_o\,(a_o - a_n)\right],$$

$$\frac{\partial \eta}{\partial L_n} = \frac{1}{|A|}\begin{vmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & a_s \iota_o - \frac{\phi}{\lambda e}\frac{\iota_o}{n_o}\frac{1}{n_o} \\ 1 & n_o\,(a_o - a_n) & \iota_o\,(a_o - a_n) \\ 0 & -2n_o + 2 - \frac{1}{n_o} & -2\iota_o + \frac{\iota_o}{n_o}\frac{1}{n_o} \end{vmatrix} = -\frac{2}{|A|}\frac{\iota_o}{n_o}\left[a_s\,(1 - n_o) + \frac{\phi}{\lambda e}\left(\frac{1}{n_o} - 2\right)\right],$$

$$\frac{\partial \iota_o}{\partial L_s} = \frac{1}{|A|}\begin{vmatrix} 0 & 1 & a_s\iota_o - \frac{\phi}{\lambda e}\frac{\iota_o}{n_o}\frac{1}{n_o} \\ a_n & 0 & \iota_o\left(a_o - a_n\right) \\ 1 & 0 & -2\iota_o + \frac{\iota_o}{n_o}\frac{1}{n_o} \end{vmatrix} = \frac{-1}{|A|}\frac{\iota_o}{n_o}\left[\frac{a_n}{n_o} - \left(a_n + a_o\right)n_o\right],$$

$$\frac{\partial \iota_o}{\partial L_n} = \frac{1}{|A|}\begin{vmatrix} 0 & 0 & a_s\iota_o - \frac{\phi}{\lambda e}\frac{\iota_o}{n_o}\frac{1}{n_o} \\ a_n & 1 & \iota_o\left(a_o - a_n\right) \\ 1 & 0 & -2\iota_o + \frac{\iota_o}{n_o}\frac{1}{n_o} \end{vmatrix} = \frac{1}{|A|}\frac{\iota_o}{n_o}\left(\frac{\phi}{\lambda e}\frac{1}{n_o} - a_s n_o\right),$$

$$\frac{\partial n_o}{\partial L_s} = \frac{1}{|A|}\begin{vmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & 1 \\ a_n & n_o\left(a_o - a_n\right) & 0 \\ 1 & -2n_o + 2 - \frac{1}{n_o} & 0 \end{vmatrix} = \frac{1}{|A|}\left[-n_o\left(a_n + a_o\right) + a_n\left(2 - \frac{1}{n_o}\right)\right] \quad \text{and}$$

$$\frac{\partial n_o}{\partial L_n} = \frac{1}{|A|}\begin{vmatrix} 0 & a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o} & 0 \\ a_n & n_o\left(a_o - a_n\right) & 1 \\ 1 & -2n_o + 2 - \frac{1}{n_o} & 0 \end{vmatrix} = \frac{1}{|A|}\left(a_s n_o + \frac{\phi}{\lambda e}\frac{1}{n_o}\right).$$